MA 113 — Calculus I Fall 2010
Final Exam December 13, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: KEY

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		12
2		8
3		12
4		8
5		10
6		10
7		10
8		15
9		15
10		15
Extra Credit		10
Total		100

(1)

(1) Compute the following limits:

(a) $\lim_{x\to 1}\frac{\ln x}{\sin(\pi x)}$ is indeterminate of type $\frac{0}{0}$. By l'Hôpital's rule,

$$\lim_{x\to 1}\frac{\ln x}{\sin(\pi x)} \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \lim_{x\to 1}\frac{1/x}{\pi\cos(\pi x)} = \lim_{x\to 1}\frac{1}{\pi x\cos(\pi x)} \underset{\substack{\uparrow \\ \text{by cont.}}}{=} -\frac{1}{\pi}. \quad (1)$$

(2)

(b) $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$ is an indeterminate form of type $\infty\cdot 0$.

$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty}\frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$ is indeterminate of type $\frac{0}{0}$. (1)

By l'Hôpital's rule, $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) = \lim_{x\to\infty}\frac{-\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} +\pi\cos\left(\frac{\pi}{x}\right) \underset{\substack{\uparrow \\ \text{by cont.}}}{=} \pi$ (1)

(2)

(c) $\lim_{t\to\infty}(\sqrt{t^2+9}-t) = \lim_{t\to\infty}\frac{(\sqrt{t^2+9}-t)(\sqrt{t^2+9}+t)}{\sqrt{t^2+9}+t}$ (2)

$= \lim_{t\to\infty}\frac{(t^2+9)-t^2}{\sqrt{t^2+9}+t} = \lim_{t\to\infty}\frac{9}{\sqrt{t^2+9}+t} = 0,$

since $\lim_{t\to\infty}\sqrt{t^2+9} = \lim_{t\to\infty} t = \infty$. (2)

(a) $\lim_{x\to 1}\frac{\ln x}{\sin(\pi x)} =$ $-1/\pi$,

(b) $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right) =$ π,

(c) $\lim_{t\to\infty}(\sqrt{t^2+9}-t) =$ 0.

(2) (a) Let $f(x) = x^9 g(2x)$. If $f'(1) = 1$ and $g(2) = -1$, determine $g'(2)$.

$$f'(x) = 9x^8 g(2x) + 2x^9 g'(2x)$$, by product & chain rules 2

$$\Rightarrow \underbrace{f'(1)}_{=1} = 9\underbrace{g(2)}_{=-1} + 2g'(2) \quad \Rightarrow \quad g'(2) = 5.$$

1 1

(b) If $h(x) = \cos(\ln x - e^{x^2})$, compute $h'(x)$.

$$h'(x) = \underbrace{-\sin(\ln x - e^{x^2})}_{2} \cdot \underbrace{\left(\frac{1}{x} - 2xe^{x^2}\right)}_{2}$$, by

chain rule.

(a) $g'(2) =$ 5 ,

(b) $h'(x) =$ $-\sin(\ln x - e^{x^2}) \cdot \left(\frac{1}{x} - 2xe^{x^2}\right)$

(3) Compute the following integrals:

(a) $$\int\left(\sin x - \frac{1}{x^2+1} + 2e^x\right)dx = \underbrace{-\cos x}_{1}\ \underbrace{-\arctan x}_{1} + \underbrace{2e^x}_{1} + \underbrace{C}_{1}$$

(b) $$\int t\sqrt{t^2+4}\,dt \underset{\underbrace{\substack{u=t^2+4\\ du=2t\,dt}}_{1}}{\overset{\uparrow}{=}} \underbrace{\frac{1}{2}\int\sqrt{u}\,du}_{1} = \underbrace{\frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C}_{1} = \underbrace{\frac{1}{3}\left(t^2+4\right)^{3/2}+C}_{1}$$

(c) $$\int_1^x \frac{1-t\cos t}{t}\,dt = \underbrace{\int_1^x\left(\frac{1}{t}-\cos t\right)dt}_{1} = \underbrace{\left(\ln|t| - \sin t\right)\Big|_1^x}_{2}$$

$$= \ln|x| - \sin x - (\underbrace{\ln 1}_{0} - \sin 1) = \underset{1}{\ln|x| - \sin x + \sin 1}$$

(a) $\int\left(\sin x - \frac{1}{x^2+1} + 2e^x\right)dx = \underline{-\cos x - \arctan x + 2e^x}$, $+\ C$

(b) $\int t\sqrt{t^2+4}\,dt = \underline{\frac{1}{3}(t^2+4)^{3/2} + C}$,

(c) $\int_1^x \frac{1-t\cos t}{t}\,dt = \underline{\ln|x| - \sin x + \sin 1}$.

(4) Consider the curve described by the equation $e^{xy} = x - y$.

(a) Find the slope of the tangent line to the curve at the point $(0, -1)$.

