Worksheet # 1: Precalculus review: functions and inverse functions

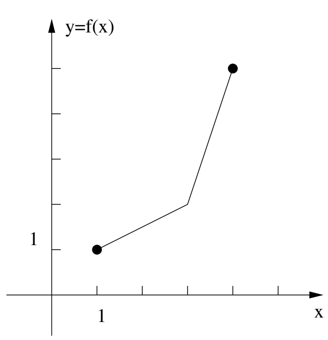
Goals for MA 113 Recitations: Recitations are not homework help sessions! There are three goals for students in MA 113 recitations:

1. to develop your ability to make sense of problems and be persistent while solving them,
2. to develop your ability to productively collaborate with peers, and
3. to develop your ability to check your own work, i.e. to decide on your own whether or not your work is correct.

To help you reach these goals, you will spend the majority of the recitation working in small groups on worksheets. You are not expected to complete all of these problems — your TA will help guide you in selecting which problems to work on. Your focus should be to *discover your misunderstandings* by doing math collaboratively. Mistakes and misunderstandings are a normal part of learning mathematics — the only path to deep learning is to learn to effectively identify and revise our mistakes and misunderstandings.

Solutions to MA 113 worksheets are not provided. Instead, you should focus on using these problems to test your self-evaluation skills. Imagine you are taking an exam, and you need to check for yourself whether or not your work is correct — this is a skill you need to practice in order to do well! By collaborating with your peers and comparing solutions, with guidance and support from your TA, your problem solving and self-evaluation skills will improve. If there are worksheet problems that you are uncertain about, you are welcome to ask about them during recitation, during your TA or instructor office hours, at the Mathskeller, or at the Study.

1. Find the domain and range of $f(x) = \dfrac{x+2}{x^2+x-6}$.
2. For each of the following conditions, find the equation of the line that satisfies those conditions.
   (a) the line passes through the point $(1, 2)$ with slope 5.
   (b) the line passes through the points (π, π) and $(-6, -3)$.
   (c) the line has y-intercept 5 and has slope -3.
3. Let f be a linear function with slope m where $m \neq 0$. What is the slope of the inverse function f^{-1}? Why is your answer correct?
4. Consider the function whose graph appears below.



   (a) Find $f(3)$, $f^{-1}(2)$ and $f^{-1}(f(2))$.
   (b) Give the domain and range of f and of f^{-1}.
   (c) Sketch the graph of f^{-1}.
5. If $f(x) = 2x+3$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. Are the functions $f \circ g$ and $g \circ f$ the same function? Why or why not?

6. Let $f(x) = 2^{\cos(2(x-1)^2+7)} - 9$.

   (a) Can you find functions g and h such that $f = g \circ h$?

   (b) Can you find functions g, h, and s such that $f = g \circ h \circ s$?

   (c) Can you do this with four functions? Five functions? What is the largest number of functions you can find so that f can be written as a composition of those functions?

7. Let $f(x) = 2 + \dfrac{1}{x-3}$. Determine the inverse function of f, which we write as f^{-1}. Give the domain and range of f and the inverse function f^{-1}. Verify that $f \circ f^{-1}(x) = x$.

8. A ball is thrown in the air from ground level. The height of the ball in meters at time t seconds is given by the function $h(t) = -4.9t^2 + 20t$. At what time does the ball hit the ground? (Be sure to use the proper units!)

9. True or False: (justify your answer!)

   (a) Every function has an inverse.

   (b) If $f \circ g(x) = x$ for all x in the domain of g, then f is the inverse of g.

   (c) If $f \circ g(x) = x$ for all x in the domain of g and $g \circ f(x) = x$ for all x in the domain of f, then f is the inverse of g.

   (d) The function $f(x) = \sin(x)$ is one to one.

   (e) The function $f(x) = 1/(x+2)^3$ is one to one.

10. We form a box by removing squares of side length x centimeters from the four corners of a rectangle of width 100 cm and length 150 cm and then folding up the flaps between the squares that were removed. a) Write a function which gives the volume of the box as a function of x. b) Give the domain for this function.

11. Create a function that is the composition of ten functions. Can you do this in a "sneaky" way so that it is hard for someone else to figure out the ten functions you used? (Hint: try using different compositions of $g(x) = x + 1$ and $h(x) = 2x + 3$, for example $f = g \circ h \circ g \circ h \circ h$. What happens?)

Worksheet # 2: The Exponential Function and the Logarithm

An Interesting Fact: The first book that gave a comprehensive discussion of both differential and integral calculus was written in 1748 by Maria Agnesi, an Italian philospher, theologian, humanitarian, and mathematician. This work was influential throughout Europe, and resulted in her election to the Bologna Academy of Sciences. Agnesi was the first woman appointed as a mathematics professor at a university.

1. Many students find statements like $2^0 = 1$ and $2^{1/3} = \sqrt[3]{2}$ a bit mysterious, even though most of us have used them for years, so let's start there. Write down the list of numbers $2^1 = 2$, $2^2 = 2 \times 2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, thus

$$2^1, 2^2, 2^3, 2^4, 2^5, \ldots .$$

   (a) What do you multiply by to get from a number on this list to the next number to the right? Starting from any number *except* 2^1, what do you divide by to get from that number to the previous number on the left?

   (b) If we start at 2^1 and move to the left following this patten, it suggests how we should define 2^0. What do you get for 2^0 if you follow the pattern?

   (c) If we now move from 2^0 another number to the left following the pattern, it suggests how we should define 2^{-1}, and then 2^{-2}, etc. What do you get for these values if you follow the pattern?

   (d) Do these patterns help you make sense of the rule $2^{a+b} = 2^a \times 2^b$? Discuss this with the students in your group. (Bonus question: discuss whether or not these patterns and this rule help us make sense of the equations $2^{1/2} = \sqrt{2}$ and $2^{1/3} = \sqrt[3]{2}$.)

   (e) Does it matter for your reasoning that the base of the exponential was the number 2? Why or why not?

2. Find the solutions to the following computational problems by using properties of exponentials and logarithms.

   (a) Solve $10^{2x+1} = 100$.

   (b) Solve $2^{(x^2)} = 16$.

   (c) Solve $2^x = 4^{x+2}$.

   (d) Find $\log_2(8)$.

   (e) Find $\ln(e^2)$.

   (f) Solve $e^{3x} = 3$.

   (g) Solve $4^x = e$.

   (h) Solve $\ln(x + 1) + \ln(x - 1) = \ln(3)$. Be sure to check your answer.

3. (a) Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ and give the domains and range of each function.
   (b) Determine if each function is one-to-one. Determine if each function is increasing or decreasing.
   (c) Graph the inverse function to f. Give the domain and range of the inverse function.

4. Since e^x and $\ln(x)$ are inverse functions, we can write $\heartsuit = e^{\ln(\heartsuit)}$ if the value of $\heartsuit$ is positive. This is super useful when dealing with exponential functions with complicated bases, because then you can use log laws to simplify the exponent on e. But, you have to be careful when you apply this rule, as the following examples show.

   (a) Explain why $b = e^{\ln(b)}$ is only true when $b > 0$. (Hint: think about the domain of natural log.)
   (b) Explain why for any $b > 0$, we have $b^a = e^{a\ln(b)}$.
   (c) Explain why $(\cos(x) + 3)^{\sin(x)+2} = e^{(\sin(x)+2)\ln(\cos(x)+3)}$ is true.
   (d) Explain why $\cos(x)^{\sin(x)} = e^{\sin(x)\ln(\cos(x))}$ is not true.
   (e) Let f be the function $f(x) = 4^x$. Find the value of k that allows you to write the function f in the form $f(x) = e^{kx}$.
   (f) Let f be the function $f(x) = 5 \cdot 3^x$. Find a k that allows you to write the function f in the form Ae^{kx}.

5. Evaluate the expressions $4^{(3^2)}$ and $(4^3)^2$. Are they equal?

6. Suppose a and b are positive real numbers and $\ln(ab) = 3$ and $\ln(ab^2) = 5$. Find $\ln(a)$, $\ln(b)$, and $\ln(a^3/\sqrt{b})$.

7. Suppose that a population doubles every two hours. If we have one hundred critters at 12 noon, how many will there be after 1 hour? after 2 hours? How many were there at 11am? Give a formula for the number of critters at t hours after 12 noon.

8. Suppose that f is a function of the form $f(x) = Ae^{kx}$. If $f(2) = 20$ and $f(5) = 10$, will we have $k > 0$ or $k < 0$? Find A and k so that $f(2) = 20$ and $f(5) = 10$.

9. The number e is mysterious and arises in many different ways.

   (a) Use your calculator to compute $(1 + \frac{1}{n})^n$ for n equal to $1, 2, 3, \ldots$. Compute this for larger and larger n until it does not make a difference in the decimal you get. What is the value you reach?
   (b) Use your calculator to compute the sums

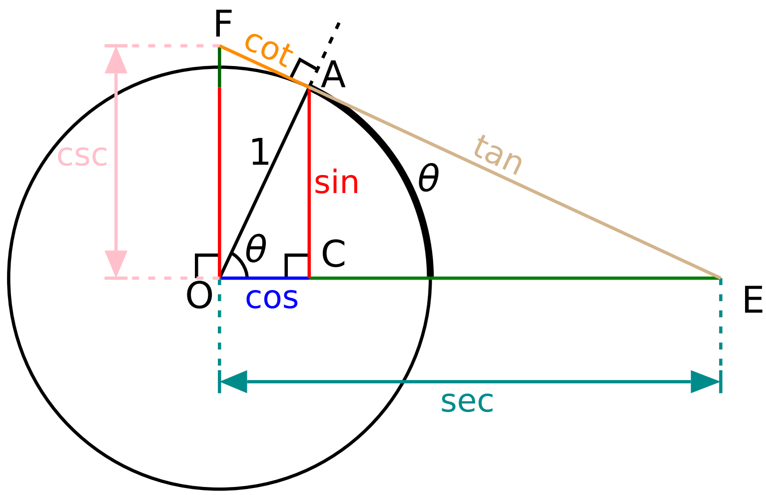
$$\frac{1}{1} + \frac{1}{1\cdot 2},$$
$$\frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3},$$
$$\frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4},$$

   and so on, until it no longer makes a difference in the decimal you get. What is the value you reach?
   (c) The first two parts of this problem showed you two ways to approximate the value of e. The first way involved limits, which we will talk about soon in this course. The second involved "infinite series," which is a topic covered in Calculus II. As we will see later in this course, there is another way to define e — we will see that the area of the region enclosed by the line $x = 1$, the x-axis, the graph of $y = 1/x$, and the line $x = a$ is equal to $\ln(a)$. So, the value of a for which this area is 1 is $a = e$. Amazing!

