Worksheet #27: Exponential growth and decay

1. Solve the following equations for α:

   (a) $500 = 1000e^{20\alpha}$

   (b) $40 = \alpha e^{10k}$, where $k = \frac{\ln(2)}{7}$.

   (c) $100,000 = 40,000e^{0.06\alpha}$.

   (d) $\alpha = 2,000e^{36k}$, where $k = \frac{\ln(0.5)}{18}$.

2. The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

3. A lucky colony of rabbits is brought to a large island where there are no predators and unlimited food. Under these conditions, they will reproduce at such a rate that the population doubles every 9 years. After 3 years, a team of scientists determines that there are 7000 rabbits on the island. How many rabbits were brought to the island originally? How many rabbits will there be 12 years after their introduction to the island?

4. A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially and 800 after 24 hours, how many cells will there be after a further 12 hours?

5. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.

$$\frac{dP}{dt} = KP$$

   Where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$

   (a) Give the units of K.

   (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$. How many mosquitoes will there be after 90 days?

6. (a) Show that $\lim_{h\to 0} \ln\left((1+hr)^{t/h}\right) = rt$. (Hint: Use L'Hospital).

   (b) From (a) show that $\lim_{h\to 0} \left((1+hr)^{t/h}\right) = e^{rt}$. (Hint: Use your previous result and properties of logarithms.)

   (c) The compound interest formula is

$$P(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

   where $P(t)$ is the value at time t, A_0 is the initial investment, r is the interest rate, t is the time in years, and n is the number of times compounded per year. Verify that

$$\lim_{h\to 0} \left((1+hr)^{t/h}\right) = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt}$$

   (Hint: Use the change of variable $h = \frac{1}{n}$)

   Use part (b) to derive a formula involving e for continuously compounded interest.

   (Hint: How can you represent continuously compounded interest as a limit where the interval of time between each compounding decreases to zero and the number of compoundings increases to infinity?)

7. (Review) Define $H(x)$ as follows

$$H(x) := \int_{x^2}^{\cos(x)} f(t)\,dt$$

Find the derivative of $H(x)$ with respect to x.

8. (Review)

(a) Evaluate the indefinite integral

$$\int (2x)^3 \sqrt{(2x)^2 + 5}\ dx$$

(b) Let k, r and m be non zero constants. Evaluate the integral

$$\int (kx)^{2m-1} \sqrt{(kx)^m + r}\ dx$$