• Differentiating w.r.t. x,

$$e^{xy} \cdot (y + xy') = 1 - y' \quad ②$$

$$\Leftrightarrow (xe^{xy} + 1)y' = 1 - ye^{xy}$$

$$\Leftrightarrow \quad y' = \frac{1 - ye^{xy}}{1 + xe^{xy}} . \quad ②$$

• The slope of the tangent to the curve @ $(0, -1)$ is

$$y' \Big|_{x=0,\, y=-1} = \frac{1+1}{1} = 2 . \quad ②$$

(b) Determine the equation of the tangent line to the curve at $(0, -1)$ in the form $y = mx + b$.

$$y + 1 = 2(x - 0)$$

$$\Leftrightarrow \quad y = 2x - 1 . \quad ②$$

(a) The slope is 2 ,

(b) The equation of the tangent line is $y = 2x - 1$.

(5) (a) Let $F(x) = \int_1^x \frac{t}{(t+3)^2}\, dt$. Determine the interval(s) on which $F(x)$ is increasing.

• By the fundamental th. of calculus, $F'(x) = \frac{x}{(x+3)^2}$. (2)

• $F(x)$ is increasing whenever $F'(x) > 0$, i.e., if $x > 0$.

$F(x)$ is therefore increasing on $(0, +\infty)$. (2)

(b) Let $G(x) = \int_x^3 \left(\frac{1-t}{t}\right)^2 dt$. Compute $G''(x)$.

$$G(x) = -\int_3^x \left(\frac{1-t}{t}\right)^2 dt. \quad (1)$$

By FTC, $G'(x) = -\left(\frac{1-x}{x}\right)^2$ (2)

$$\Rightarrow G''(x) = -2\left(\frac{1-x}{x}\right) \cdot \frac{-\cancel{x} - (1-\cancel{x})}{x^2}$$ by chain & quotient rules

$$= \frac{2(1-x)}{x^3}. \quad (3)$$

(a) $F(x)$ is increasing on $(0, \infty)$,

(b) $G''(x) =$ $2(1-x)/x^3$.

(6) Consider the function $f(x) = \frac{1}{x^2}$.

(a) Compute the Riemann sum for $f(x)$ on the interval $[1,5]$ with $n = 4$ subintervals and by taking the left endpoints as your sample points. (Give your answer as a rational number.)

$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$. The left endpoints are $1, 2, 3, \& 4$. (1)

Hence the Riemann sum for $f(x)$ is:

$L_4 = f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{205}{144}$. (1) (1)

(b) Determine whether $f(x)$ is increasing or decreasing on $[1,5]$.

$f'(x) = -\frac{2}{x^3} < 0$ for all x in $[1,5]$ (2)

$\Rightarrow f(x)$ is decreasing on $[1,5]$ – (1)

(c) Without computing the integral $\int_1^5 f(x)\,dx$, determine whether the Riemann sum of Part (a) overestimates or underestimates this integral.

- $f(x)$ is positive $\Rightarrow \int_1^5 f(x)\,dx$ is the area of the region under the graph of $f(x)$, above the x-axis, to the right of 1 & left of 5. (2)
- $f(x)$ is decreasing $\Rightarrow$ the Riemann sum, being the sum of areas of rectangles with heights f evaluated at the left endpoints, overestimates $\int_1^5 f(x)\,dx$. (2)

(a) The Riemann sum is 205/144,

(b) $f(x)$ is decreasing,

(c) The Riemann sum overestimates the integral.

(7) Let a and b be positive numbers whose product is 8. Find the minimum value of $a^2 + 2b$. Determine the values of a and b for which the minimum is attained. (As usual, **justify** your answers.)

$ab = 8$, where a & $b > 0 \Rightarrow b = \frac{8}{a}$

$\Rightarrow a^2 + 2b = a^2 + \frac{16}{a}$. 2

Hence, we want to minimize $F(a) = a^2 + \frac{16}{a}$, where $a > 0$. 1

Differentiating w.r.t. a, we get $F'(a) = 2a - \frac{16}{a^2}$. 1

The derivative equals 0 when $2a = \frac{16}{a^2} \Leftrightarrow a^3 = 8 \Leftrightarrow a = 2$.

a	0		2		$+\infty$
$F'(a)$		$-$		$+$	
$F(a)$		$\searrow$		$\nearrow$	

2 2

The absolute minimum of $F(a)$ is therefore attained @ $a = 2$ & $b = \frac{8}{a} = 4$, & equals $F(2) = 4 + \frac{16}{2} = 12$.

2

The minimum value of $a^2 + 2b$ is 12. It is attained if $a =$ 2 and $b =$ 4.

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) State both parts of the fundamental theorem of calculus. Use complete sentences and be sure to include all assumptions.

Let $f(x)$ be continuous on $[a,b]$. ①

- Then $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$, differentiable on (a,b), & $g'(x) = f(x)$ ②
- Moreover, $\int_a^b f(x)\,dx = F(b) - F(a)$, with $F(x)$ an antiderivative of $f(x)$.