Worksheet # 3: Review of Trigonometry

An Interesting Fact: The study of trigonometry began in both ancient Greece and ancient India, and these two traditions were merged into roughly our modern form by Islamic mathematicians, notably Abu Nasr Mansur, around the year 1000. Islamic mathematicians established the connection between trig functions and the unit circle, and systematically studied the six fundamental trig functions.

> The etymology of the word 'sine' is instructive, for it shows what can happen as a result of imperfect linguistic and cultural filtering. The Sanskrit term for sine...was *jya-ardha* (half chord), which was later abbreviated to *jya*. From this came the phonetically derived Arabic word *jiba*...written as *jyb*. Early Latin translators, coming across this word, mistook it for anothe word, *jaib*...which was translated as *sinus*, which in Latin had a number of meanings...And hence the present word 'sine'. — George Gheverghese Joseph.



1. The key to understanding trig functions is to understand the unit circle — given an angle θ between 0 and $\pi/2$ (measured in radians!), each of the six trig functions measures a length related to the unit circle.

   (a) The word *radian* is an abbreviation of the phrase "radial angle." In a circle of radius r, one radian is defined to be the angle given by an arc of the circle having length r. Draw the unit circle in the plane and for each $m = 1, 2, 3, 4, 5, 6$, place a dot at the *approximate* point on the circle that is m radians counterclockwise from the point $(1, 0)$ (you won't be able to measure this precisely, just estimate it as best you can).

   (b) Recall that the definition of π is the ratio of the circumference to the diameter in a circle. Use your picture from the previous problem to explain why the value of π is greater than 3 but less than 3.5.

   (c) Define the functions $\sin(\theta)$ and $\cos(\theta)$ to be the lengths of the arcs AC and OC, respectively, on the diagram above. Explain why this definition of $\sin(\theta)$ and $\cos(\theta)$ agrees with the usual triangle-based definitions, i.e. that $\sin(\theta)$ is equal to "opposite" over "hypotenuse" in a right triangle.

   (d) Use different pairs of similar triangles to explain why each of the functions $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, and $\sec(\theta)$ measure the correspondingly labeled length in the picture above.

(e) Explain why $\sin^2(\theta)+\cos^2(\theta) = 1$ is equivalent to the Pythagorean theorem applied to the triangle OAC above, and why $1 + \tan^2(\theta) = \sec^2(\theta)$ is equivalent to the Pythagorean theorem applied to the triangle OAE above.

(f) For the unit circle, the radial angle of θ corresponds to an arc of length θ. Inverse trig functions are sometimes written $\arcsin(x)$ and $\arccos(x)$. Discuss with the other students in your group why it makes sense that the function $\sin(\theta)$ gives the length of the vertical line AC in the diagram above, while the function $\arcsin(x)$ is equal to the length of the circular arc for which the vertical line AC has length x; conclude that if $\sin(\theta) = x$, then $\arcsin(x) = \theta$. Discuss arccos similarly.

2. When θ is not between 0 and $\pi/2$, then we extend the definition of the trig functions as you have seen in your previous courses, allowing us to answer the following questions.

   (a) Suppose that $\sin(\theta) = 5/13$ and $\cos(\theta) = -12/13$. Find the values of $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, $\sec(\theta)$, and $\tan(2\theta)$.

   (b) If $\pi/2 \leq \theta \leq 3\pi/2$ and $\tan\theta = 4/3$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

   (c) Find all solutions of the equations (a) $\sin(x) = -\sqrt{3}/2$ and (b) $\tan(x) = 1$.

   (d) A ladder that is 6 meters long leans against a wall so that the bottom of the ladder is 2 meters from the base of the wall. Make a sketch illustrating the given information and answer the following questions: How high on the wall is the top of the ladder located? What angle does the top of the ladder form with the wall?

3. Let O be the center of a circle whose circumference is 48 centimeters. Let P and Q be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments OQ and OP. Express your answer in radians.

   Find the distance between P and Q.

4. Show that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

5. Simplify the expressions

   (a) $\tan(\sin^{-1}(x))$

   (b) $\sin(\tan^{-1}(x))$

   (c) $\sin(2\cos^{-1}(x))$

6. Find the exact values of the following expressions. Do not use a calculator.

   (a) $\tan^{-1}(1)$

   (b) $\tan(\tan^{-1}(10))$

   (c) $\sin^{-1}(\sin(7\pi/3))$

   (d) $\tan(\sin^{-1}(0.8))$

   (e) $\cos(\sin^{-1}(-0.6))$

7. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

   (a) $\sqrt{3}\cos(x) + 2\tan(x)\cos^2(x) = 0$

   (b) $3\cot^2(x) = 1$

   (c) $2\cos(x) + \sin(2x) = 0$

   (d) $\sin x = \tan x$

   (e) $2 + \cos(2x) = 3\cos x$

   (f) $2\sin^2(x) = 1$

Worksheet # 4: Average and Instantaneous Velocity; Limits

1. A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$.

   (a) How far does the ball travel during the time interval $[1, 3]$?

   (b) Compute the ball's average velocity over the time interval $[1, 3]$.

   (c) Graph the curve $y = h(t)$ and the line between the points $(1, h(1))$ and $(3, h(3))$. How does the slope of this line relate to your answer in part (b)?

   (d) Compute the ball's average velocity over the time intervals $[1, 1.01]$, $[1, 1.001]$,$[0.99, 1]$, and $[0.999, 1]$.

   (e) Estimate the instantaneous velocity when $t = 1$.

2. A particle moves along a line and its position $p(t)$ in meters after time t seconds is given by the following table.

t	0.0	0.2	0.5	0.65	0.9	1.1	1.15	1.3
$p(t)$	3	4.2	5.7	8.8	7.6	8.0	9.0	9.5

   (a) Describe the motion of the particle between 0 seconds and 1.3 seconds. Justify your description by representing the table as points $(t, p(t))$ plotted in the plane.

   (b) Consider the average velocity of the particle across different time intervals. Based on your calculations, can you conclude at what point in time the particle is moving with the largest positive *instantaneous* velocity? Why or why not? Based on this data, when do you believe that the particle is likely to be moving with the largest positive instantaneous velocity? Why?

3. Let $p(t) = t^3 - 45t$ denote the distance (in meters) to the right of the origin of a particle at time t minutes after noon.

   (a) Find the average velocity of the particle on the intervals $[2, 2.1]$ and $[2, 2.01]$.

   (b) Use this information to guess a value for the instantaneous velocity of particle at 12:02pm.

4. A particle is moving along a straight line so that its position at time t seconds is given by $s(t) = 4t^2 - t$ meters.

   (a) Find the average velocity of the particle over the time interval $[1, 2]$.

   (b) Determine the average velocity of the particle over the time interval $[2, t]$ where $t > 2$. Simplify your answer. [Hint: Factor the numerator.]

   (c) Based on your answer in (b) can you guess a value for the instantaneous velocity of the particle at $t = 2$?

5. For each task or question below, provide a specific example of a function $f(x)$ that supports your answer.

   (a) In words, briefly describe what "$\lim_{x\to a} f(x) = L$" means.

   (b) In words, briefly describe what "$\lim_{x\to a} f(x) = \infty$" means.

   (c) Suppose $\lim_{x\to 1} f(x) = 2$. Does this imply $f(1) = 2$?

   (d) Suppose $f(1) = 2$. Does this imply $\lim_{x\to 1} f(x) = 2$?

6. Compute the value of the following functions near the given $x-$value. Use this information to guess the value of the limit of the function (if it exists) as x approaches the given value.

(a) $f(x) = 2^{x-1} + 3$, $x = 1$

(b) $f(x) = \frac{\sin(2x)}{x}$, $x = 0$

(c) $f(x) = \sin\left(\frac{\pi}{x}\right)$, $x = 0$

(d) $f(x) = \frac{x^2-3x+2}{x^2+x-6}$, $x = 2$

7. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x-1 & \text{if } 0 < x \text{ and } x \neq 2 \\ -3 & \text{if } x = 2 \end{cases}$.

   (a) Sketch the graph of f.

   (b) Compute the following:

      i. $\lim_{x\to 0^-} f(x)$

      ii. $\lim_{x\to 0^+} f(x)$

      iii. $\lim_{x\to 0} f(x)$

      iv. $f(0)$

      v. $\lim_{x\to 2^-} f(x)$

      vi. $\lim_{x\to 2^+} f(x)$

      vii. $\lim_{x\to 2} f(x)$

      viii. $f(2)$

8. In the following, sketch the functions and use the sketch to compute the limit.

   (a) $\lim_{x\to\pi} x$

   (b) $\lim_{x\to 3} \pi$

   (c) $\lim_{x\to a} |x|$

   (d) $\lim_{x\to 3} 2^x$

Worksheet # 5: Limit Laws

An Interesting Fact: Mathematicians did not use a formal theory of limits between the invention of calculus in the 1660's and the formal definition of a limit in the 1820's. Even after the 1820's, mathematicians and scientists wrote lim without writing $x \to a$ below it. It appears that the widespread use of $\lim_{x \to a}$ was only adopted in the early 1900's after being used in several books, including one by G. H. Hardy titled "A Course of Pure Mathematics."

Remark on Notation: When working through a limit problem, your answers should be a chain of true equalities. Make sure to keep the $\lim_{x \to a}$ operator until the very last step.

1. Show that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist by examining one-sided limits. Then sketch the graph of $\frac{|h|}{h}$ to verify your reasoning.
2. Compute the following limits or explain why they fail to exist.

