Worksheet # 14: Implicit differentiation and Inverse Functions

1. Find the derivative of y with respect to x:
   (a) $\ln(xy) = \cos(y^4)$.
   (b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.
   (c) $\sin(xy) = \ln\left(\frac{x}{y}\right)$.

2. Consider the ellipse given by the equation $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{81} = 1$.
   (a) Find the equation of the tangent line to the ellipse at the point (u, v) where $u = 4$ and $v > 0$.
   (b) Sketch the ellipse and the line to check your answer.

3. Find the derivative of $f(x) = \pi^{\tan^{-1}(\omega x)}$, where ω is a constant.

4. Let (a, b) be a point in the circle $x^2 + y^2 = 144$. Use implicit differentiation to find the slope of the tangent line to the circle at (a, b).

5. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

6. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

7. Find the derivative
$$\frac{d}{dx}\left((\sqrt{2})^{-\ln(2x)}\right)$$

8. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. Use this information to find $G'(2)$.
$$G(x) = \frac{x}{4f(x)}$$

9. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

10. Use implicit differentiation to find the derivative of $\tan^{-1}(x)$. Thus if $x = \tan(y)$, use implicit differentiation to compute dy/dx. Can you simplify to express dy/dx in terms of x?

11. (a) Compute $\dfrac{d}{dx}\sin^{-1}(\cos(x))$.
    (b) Compute $\frac{d}{dx}\left(\sin^{-1}(x) + \cos^{-1}(x)\right)$. Give a geometric explanation as to why the answer is 0.
    (c) Compute $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}(x)\right)$ and simplify to show that the derivative is 0. Give a geometric explanation of your result.

12. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

13. Let f be defined by $f(x) = e^{-x^2}$.
    (a) For which values of x is $f'(x) = 0$
    (b) For which values of x is $f''(x) = 0$