②

(b) Compute the derivative of $F(x) = \int_0^x \sin(\sqrt{t})\,dt$ at $x = \frac{\pi^2}{4}$.

$$F'(x) = \sin(\sqrt{x}) \Rightarrow F'\left(\frac{\pi^2}{4}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

① ②

(c) Find the derivative of $G(x) = \int_1^{x^2} (\ln t)^3\,dt$.

$$G(x) = \int_1^u (\ln t)^3\,dt, \text{ with } u = x^2. \quad ②$$

$$\Rightarrow G'(x) = \frac{dG}{dx} \underset{\substack{\uparrow \\ \text{(chain rule)}}}{=} \frac{dG}{du} \cdot \frac{du}{dx} \underset{\substack{\uparrow \\ \text{(FTC 1)}}}{=} (\ln u)^3 \cdot 2x = 2x(\ln(x^2))^3$$

② ② ①

(b) $F'(x) =$ $\sin(\sqrt{x})$,

(c) $G'(x) =$ $2x(\ln(x^2))^3$.

(9) (a) What does it mean for a function $f(x)$ to be continuous at a point a? Use complete sentences.

$f(x)$ is continuous @ $x=a$ if $\lim_{x\to a} f(x) = f(a)$.

3

(b) Consider the piecewise defined function

$$f(x) = \begin{cases} 2 & \text{if } 0 \leq x \leq 1, \\ ax+3 & \text{if } 1 < x \leq 2, \\ x^2 - 2x + b & \text{if } 2 \leq x \leq 3, \end{cases}$$

where a and b are constants. Find the values of a and b for which $f(x)$ is continuous on $[0,3]$.

- Since $f(x)$ is continuous on $[0,1)$, $(1,2)$, & $(2,3]$,

$\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = a+3$, 2

$\lim_{x\to 2^-} f(x) = 2a+3$, & $\lim_{x\to 2^+} f(x) = 4-4+b = b$.

2

- For $f(x)$ to be cont. @ 1, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$ 2

$\Leftrightarrow 2 = a+3 \Leftrightarrow a = -1$. 2

Similarly, for $f(x)$ to be cont. @ 2, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$ 2

(b) $a =$ ___-1___ and $b =$ ___1___.

$\Leftrightarrow 2a+3 = b \Leftrightarrow b = 1$.

2

(10) A particle is moving along the x axis with velocity $v(t) = 3t^2 + 3t - 6$ at time t (measured in meters per second).

(a) Find the acceleration $a(t)$ of the particle at time $t = 2$.

$$a(t) = v'(t) = 6t + 3 \quad (m/s^2)$$

total ③

$$\Rightarrow a(2) = 15\ m/s^2 \quad ②$$

units. ①

(b) Is the particle speeding up or slowing down at time $t = 2$? Justify your answer.

$$v(2) = 12\ m/s > 0 \ \& \ a(2) = 15\ m/s^2 > 0$$

total ③

units ①

Since $v(2)$ & $a(2)$ are of the same sign, the particle is speeding up. ②

(c) Find all times t at which the particle changes direction. Again, justify your answer.

The particle changes dir. when the velocity changes sign. ①

$v(t) = 3(t-1)(t+2) = 0 \Leftrightarrow t = 1$ or $t = -2$. ① Since $t > 0$, we retain $t = 1$ s. only. ①

Hence, the particle changes dir. @ $t = 1$s. ①

total ④

(d) Find the total distance traveled by the particle during the first 3 seconds.

total distance traveled between $t = 0$ & $t = 3$s. is

$$\left|\int_0^1 v(t)\,dt\right| + \left|\int_1^3 v(t)\,dt\right| = \left|\left(t^3 + \tfrac{3}{2}t^2 - 6t\right)\Big|_0^1\right| + \left|\left(t^3 + \tfrac{3}{2}t^2 - 6t\right)\Big|_1^3\right|$$

② ②

$$= \left|-\frac{7}{2}\right| + |26| = \frac{7}{2} + 26 = \frac{59}{2} = 29.5\ m.$$

①

total ⑤

(a) $a(2) =$ 15 m/s^2,

(b) the particle is speeding up,

(c) the particle changes direction at $t = 1$s.,

(d) the total distance traveled is 29.5 m.

Extra Credit Problem:

Check the correct answers below. For each correct answer you earn 2 points, and for each incorrect answer 1 point will be subtracted. Therefore, it might be wise to skip a question rather than risking losing a point. However, your final score on this problem will not be negative! You need not justify your answer.

True	False	
☐	☑	If $f(2) < 0$ and $f(5) > 0$, then there exists a number c between 2 and 5 such that $f(c) = 0$.
☐	☑	If $f'(c) = 0$, then $f(x)$ has a local maximum or minimum at $x = c$.
☑	☐	If f is differentiable function and $f(-2) = f(2)$, then there exists a number c such that $\lvert c\rvert < 2$ and $f'(c) = 0$.
☑	☐	$\int \ln x \, dx = x \ln x - x + C$.
☑	☐	$\int_{-3}^{3} (ax^2 + c) \, dx = 2 \int_{0}^{3} (ax^2 + c) \, dx$ for every choice of a and c.