   (a) $\lim_{x \to -3^+} \frac{x+2}{x+3}$

   (b) $\lim_{x \to -3^-} \frac{x+2}{x+3}$

   (c) $\lim_{x \to -3} \frac{x+2}{x+3}$

   (d) $\lim_{x \to 0^-} \frac{1}{x^3}$

3. In the theory of relativity, the mass of a particle with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

   where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

4. Given $\lim_{x \to 2} f(x) = 5$ and $\lim_{x \to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit.

   (a) $\lim_{x \to 2} f(x)^2 + x \cdot g(x)^2$

   (b) $\lim_{x \to 2} \frac{f(x) - 5}{g(x) - 2}$

   (c) $\lim_{x \to 2} \frac{f(x)g(x)}{x}$

   (d) $\lim_{x \to 2} (f(x)g(2))$

5. For each limit, evaluate the limit or or explain why it does not exist. Use the limit rules to justify each step. It is good practice to sketch a graph to check your answers.

   (a) $\lim_{x \to 2} \frac{x+2}{x^2-4}$

   (b) $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{3}{x^2-x-2} \right)$

   (c) $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$

   (d) $\lim_{x \to 0} \frac{(2+x)^3 - 8}{x}$

6. Let $f(x) = 1 + x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. Consider $\lim_{x \to 0} f(x)$.

   (a) Find two simpler functions, g and h, that satisfy the hypothesis of the Squeeze Theorem.

   (b) Determine $\lim_{x \to 0} f(x)$ using the Squeeze Theorem.

   (c) Use a calculator to produce a graph that illustrates this application of the Squeeze Theorem.

Worksheet # 6: Continuity; Limits at Infinity & Asymptotes

1. For each of the following tasks/problems, provide a specific example of a function $f(x)$ that supports your answer.
   (a) State the definition of continuity.
   (b) List the three things required to show f is continuous at a.
   (c) What does it mean for $f(x)$ to be continuous on the interval $[a, b]$? What does it mean to say only that "$f(x)$ is continuous"?
   (d) Identify the three possible types of discontinuity of a function at a point. Provide a sketch of each type.
   (e) True or false? Every function is continuous on its domain.
   (f) True or false? The sum, difference, and product of continuous functions are all continuous.
   (g) If $f(x)$ is continuous at $x = a$, what can you say about $\lim_{x\to a^+} f(x)$?

2. Show that the following functions are continuous at the given point a.
   (a) $f(x) = \pi$, $a = 1$
   (b) $f(x) = \dfrac{x^2 + 3x + 1}{x + 3}$, $a = -1$
   (c) $f(x) = \sqrt{x^2 - 9}$, $a = 4$

3. Give the intervals of continuity for the following functions.
   (a) $f(x) = \dfrac{x + 1}{x^2 + 4x + 3}$
   (b) $f(x) = \dfrac{x}{x^2 + 1}$
   (c) $f(x) = \sqrt{2x - 3} + x^2$
   (d) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ x + 1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \ge 2 \end{cases}$

4. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.
   (a) Find all numbers c such that $\lim_{x\to 1} f(x)$ exists.
   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

5. Find parameters a and b so that the following function is continuous

$$f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \le -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \le x \end{cases}$$

6. Suppose that

$$f(x) = \begin{cases} \dfrac{x-6}{|x-6|} & \text{if } x \ne 6, \\ 1 & \text{if } x = 6 \end{cases}$$

   Determine the points at which the function $f(x)$ is discontinuous and state the type of discontinuity.

7. State the Intermediate Value Theorem. Show $f(x) = x^3 + x - 1$ has a zero in the interval $(0, 1)$.

8. Using the Intermediate Value Theorem, find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

9. Let $f(x) = \dfrac{e^x}{e^x - 2}$.

   (a) Show that $f(0) < 1 < f(\ln(4))$.

   (b) Can you use the Intermediate Value Theorem to conclude that there is a solution of $f(x) = 1$?

   (c) Can you find a solution to $f(x) = 1$?

10. (a) Show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$.

    (b) Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.

    (c) Continue in this manner to find an interval of length $1/8$ which contains a solution of the equation $xe^x = 2$.

11. Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.

12. Explain the difference between "$\lim_{x\to-3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".

13. Evaluate the following limits, or explain why the limit does not exist:

    (a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 7x}{x - 8}$

    (b) $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 6}{x^4 - 8x + 9}$

    (c) $\displaystyle\lim_{x\to-\infty} \frac{x}{x^6 - 4x^2}$

    (d) $\displaystyle\lim_{x\to-\infty} 3$

    (e) $\displaystyle\lim_{x\to\pm\infty} \frac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$

    (f) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^{10} + 2x}}{x^5}$

14. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \left(\dfrac{x^2}{x+1} - \dfrac{x^2}{x-1}\right)$.

15. Sketch a graph with all of the following properties:

    - $\lim_{t\to\infty} f(t) = 2$
    - $\lim_{t\to-\infty} f(t) = 0$
    - $\lim_{t\to0^+} f(t) = \infty$
    - $\lim_{t\to0^-} f(t) = -\infty$
    - $\lim_{t\to4} f(t) = 3$
    - $f(4) = 6$

Worksheet # 7: Derivatives

1. Comprehension check:

   (a) What is the definition of the derivative $f'(a)$ at a point a?

   (b) What is the geometric meaning of the derivative $f'(a)$ at a point a?

   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$?

2. Write each of these questions as a problem about the graph of $h(t)$, secant lines to the graph of $h(t)$, and/or tangent lines to the graph of $h(t)$.

   A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$.

   (a) How far does the ball travel during the time interval $[1, 3]$?

   (b) Compute the ball's average velocity over the time interval $[1, 3]$.

   (c) Compute the ball's average velocity over the time intervals $[1, 1.01]$, $[1, 1.001]$,$[0.99, 1]$, and $[0.999, 1]$.

   (d) Estimate the instantaneous velocity when $t = 1$.

3. (a) Find a function f and a number a so that the following limit represents a derivative $f'(a)$.

   $$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h}$$

   (b) Using your function f, set $h = -2$ and draw the graph of f and the secant line whose slope is given by $\frac{(4+h)^3-64}{h}$.

   (c) Create a real-world scenario that is modeled by f, and write a problem about this scenario for which the answer is given by $\lim_{h\to 0} \dfrac{(4+h)^3 - 64}{h}$.

4. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

5. The point $P = (3, 1)$ lies on the curve $y = \sqrt{x-2}$.

   (a) If Q is the point $(x, \sqrt{x-2})$, find a formula for the slope of the secant line PQ.

   (b) Using your formula from part (a) and a calculator, find the slope of the secant line PQ for the following values of x (do not round until you get to the final answer):

   2.9, 2.99, 2.999, 3.1, 3.01, and 3.001

   TI-8x Calculator Tip: Enter the formula under "y=" and then use "Table".

   (c) Using the results of part (b), guess the value of the slope of the tangent line to the curve at $P = (3, 1)$.

   (d) Verify that your guess is correct by computing an appropriate derivative.

   (e) Using the slope from part (d), find the equation of the tangent line to the curve at $P = (3, 1)$.

6. Let

   $$g(t) = \begin{cases} at^2 + bt + c & \text{if } t \le 0 \\ t^2 + 1 & \text{if } t > 0 \end{cases}.$$

   Find all values of a, b, and c so that g is differentiable at $t = 0$.

7. Let $f(x) = e^x$ and estimate the derivative $f'(0)$ by considering difference quotients $(f(h) - f(0))/h$ for small values of h.

8. Suppose that $f'(0)$ exists. Does the limit

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{h}$$

exist? Can you express the limit in terms of $f'(0)$?

9. Can you find a function f so that

$$\lim_{h \to 0} \frac{f(h) - f(-h)}{h}$$

exists, but f is not differentiable at 0?

10. Find A and B so that the limit

$$\lim_{x \to 1} \frac{x^2 + 2x - (Ax + B)}{(x - 2)^2}$$

is finite. Give the value of the limit.

11. Find the specified derivative for each of the following using the limit definition of derivative.

    (a) If $f(x) = 1/x$, find $f'(2)$.
    (b) If $g(x) = \sqrt{x}$, find $g'(2)$.
    (c) If $h(x) = x^2$, find $h'(s)$.
    (d) If $f(x) = x^3$, find $f'(-2)$.
    (e) If $g(x) = 1/(2 - x)$, find $g'(t)$.

Worksheet # 8: Review for Exam I

1. Find all real numbers of the constant a and b for which the function $f(x) = ax + b$ satisfies:
   (a) $f \circ f(x) = f(x)$ for all x.
   (b) $f \circ f(x) = x$ for all x.

2. Simplify the following expressions.
   (a) $\log_5 125$
   (b) $(\log_4 16)(\log_4 2)$
   (c) $\log_{15} 75 + \log_{15} 3$
   (d) $\log_x(x(\log_y y^x))$
   (e) $\log_\pi(1 - \cos x) + \log_\pi(1 + \cos x) - 2\log_\pi \sin x$

3. Suppose that $\tan(x) = \frac{3}{4}$ and $-\pi < x < 0$. Find $\cos(x)$, $\sin(x)$, and $\sin(2x)$.

4. (a) Solve the equation $3^{2x+5} = 4$ for x. Show each step in the computation.
   (b) Express the quantity $\log_2(x^3 - 2) + \dfrac{1}{3}\log_2(x) - \log_2(5x)$ as a single logarithm. For which x is the resulting identity valid?

5. Suppose that the height of an object at time t is $h(t) = 5t^2 + 40t$.
   (a) Find the average velocity of the object on the interval $[3, 3.1]$.
   (b) Find the average velocity of the object on the interval $[a, a + h]$.
   (c) Find the instantaneous velocity of the object time a.

6. Calculate the following limits using the limit laws. Carefully show your work and use only one limit law per step.
   (a) $\lim\limits_{x\to 0}(2x - 1)$
   (b) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4} - 2}{x}$

7. Calculate the following limits if they exist or explain why the limit does not exist.
   (a) $\lim\limits_{x\to 1} \dfrac{x-2}{\frac{1}{x} - \frac{1}{2}}$
   (b) $\lim\limits_{x\to 2} \dfrac{x-2}{\frac{1}{x} - \frac{1}{2}}$
   (c) $\lim\limits_{x\to 2^+} \dfrac{x^2 - 1}{x - 2}$
   (d) $\lim\limits_{x\to a}(xa - a^2)$
   (e) $\lim\limits_{x\to 4} \dfrac{\sqrt{x} - 2}{x^2 - 16}$
   (f) $\lim\limits_{x\to 2} \dfrac{x+1}{x-2}$
   (g) $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1}$
   (h) $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 2}$
   (i) $\lim\limits_{x\to 2} \dfrac{x^3 - 8}{x - 2}$
   (j) $\lim\limits_{a\to x}(xa - a^2)$

8. State the Squeeze Theorem. Use it to find the following limits.
   (a) $\lim\limits_{x\to 0} x \sin \dfrac{1}{x^2}$
   (b) $\lim\limits_{x\to \frac{\pi}{2}} \cos x \cos(\tan x)$

9. If $f(x) = \dfrac{|x-3|}{x^2 - x - 6}$, find $\lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3} f(x)$.

10. If $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = 5$, find the following or explain why we cannot find the limit.

(a) $\lim_{x\to 2}(2f(x) + 3g(x))$

(b) $\lim_{x\to 2} \dfrac{f(x)}{g(x)+1}$

(c) $\lim_{x\to 2} f(x)g(x)$

(d) $\lim_{x\to 2} \dfrac{x-2}{2f(x)-6}$

11. (a) State the definition of the continuity of a function $f(x)$ at $x = a$.

(b) Find the constant a so that the function is continuous on the entire real line.

$$f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a} & \text{if } x \neq a \\ 8 & \text{if } x = a \end{cases}$$

12. If $f(x) = x^2 + 5^x - 3$, use the Intermediate Value Theorem to show that there is a number a such that $g(a) = 10$.

13. (a) A function $f(x)$ passes the horizontal line test, if the function f is ________

(b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{ provided } \_\_\_\_\_\_\_\_$$

(c) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ if and only if ________

(d) Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ be a piecewise function. The function $g(x)$ is not continuous at $x = 2$ because ________

(e) Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$ be a piecewise function. The function $f(x)$ is NOT continuous at $x = 0$ because ________

14. Find the horizontal asymptotes for the function $f(x) = \dfrac{1-2x}{\sqrt{1+x^2}}$.

Find the following limits.

15. $\lim_{x\to -\infty} \dfrac{4x^3 + 6x^2 - 2}{2x^3 - 4x + 5}$

16. $\lim_{x\to \infty} \dfrac{t - t\sqrt{t}}{2t^{3/2} + 3t - 5}$

17. $\lim_{x\to \infty} \dfrac{(2x^2+1)^2}{(x-1)^2(x^2+x)}$

18. $\lim_{x\to \infty} \dfrac{\sqrt{x + 3x^2}}{4x - 1}$

19. (a) State the definition of the derivative of a function $f(x)$ at a point a.

(b) Find the derivative of $f(x) = 4x - x^2$ at the point $a = 1$ using your definition.

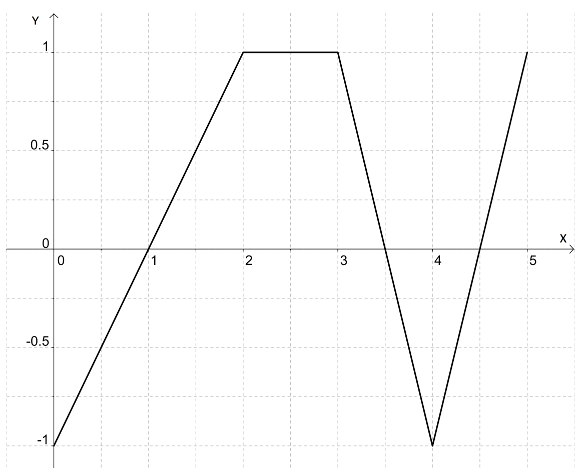
20. (a) Find the slope of the tangent line to the parabola $y = x - x^3$ at the point $(1, 0)$.

(b) Find the equation of the tangent line in part (a).

Worksheet # 9: The Derivative as a Function

Interesting Fact: Derivatives play a fundamental role in modeling physical problems. As one example, to mathematically describe elastic surfaces requires careful consideration of certain equations involving derivatives called "differential equations." One of the pioneers in elasticity theory was Sophie Germain, who in 1816 was the first woman to win a prize from the Paris Academy of Sciences for her pioneering paper "Recherches sur la théorie des surfaces élastiques." Germain is most famous for her work in number theory, and her award of the Paris Academy Prize inspired her to continue her work on what is now called "Fermat's Last Theorem". Germain's achievements are particularly impressive because her parents did not approve of women studying mathematics. At night, they would deny her warm clothing or a fire in her bedroom to prevent her from studying, but she hid candles and quilts and taught herself anyway.

1. Consider the graph below of the function $f(x)$ on the interval $[0, 5]$.



   (a) For which x values would the derivative $f'(x)$ not be defined?

   (b) Sketch the graph of the derivative function f'.

2. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (Celsius)	15.5	17.7	20.0	22.4	24.4
Weight gained (grams)	37.2	31.0	19.8	9.7	-9.8

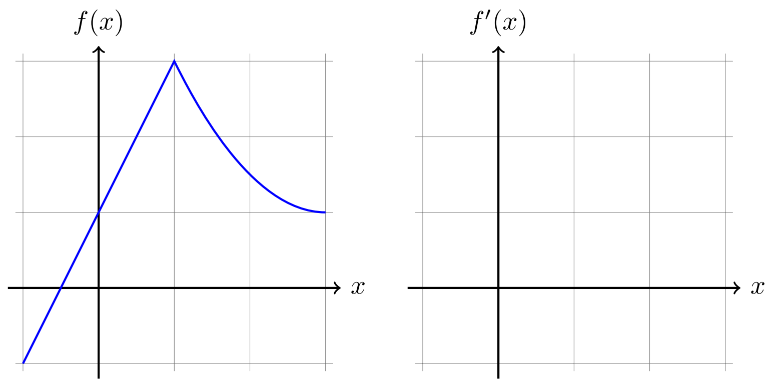
   (a) If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W'.

(b) Plot the points for $W(x)$ and $W'(x)$ in the $x-y$-plane. Sketch one possible version of the graph for W, and use this to sketch a possible version of the graph for W'.

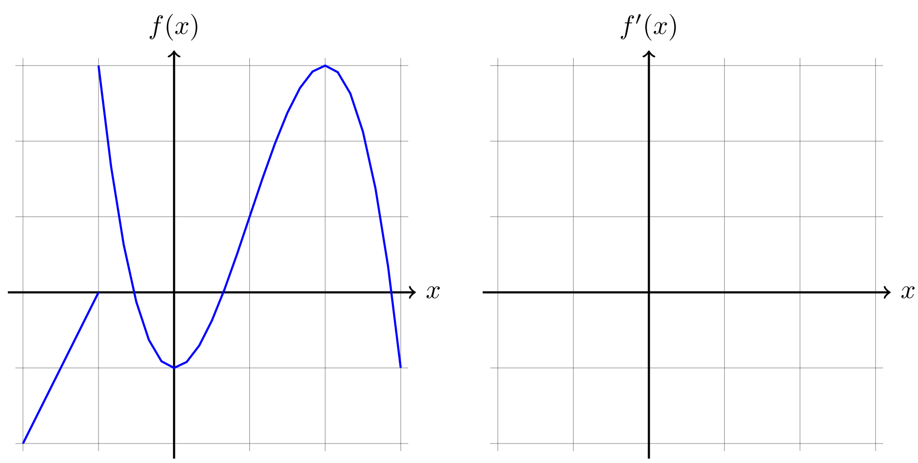
(c) What are the units for W'?

3. For each function f whose graph is given below, identify points where $f'(x)$ does not exist and sketch the graph of f'.

(a)



(b)



Worksheet # 10: Polynomials, Exponentials and Product and Quotient Rules

1. Compute the derivative of the following functions using both the limit definition and the rules for polynomials and exponentials.

   (a) $f(x) = 4 + 8x - 10x^3$

   (b) $g(x) = -7x^2 + x - 2$

2. Suppose N is the number of people in the United States who travel by car to another state for a vacation this year when the average price of gasoline is p dollars per gallon. Do you expect dN/dp to be positive or negative? Explain your answer. What about d^2N/dp^2?

3. Find a formula for the n-th derivative of x^n.

4. Find the first, second, and third derivatives of the following functions using the rules for polynomials and exponentials.

   (a) $f(x) = 3^{30}$

   (b) $g(t) = (t+1)(t+2)(t+3)$

   (c) $h(a) = \dfrac{\sqrt{a} + a}{a^3}$

   (d) $y = e^{x+2} + 1$

   (e) $F(x) = \dfrac{2}{x^3} + 3e^x - x^7$

5. (a) If $f'(x) = g'(x)$ for all x, then does $f = g$? Explain your answer.

   (b) If $f(x) = g(x)$ for all x, then does $f' = g'$? Explain your answer.

   (c) How is the number e defined?

   (d) Are differentiable functions also continuous? Are continuous functions also differentiable? Provide several concrete examples to support your answers.

6. Calculate the derivatives of the following functions in the two ways that are described.

   (a) $f(r) = r^3/3$

      i. using the constant multiple rule and the power rule

      ii. using the quotient rule and the power rule

      Which method should we prefer?

   (b) $f(x) = x^5$

      i. using the power rule

      ii. using the product rule by considering the function as $f(x) = x^2 \cdot x^3$

   (c) $g(x) = (x^2 + 1)(x^4 - 1)$

      i. first multiply out the factors and then use the power rule

      ii. by using the product rule

7. State the quotient and product rule and be sure to include all necessary hypotheses.

8. Compute the first derivative of each of the following:

(a) $f(x) = (3x^2 + x)e^x$

(b) $f(x) = \dfrac{\sqrt{x}}{x-1}$

(c) $f(x) = \dfrac{e^x}{2x^3}$

(d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x-1}{\sqrt{x}}\right)$

(e) $f(x) = \dfrac{2x}{4+x^2}$

(f) $f(x) = \dfrac{ax+b}{cx+d}$

(g) $f(x) = \dfrac{(x^2+1)(x^3+2)}{x^5}$

(h) $f(x) = (x-3)(2x+1)(x+5)$

9. Find an equation of the tangent line to the given curve at the specified point. Sketch the curve and the tangent line to check your answer.

   (a) $y = x^2 + \dfrac{e^x}{x^2+1}$ at the point $x = 3$.

   (b) $y = 2xe^x$ at the point $x = 0$.

10. Let $f(x) = (3x-1)e^x$. For which x is the slope of the tangent line to f positive? Negative? Zero?

11. Suppose that $f(2) = 3,\ g(2) = 2,\ f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

    (a) $h(x) = 5f(x) + 2g(x)$

    (b) $h(x) = f(x)g(x)$

    (c) $h(x) = \dfrac{f(x)}{g(x)}$

    (d) $h(x) = \dfrac{g(x)}{1+f(x)}$

12. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the n-th derivative of f.

Worksheet # 11: Derivatives of Trigonometric Functions

Interesting Fact: The followers of Newton (in England) and Leibniz (in continental Europe) argued and fought regarding which of them should receive "credit" for inventing calculus. The reality is that both made important and independent contributions to the development of calculus. Émilie du Châtelet was a French mathematician and physicist who in 1749 completed her translation of Newton's *Principia* into French, expanding the influence of Newton's work in Europe — to this day, du Châtelet's translation is considered the definitive French translation of this work. One of the reasons this translation was so influential was that she did more than just translate. In her extensive commentary on the translation, du Châtelet made a crucial contribution to Newtonian mechanics via her conservation law for total energy. In addition to her translation of Newton, she also wrote an influential treatise on physics, *Institutions de Physique*, in 1740.

1. For each of these problems, explain why it is true or give an example showing it is false.
   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.
   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \frac{2}{3}$, then the hypotenuse of the triangle must have length 3.
2. Calculate the first five derivatives of $f(x) = \sin(x)$. Then determine $f^{(8)}$ and $f^{(37)}$
3. Let $f(t) = t + 2\cos(t)$.
   (a) Find all values of t where the tangent line to f at the point $(t, f(t))$ is horizontal.
   (b) What are the largest and smallest values for the slope of a tangent line to the graph of f?
4. Differentiate each of the following and simplify your answer.
   (a) $r(\theta) = \theta^3 \sin(\theta)$
   (b) $s(t) = \tan(t) + \csc(t)$
   (c) $h(x) = \sin(x)\csc(x)$
   (d) $g(x) = \sec(x) + \cot(x)$
5. Find an equation of the tangent line to the curve $f(x) = \sin(x) + \tan(x)$ at the point $x = 0$.
6. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.
   (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.
   (b) Is the particle moving towards or away from the origin at π seconds? Explain.
   (c) Now, find the velocity of the particle at time $t = \frac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?
   (d) Is the particle increasing speed at $\frac{\pi}{2}$ seconds?

Worksheet # 12: Chain Rule, Implicit Differentiation and Inverse Functions

1. (a) Carefully state the chain rule using complete sentences.

   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \dfrac{1}{4}$, and $g'(2) = 2$. Find each of the following:

      i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.

      ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

2. Differentiate both sides of the double-angle formula for the cosine function, $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar identity?

3. Differentiate each of the following and simplify your answer.

   (a) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$

   (b) $g(t) = \tan(\sin(t))$

   (c) $h(u) = \sec^2(u) + \tan^2(u)$

   (d) $f(x) = xe^{(3x^2+x)}$

4. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 e^{3x}$, $x = 2$

   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

5. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

   Find: $h'(-1), h'(3)$ and $k'(2)$.

6. Find the derivative of y with respect to x:

   (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.

   (b) $e^y \sin(x) = x + xy$.

   (c) $\cos(xy) = 1 + \sin(y)$.

7. Consider the ellipse given by the equation $\dfrac{(x-2)^2}{25} + \dfrac{(y-3)^2}{81} = 1$.

   (a) Find the equation of the tangent line to the ellipse at the point (u, v) where $u = 4$ and $v > 0$.

   (b) Sketch the ellipse and the line to check your answer.

8. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

9. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

10. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

11. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

12. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

Worksheet # 13: Derivatives of Logarithms

An Interesting Fact: While several people independently developed the idea of the logarithm, the most influential of these was John Napier through his 1614 book *Mirifici Logarithmorum Canonis Descriptio*. Napier developed the theory of logarithms to allow faster calculation, and it was so successful that within a few decades his logarithms had spread across the globe due to the promotion of Henry Briggs and Edward Wright in England, Bonaventura Cavalieri in Italy, Johannes Kepler in Germany, and Xue Fengzuo in China. Through reprintings of the book *Shu Li Ching Yün*, originally published in Beijing by Emperor K'ang Hsi, Napier's theory of logarithms reached Japanese mathematicians in the early 1700's.

1. Find the derivative of $f(x) = 3^x$. Compute the derivative of $\log_3(x)$. Explain the relationship between your answers.

2. Find the derivatives of the following functions.
   (a) $f(x) = \ln(\sin^2 x)$
   (b) $g(x) = \ln(xe^{-2x})$

3. Find y' and y''.
   (a) $y = \sqrt{x}\ln(x)$
   (b) $y = \ln(1 + \ln(x))$
   (c) $y = \dfrac{\ln(x)}{1 + \ln(x)}$

4. Find an equation of the tangent line to the curve at the given point.
   (a) $y = \ln(x^2 - 3x + 1)$, $(3, 0)$
   (b) $y = x^2 \ln(x)$, $(1, 0)$

5. If $f(x) = \ln(x + \ln(x))$, find $f'(1)$.

6. Find y' if $y = \ln(x^2 + y^2)$.

7. Find $\dfrac{d^9}{dx^9}(x^8 \ln(x))$.

Worksheet # 14: Rates of Change; Exponential Growth & Decay

1. Given that $G(t) = 4t^2 - 3t + 42$ find the instantaneous rate of change when $t = 3$.

2. A particle moves along a line so that its position at time t is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin. Recall that *speed* is given by the absolute value of velocity.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

3. An object is thrown upward so that its height at time t seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time t.

4. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s^2.

5. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s^2.

6. Suppose that height of a triangle is equal to its base b. Find the instantaneous rate of change in the area with respect to the base b when the base is 7.

7. Suppose that an object is shot into the air vertically with an initial velocity v_0 and initial height s_0, with acceleration due to gravity denoted by g. Let $s(t)$ denote the height of the object after t time units.

   (a) Explain why $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$ (this is sometimes called "Galileo's formula").

   (b) If time is measured in seconds and distance in meters, what are the units for s_0, v_0, and g?

8. The cost in dollars of producing x bicycles is $C(x) = 4000 + 210x - x^2/1000$.

   (a) Explain why $C'(40)$ is a good approximation for the cost of the 41st bicycle.

   (b) How can you use the values of $C(40)$ and $C'(40)$ to approximate the cost of 42 bicycles?

   (c) Explain why the model for $C(x)$ is not a good model for cost. What happens if x is very large?

9. Suppose that a population of bacteria triples every hour and starts with 400 bacteria.

   (a) Find an expression for the number n of bacteria after t hours.

   (b) Use this expression to estimate the rate of growth of the bacteria population after 2.5 hours.

10. Think about the other science courses you are currently taking (or have taken in the past). Identify three to five examples of problems from those courses that involve computing rates of change where methods from calculus might be useful.

11. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

    (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.

    (b) Is the particle moving towards or away from the origin at π seconds? Explain.

    (c) Now, find the velocity of the particle at time $t = \frac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?

(d) Is the particle speeding up at $\frac{\pi}{2}$ seconds?

12. Solve the following equations for α:

    (a) $500 = 1000e^{20\alpha}$

    (b) $40 = \alpha e^{10k}$, where $k = \frac{\ln(2)}{7}$.

    (c) $100,000 = 40,000e^{0.06\alpha}$.

    (d) $\alpha = 2,000e^{36k}$, where $k = \frac{\ln(0.5)}{18}$.

13. The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

14. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.
$$\frac{dP}{dt} = KP$$
where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$

    (a) Give the units of K.

    (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$. How many mosquitoes will there be after 90 days?

15. A lucky colony of rabbits is brought to a large island where there are no predators and unlimited food. Under these conditions, they will reproduce at such a rate that the population doubles every 9 years. After 3 years, a team of scientists determines that there are 7000 rabbits on the island.

    (a) How many rabbits were brought to the island originally?

    (b) How many rabbits will there be 12 years after their introduction to the island?

Worksheet # 15: Related Rates

1. Let a and b denote the length in meters of the two legs of a right triangle. At time $t = 0$, $a = 20$ and $b = 20$. If a is decreasing at a constant rate of 2 meters per second and b is increasing at a constant rate of 3 meters per second. Find the rate of change of the area of the triangle at time $t = 5$ seconds.

2. A spherical snow ball is melting. The rate of change of the surface area of the snow ball is constant and equal to -7 square centimeters per hour. Find the rate of change of the radius of the snow ball when $r = 5$ centimeters.

3. The height of a cylinder is a linear function of its radius (i.e. $h = ar + b$ for some a, b constants). The height increases twice as fast as the radius r and $\frac{dr}{dt}$ is constant. At time $t = 1$ seconds the radius is $r = 1$ feet, the height is $h = 3$ feet and the rate of change of the volume is 16π cubic feet/second.
   (a) Find an equation to relate the height and radius of the cylinder.
   (b) Express the volume as a function of the radius.
   (c) Find the rate of change of the radius when the radius is 4 feet.

4. A water tank is shaped like a cone with the vertex pointing down. The height of the tank is 5 meters and diameter of the base is 2 meters. At time $t = 0$ the tank is full and starts to be emptied. After 3 minutes the height of the water is 4 meters and it is decreasing at a rate of 0.5 meters per minute. At this instant, find the rate of change of the volume of the water in the tank. What are the units for your answer? Recall that the volume of a right-circular cone whose base has radius r and of height h is given by $V = \frac{1}{3}\pi r^2 h$.

5. A plane flies at an altitude of 5000 meters and a speed of 360 kilometers per hour. The plane is flying in a straight line and passes directly over an observer.
   (a) Sketch a diagram that summarizes the information in the problem.
   (b) Find the angle of elevation 2 minutes after the plane passes over the observer.
   (c) Find rate of change of the angle of elevation 2 minutes after the plane passes over the observer.

6. Let $f(x) = \dfrac{1}{1 + x^3}$ and $h(x) = \dfrac{1}{1 + f(x)}$
   (a) Find $f'(x)$.
   (b) Use the previous result to find $h'(x)$.
   (c) Let $x = x(t)$ be a function of time t with $x(1) = 1$ and set $F(t) = h(x(t))$. If $F'(1) = 18$, find $x'(1)$.

Worksheet # 16: Review for Exam II

1. State the following rules with the hypotheses and conclusion.

   (a) The product rule and quotient rule.

   (b) The chain rule.

2. A particle is moving along a line so that at time t seconds, the particle is $s(t) = \frac{1}{3}t^3 - t^2 - 8t$ meters to the right of the origin.

   (a) Find the time interval(s) when the particle's velocity is negative.

   (b) Find the time(s) when the velocity is zero.

   (c) Find the time interval(s) when the particle's acceleration is positive.

   (d) Find the time interval(s) when the particle is speeding up. Hint: What do we need to know about velocity and acceleration in order to know that the derivative of the speed is positive?

3. Compute the first derivative of each of the following functions:

   (a) $f(x) = \cos(4\pi x^3) + \sin(3x+2)$

   (b) $b(x) = x^4 \cos(3x^2)$

   (c) $y(\theta) = e^{\sec(2\theta)}$

   (d) $k(x) = \ln(7x^2 + \sin(x) + 1)$

   (e) $u(x) = (\sin^{-1}(2x))^2$

   (f) $h(x) = \dfrac{8x^2 - 7x + 3}{\cos(2x)}$

   (g) $m(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

   (h) $q(x) = \dfrac{e^x}{1+x^2}$

   (i) $n(x) = \cos(\tan(x))$

   (j) $w(x) = \arcsin(x) \cdot \arccos(x)$

4. Let $f(x) = \cos(2x)$. Find the fourth derivative at $x = 0$, $f^{(4)}(0)$.

5. Let f be a one to one, differentiable function such that $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$ and $f'(2) = 5$. Find the derivative of the inverse function, $(f^{-1})'(2)$.

6. The tangent line to $f(x)$ at $x = 3$ is given by $y = 2x - 4$. Find the tangent line to $g(x) = \dfrac{x}{f(x)}$ at $x = 3$. Put your answer in slope-intercept form.

7. Consider the curve $xy^3 + 12x^2 + y^2 = 24$. Assume this equation can be used to define y as a function of x (i.e. $y = y(x)$) near $(1, 2)$ with $y(1) = 2$. Find the equation of the tangent line to this curve at $(1, 2)$.

8. Suppose f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = \frac{1}{4}$, and $g'(2) = 2$. Find:

   (a) $h'(2)$ where $h(x) = \ln([f(x)]^2)$;

   (b) $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

9. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

10. Suppose $y = \sqrt{2x+1}$, where x and y are functions of t.

    (a) If $dx/dt = 3$, find dy/dt when $x = 4$.

    (b) If $dy/dt = 5$, find dx/dt when $x = 12$.

11. Strontium-90 has a half-life of 28 days.

    (a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.
    (b) Find the mass remaining after 40 days.
    (c) How long does it take the sample to decay to a mass of 2 mg?

Worksheet # 17: Extreme Values

Figure 1: An excerpt from *Hisab al-jabr w'al-muqabala*, courtesy of the Bodleian library.

An Interesting Fact: When we study extreme values of functions, we rely on our ability to do numerical calculations and solve algebraic equations. Our modern understanding of number systems and algebra can be traced to Baghdad in the 9th century, with the work of Muhammad ibn Musa al-Khwarizmi, who authored two incredibly influential books.

> [One of these books was] *Hisab al-jabr w'al-muqabala* (which may be loosely translated as Calculation by Reunion and Reduction). . . In the twelfth century the book was translated into Latin under the title *Liber algebrae et almucabola*, thus giving a name to a central area of mathematics. . . [Another of al-Khwarizmi's books was] *Algorithmi de numero indorum*, which explained the Indian number system. While al-Khwarizmi was at pains to point out the Indian origin of this number system, subsequent translations of the book [were attributed to] the author. Hence, in Europe any scheme using these numerals came to be known as an "algorism", or, later, "algorithm" (a corruption of the name al-Khwarizmi). — George Gheverghese Joseph

1. Comprehension check:

   (a) True or False: If $f'(c) = 0$ then f has a local maximum or local minimum at c.

   (b) True or False: If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.

   (c) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.

   (d) True or False: If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.

2. Sketch the following:

   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.

   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.

   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.

   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.

   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.

3. State the definition of a critical number. Use this definition to find the critical numbers for the following functions:

   (a) $f(x) = x^4 + x^3 + 1$

   (b) $g(x) = e^{3x}(x^2 - 7)$

(c) $h(x) = |5x - 1|$

(d) $j(x) = (4 - x^2)^{1/3}$

4. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.

   (a) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (b) $h(x) = x + \sqrt{1 - x^2}$, $[-1, 1]$

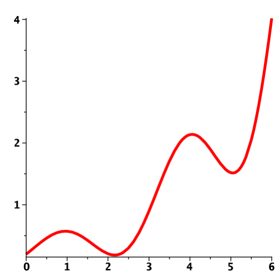
   (c) $f(x) = 2\cos(x) + \sin(2x)$, $[0, \frac{\pi}{2}]$

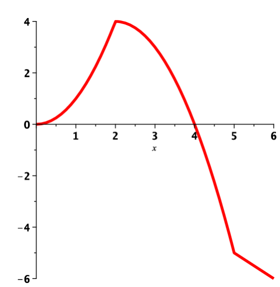
   (d) $f(x) = x^{-2} \ln x$, $[\frac{1}{2}, 4]$

5. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1 - x)^b$ on the interval $0 \leq x \leq 1$.

Worksheet # 18: The Mean Value Theorem & How Derivatives Affect the Shape of a Graph

1. State the Mean Value Theorem. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.
   (a) $f(x) = \dfrac{x}{x+2}$ on the interval $[1, 4]$
   (b) $f(x) = \sin(x) - \cos(x)$ on the interval $[0, 2\pi]$
2. Use the Mean Value Theorem to show that $\sin(x) \leq x$ for $x \geq 0$. What can you say for $x \leq 0$?
3. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.
4. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why did she deserve the ticket?
5. Suppose that we know that $1 \leq f(2) \leq 4$ and that $2 \leq f'(x) \leq 5$ for all x. What are the largest and smallest possible values for $f(6)$? What about $f(-1)$?
6. Comprehension Check:
   (a) Explain what the First Derivative Test reveals about a continuous function $f(x)$ including when and how to use it.
   (b) Explain what the Second Derivative Test reveals about a twice differentiable function $f(x)$ and include how to use it. Does the test always work? What should you do if it fails?
   (c) Identify the similarities and differences between these two tests.
7. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.
      i. Find the critical number(s) of $f(x)$.
      ii. Find the intervals on which $f(x)$ is increasing or decreasing.
      iii. Find the local extrema of $f(x)$.
   (b) Repeat with the function $f(x) = \dfrac{x}{x^2+4}$ on $(-\infty, \infty)$.
8. Below are the graphs of two functions.





   (a) Find the intervals where each function is increasing and decreasing respectively.
   (b) Find the intervals of concavity for each function.
   (c) For each function, identify all local extrema and inflection points on the interval (0,6).
9. (a) Consider the function $f(x) = x^4 - 8x^3 + 5$.
      i. Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
      ii. Find the inflection points of $f(x)$.
      iii. Find the intervals of concavity of $f(x)$.
   (b) Repeat with the function $f(x) = 2x + \sin(x)$ on $\left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

(c) Repeat with the function $f(x) = xe^x$.

10. Find the local extrema of the following functions using the second derivative test (if possible):

    (a) $f(x) = x^5 - 5x + 4$

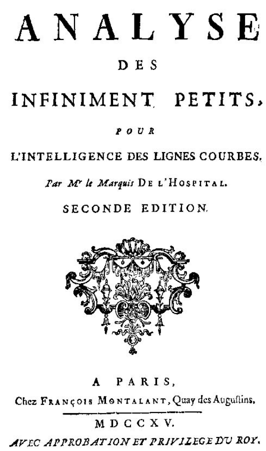
    (b) $g(x) = 5x - 10\ln(2x)$

    (c) $h(x) = 3x^5 - 5x^3 + 10$

11. Sketch a graph of a continuous function $f(x)$ with the following properties:

    - f is increasing on $(-\infty, -3) \cup (1, 7) \cup (7, \infty)$
    - f is decreasing on $(-3, 1)$
    - f is concave up on $(0, 3) \cup (7, \infty)$
    - f is concave down on $(-\infty, 0) \cup (3, 7)$

Worksheet # 19: l'Hospital's Rule


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par M.r le Marquis DE L'HOSPITAL.
SECONDE EDITION.
A PARIS,
Chez FRANÇOIS MONTALANT, Quay des Augustins.
M D C C X V.
AVEC APPROBATION ET PRIVILEGE DU ROY.

An Interesting Fact: L'Hospital's Rule was probably not discovered by l'Hospital!

> The Marquis de l'Hospital was a French nobleman and amateur mathematician who [wanted to learn] calculus. [He] employed Johann Bernoulli to supply him with tracts on various aspects of the subject, as well as to provide him with any new mathematical discoveries of note. In a sense, it appears that l'Hospital bought the rights to Bernoulli's mathematical research [and published it under his own name as] *Analyse de infiniment petits.* — William Dunham

1. Suppose we know:

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

   Which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

   (a) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$

   (b) $\lim\limits_{x\to a} \dfrac{f(x)}{p(x)}$

   (c) $\lim\limits_{x\to a} f(x)p(x)$

   (d) $\lim\limits_{x\to a} \dfrac{p(x)}{f(x)}$

   (e) $\lim\limits_{x\to a} p(x)q(x)$

   (f) $\lim\limits_{x\to a} \dfrac{p(x)}{q(x)}$

2. Carefully state l'Hospital's Rule.

3. Compute the following limits. Use l'Hospital's Rule where appropriate, but first check that no easier method will solve the problem.

   (a) $\lim\limits_{x\to 1} \dfrac{x^9 - 1}{x^5 - 1}$

   (b) $\lim\limits_{x\to \infty} \dfrac{3x + 2\sqrt{x}}{1 - x}$

   (c) $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{\tan(5x)}$

   (d) $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$

(e) $\lim_{x \to \infty} \frac{e^x}{x^3}$

(f) $\lim_{x \to -\infty} \frac{2x - 5}{|3x + 2|}$

(g) $\lim_{x \to \infty} \frac{5x^2 + \sin x}{3x^2 + \cos x}$

(h) $\lim_{x \to 1} \frac{x^2 + 2x - 2}{x^2 - 2x + 2}$

(i) $\lim_{x \to -\infty} x^2 e^x$

(j) $\lim_{x \to \infty} x^3 e^{-x^2}$

(k) $\lim_{x \to \pi} \frac{\cos(x) + 1}{x^2 - \pi^2}$

(l) $\lim_{x \to \infty} x \cdot \left(\arctan(x) - \frac{\pi}{2}\right)$

4. Find the value A for which we can use l'Hospital's rule to evaluate the limit

$$\lim_{x \to 2} \frac{x^2 + Ax - 2}{x - 2}.$$

For this value of A, give the value of the limit.

Worksheet # 20: Optimization

1. Find the dimensions of x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating x and y?

   (b) Find a formula for the area in terms of x alone.

   (c) Solve the optimization problem.

2. Find two numbers whose difference is 100 and whose product is a minimum.

3. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

4. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

5. Find the dimensions x and y of the rectangle inscribed in a circle of radius r that maximizes the quantity xy^2.

6. Suppose that f is a function on an open interval $I = (a, b)$ and c is in I. Suppose that f is continuous at c, $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$. Is $f(c)$ an absolute minimum value for f on I? Justify your answer.

7. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

8. A hockey team plays in an arena with a seating capacity of 15000 spectators. With the ticket price set at \$12, average attendance at a game has been 11000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

9. An oil company needs to run a pipeline to a nearby station. The station and oil company are on opposite sides of a river that is 1 km wide, and that runs exactly west-east and the station is 10 km east along the river from the the oil company. The cost of building pipe on land is \$200 per meter and the cost of building pipe in water is \$300 per meter. Set up an equation whose solution(s) are the critical points of the cost function for this problem.

   Find the least expensive way to construct the pipe.

10. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

11. A 10 meter length of rope is to be cut into two pieces to form a square and a circle. How should the rope be cut to maximize the enclosed area?

12. Find the point(s) on the hyperbola $y = \dfrac{16}{x}$ that is (are) closest to $(0, 0)$. Be sure to clearly state what function you choose to minimize or maximize and why.

13. Consider a can in the shape of a right circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter, and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72 π cubic centimeters, and whose cost is as small as possible.

    (a) Find a function $f(r)$ which gives the cost of the can in terms of radius r. Be sure to specify the domain.

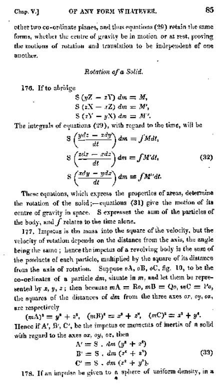
(b) Give the radius and height of the can with least cost.

(c) Explain how you known you have found the can of least cost.

14. Find the point on the line $y = x$ closest to the point $(1, 0)$. Find the point on the line $y = x$ closest to the point $(r, 1 - r)$. What does the collection of points $(r, 1 - r)$ look like graphically?

15. A box is to have a square base, no top, and a volume of 10 cubic centimeters. What are the dimensions of the box with the smallest possible total surface area? Provide an exact answer; do not convert your answer to decimal form. Make a sketch and introduce all the notation you are using.

Worksheet # 21: Antiderivatives



Chap. V.] OF ANY FORM WHATEVER. 85

other two co-ordinate planes, and thus equations (29) retain the same forms, whether the centre of gravity be in motion or at rest, proving the motions of rotation and translation to be independent of one another.

Rotation of a Solid.

176. If to abridge

$$S\,(yZ - zY)\,dm = M,$$
$$S\,(zX - xZ)\,dm = M',$$
$$S\,(xY - yX)\,dm = M''.$$

The integrals of equations (29), with regard to the time, will be

$$S\left(\frac{ydz - zdy}{dt}\right)dm = \int Mdt,$$
$$S\left(\frac{zdx - xdz}{dt}\right)dm = \int M'dt, \qquad (32)$$
$$S\left(\frac{xdy - ydx}{dt}\right)dm = \int M''dt.$$

These equations, which express the properties of areas, determine the rotation of the solid;—equations (31) give the motion of its centre of gravity in space. S expresses the sum of the particles of the body, and $\int$ relates to the time alone.

177. Impetus is the mass into the square of the velocity, but the velocity of rotation depends on the distance from the axis, the angle being the same; hence the impetus of a revolving body is the sum of the products of each particle, multiplied by the square of its distance from the axis of rotation. Suppose oA, oB, oC, fig. 10, to be the co-ordinates of a particle *dm*, situate in *m*, and let them be represented by *x*, *y*, *z*; then because *m*A = Ro, *m*B = Qo, *m*C = Po, the squares of the distances of *dm* from the three axes *ox*, *oy*, *oz*, are respectively

$$(mA)^2 = y^2 + z^2, \quad (mB)^2 = x^2 + z^2, \quad (mC)^2 = x^2 + y^2.$$

Hence if A', B', C', be the impetus or moments of inertia of a solid with regard to the axes *ox*, *oy*, *oz*, then

$$A' = S\,.\,dm\,(y^2 + z^2)$$
$$B' = S\,.\,dm\,(x^2 + z^2) \qquad (33)$$
$$C' = S\,.\,dm\,(x^2 + y^2).$$

178. If an impulse be given to a sphere of uniform density, in a

An Interesting Fact: Antiderivatives, areas, and distances are fundamental in physics and mathematics. Pierre-Simon Laplace published *Mécanique Céleste* in five volumes between 1798 and 1825, and this is generally considered the next major work on gravitational mathematics and celestial mechanics after Newton's *Principia*. Eager to have a version in English, the Society for the Diffusion of Useful Knowledge commissioned a translated and expanded version from Mary Somerville, a famous Scottish mathematician and astronomer, which was published in 1831 under the title *The Mechanism of the Heavens*. In part due to the phenomenal success of her translation and extensions of the work of Laplace, in 1835 Somerville was one of the first two women (jointly with Caroline Herschel) to become a member of the Royal Astronomical Society.

1. Comprehension Check:
   (a) If F is an antiderivative of a continuous function f, is F continuous? What if f is not continuous?
   (b) Let $g(x) = \dfrac{x^3}{3} + 1$. Find $g'(x)$. Now give two antiderivatives of $g'(x)$.
   (c) Let $h(x) = x^2 + 1$, and let $H(x)$ be any antiderivative of h. What is $H'(x)$?

2. Find the most general antiderivatives of the following functions
   (a) $f(x) = x^2 - 3x + 2 - \frac{5}{x}$.
   (b) $g(x) = (x-5)^2$.
   (c) $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$.
   (d) $h(x) = 2\sin(x) - \sec^2(x)$.
   (e) $r(x) = \sec(x)\tan(x) - 2e^x$.
   (f) $f(x) = 1 + 2\sin(x) + \dfrac{3}{\sqrt{x}}$.

3. Find f given that
$$f'(x) = \sin(x) - \sec(x)\tan(x), \qquad f(\pi) = 1.$$

4. Find g given that
$$g''(t) = -9.8, \qquad g'(0) = 1, \qquad g(0) = 2.$$
   On the surface of the earth, the acceleration of an object due to gravity is approximately $-9.8\,\text{m/s}^2$. What situation could we describe using the function g? Be sure to specify what g and its first two derivatives represent.

5. A stone is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground.
   (a) Find the distance of the stone above the ground at time t.
   (b) How long does it take the stone to reach the ground?
   (c) With what velocity does it strike the ground?
   (d) If the stone is thrown downward with a speed of 5 m/s, how long does it take to reach the ground?
6. Two balls are thrown upward from the edge of a cliff 432 feet above the ground. The first is thrown with a speed of 48 ft/sec and the second is thrown a second later with a speed of 24 ft/sec. Do the balls ever pass each other?
7. A high-speed bullet train accelerates and decelerates at the rate of 4 ft/sec^2. Its maximum cruising speed is 90 mph.
   (a) What is the maximum distance the train can travel if it accelerates from rest until it reaches its cruising speed and then runs at that speed for 15 minutes?
   (b) Suppose that the train starts from rest and must come to a complete stop in 15 minutes. What is the maximum distance it can travel under these conditions?
   (c) Find the minimum time that the train takes to travel between two consecutive stations that are 45 miles apart.
   (d) The trip from one station to the next takes 37.5 minutes. How far apart are the stations?

Worksheet # 22: Areas and Distances & Definite Integrals

The following summation formulas will be useful below.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

1. The velocity of a train at several times is shown in the table below. Assume that the velocity changes linearly between each time given.

t=time in minutes	0	3	6	9
v(t)=velocity in Km/h	20	80	100	140

   (a) Plot the velocity of the train versus time.

   (b) Compute the left and right-endpoint approximations to the area under the graph of v.

   (c) Explain why these approximate areas are also an approximation to the distance that the train travels.

2. Let $f(x) = \frac{1}{x}$. Divide the interval $[1, 3]$ into five subintervals of equal length and compute R_5 and L_5, the left and right endpoint approximations to the area under the graph of f in the interval $[1, 3]$. Is R_5 larger or smaller than the true area? Is L_5 larger or smaller than the true area?

3. Let $f(x) = \sqrt{1 - x^2}$. Divide the interval $[0, 1]$ into four equal subintervals and compute L_4 and R_4, the left and right-endpoint approximations to the area under the graph of f. Is R_4 larger or smaller than the true area? Is L_4 larger or smaller than the true area? What can you conclude about the value π?

4. Write each of following in summation notation:

   (a) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

   (b) $2 + 4 + 6 + 8 + 10 + 12 + 14$

   (c) $2 + 4 + 8 + 16 + 32 + 64 + 128$.

5. Compute $\displaystyle\sum_{i=1}^{4} \left(\sum_{j=1}^{3} (i + j) \right)$.

6. Let $f(x) = x^2$.

   (a) If we divide the interval $[0, 2]$ into n equal intervals of equal length, how long is each interval?

   (b) Write a sum which gives the right-endpoint approximation R_n to the the area under the graph of f on $[0, 2]$.

   (c) Use one of the formulae for the sums of powers of k to find a closed form expression for R_n.

   (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.

7. Find the number n such that $\sum_{i=1}^{n} i = 78$.

8. Give the value of the following sums.

   (a) $\displaystyle\sum_{k=1}^{20} (2k^2 + 3)$

(b) $\sum_{k=11}^{20}(3k+2)$

9. Recognize the sum as a Riemann sum and express the limit as an integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4}.$$

10. Let $f(x)=x$ and consider the partition $P=\{x_0,x_1,\dots,x_n\}$ which divides the interval $[1,3]$ into n subintervals of equal length.

   (a) Find a formula for x_k in terms of k and n.

   (b) We form a rectangle whose width is $\Delta x=(x_k-x_{k-1})$ and whose height is $f(x_k)$. Give the area of the rectangle.

   (c) Choose the sample points to be the right endpoint of each subinterval. Form the Riemann sum, and use the formula for sums of powers to simplify the Riemann sum.

   (d) Take the limit as n tends to infinity to find the area of the region under $f(x)$ for $1\le x\le 3$.

   (e) Find the area above using geometry to check your answer.

11. Suppose $\int_0^1 f(x)\,dx=2$, $\int_1^2 f(x)\,dx=3$, $\int_0^1 g(x)\,dx=-1$, and $\int_0^2 g(x)\,dx=4$.

   Compute the following using the properties of definite integrals:

   (a) $\int_1^2 g(x)\,dx$

   (b) $\int_0^2 [2f(x)-3g(x)]\,dx$

   (c) $\int_1^1 g(x)\,dx$

   (d) $\int_1^2 f(x)\,dx+\int_2^0 g(x)\,dx$

   (e) $\int_0^2 f(x)\,dx+\int_2^1 g(x)\,dx$

12. Suppose that f is a continuous function and $6\le f(x)\le 7$ for x in the interval $[3,9]$.

   (a) Find the largest and smallest possible values for $\int_3^9 f(x)\,dx$.

   (b) Find the largest and smallest possible values for $\int_8^4 f(x)\,dx$.

13. Suppose that we know $\int_0^x f(t)\,dt=\sin(x)$, find the following integrals.

   (a) $\int_0^{\pi} f(t)\,dt$

   (b) $\int_{\pi/2}^{\pi} f(t)\,dt$

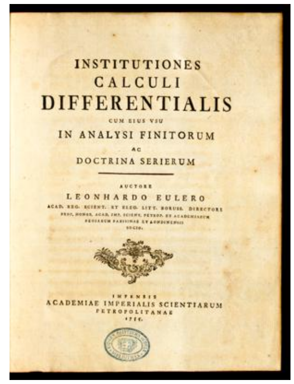
   (c) $\int_{-\pi}^{\pi} f(t)\,dt$

14. Find $\int_0^5 f(x)\,dx$ where $f(x)=\begin{cases}3 & \text{if } x<3\\ x & \text{if } x\ge 3\end{cases}$.

15. Simplify

$$\int_a^b f(t)\,dt+\int_b^c f(u)\,du+\int_c^a f(v)\,dv.$$

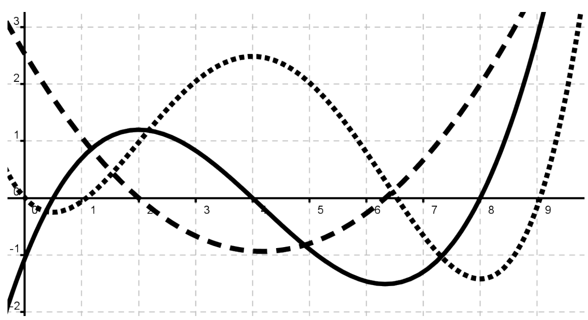
Worksheet # 23: The Fundamental Theorem of Calculus, Part 1



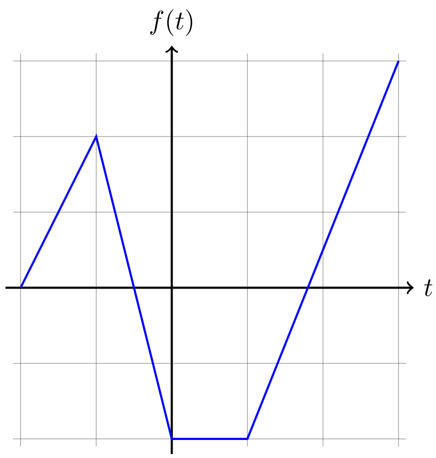
An Interesting Fact: One of the most influential mathematicians of all time is Leonhard Euler, who lived from 1707 to 1783. He worked in every branch of mathematics, and also in physics, astronomy, engineering, and music theory. Euler (pronounced "OIL-ER") was so productive in so many areas, it has been estimated that he wrote roughly one-third of the research publications in the mathematical sciences in Europe between 1725 and 1800.

In addition to his research work, Euler wrote several textbooks such as *Institutiones calculi differentialis* (1755) and *Institutionum calculi integralis* (1768–1770) (Foundations of Differential and Integral Calculus). Because Euler's textbooks were so influential, his notation and conventions are what we use today, for example writing $f(x)$ for functions, π for the ratio of the circumference to diameter of a circle, and e for the base of the natural logarithm.

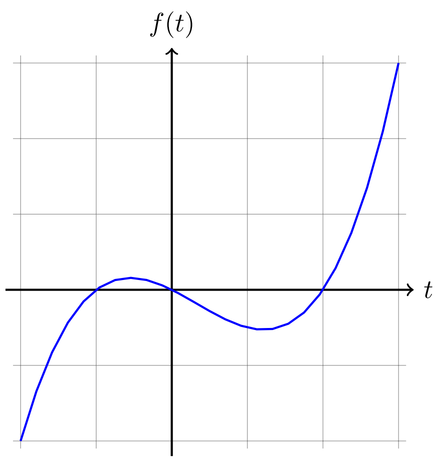
1. Below is pictured the graph of the function $f(x)$, its derivative $f'(x)$, and the integral $\int_0^x f(t)\,dt$. Identify $f(x), f'(x)$ and $\int_0^x f(t)\,dt$ and explain your reasoning.



2. Let $g(x) = \int_{-2}^x f(t)\,dt$ where f is the function whose graph is shown below.
   (a) Evaluate $g(-1)$, $g(0)$, $g(1)$, $g(2)$, and $g(3)$.
   (b) On what interval is g increasing? Why?
   (c) Where does g have a maximum value? Why?



3. Let $g(x) = \int_{-2}^{x} f(t)\,dt$ where f is the function whose graph is shown below. Where is $g(x)$ increasing and decreasing? Explain your answer.



4. Let $F(x) = \int_{2}^{x} e^{t^2}\,dt$. Find an equation of the tangent line to the curve $y = F(x)$ at the point with x-coordinate 2.

5. Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of the following functions:

(a) $g(x) = \displaystyle\int_1^x \left(2 + t^4\right)^5 \, dt$

(b) $F(x) = \displaystyle\int_x^4 \cos\left(t^5\right) \, dt$

(c) $h(x) = \displaystyle\int_0^{x^2} \sqrt[3]{1 + r^3}\, dr$

(d) $y(x) = \displaystyle\int_{\frac{1}{x^2}}^{0} \sin^3(t)\, dt$

(e) $G(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \sqrt{t}\sin(t)\, dt$

6. Find a function $f(t)$ and a number a such that

$$6 + \int_a^x \frac{f(t)}{t^2}\, dt = 2\sqrt{x}$$

for all $x > 0$.

Worksheet # 22: Review for Exam III

1. Describe in words and diagrams how to use the first and second derivative tests to identify and classify extrema of a function $f(x)$.

2. Find the absolute minimum of the function $f(t) = t + \sqrt{1-t^2}$ on the interval $[-1, 1]$. Be sure to specify the value of t where the minimum is attained.

3. (a) Consider the function $f(x) = 2x^3 + 3x^2 - 72x - 47$ on $(-\infty, \infty)$.
   i. Find the critical number(s) of f.
   ii. Find the intervals on which f is increasing or decreasing.
   iii. Find the local maximum and minimum values of f.
   iv. Find the intervals of concavity and the inflection points.

   (b) Repeat with the function $f(x) = x^4 - 2x^2 + 3$.

   (c) Repeat with the function $f(x) = e^{2x} + e^{-x}$.

4. For what values of c does the polynomial $p(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? No inflection points?

5. (a) State the Mean Value Theorem. Use complete sentences.

   (b) Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x?

6. Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x?

7. State L'Hospital's Rule for limits in indeterminate form of type 0/0. Use complete sentences, and include all necessary assumptions. Then evaluate the following limits:

   (a) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

   (b) $\lim_{x\to 0^+} x^3 \ln(x)$

   (c) $\lim_{x\to -\infty} \frac{x+2}{\sqrt{9x^2+1}}$

   (d) $\lim_{x\to 2} \frac{e^{2x}}{x+2}$

8. A poster is to have an area of 180 cm^2 with 1 cm margins at the bottom and sides and 2 cm margins at the top. What dimensions will give the largest printed area? Be sure to explain how you know you have found the largest area.

   (a) Draw a picture and write the constraint equation.

   (b) Write the function you are asked to maximize or minimize and determine its domain.

   (c) Find the maximum or minimum of the function that you found in part (b).

9. Find a positive number such that the sum of the number and twice its reciprocal is as small as possible.

Find the most general antiderivative for the following.

10. $f(x) = 6x^5 - 8x^4 - 9x^3$

11. $f(x) = (x-5)^2$

12. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x}$

13. $f(x) = \frac{1}{5} - \frac{2}{x}$

14. $f(x) = 2\sin x - \sec^2 x$

15. $f(x) = \frac{2x^4 + 4x^3 - x}{x^3}$

16. A stone was dropped off a cliff and hit the ground with a speed of 120 ft/sec. What is the height of the cliff?

17. Oil leaked from a tnak at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two-hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (hr)	0	2	4	6	8	10
$r(t)$ (L/hr)	8.7	7.6	6.8	6.2	5.7	5.3

Using Part 1 of the Fundamental Theorem of Calculus find the derivatives of the following functions.

18. $g(x) = \int_1^x \ln(1+t^2)\,dt$

19. $f(x) = \int_x^0 \sqrt{1+\sec t}\,dt$

20. $h(x) = \int_1^{\sqrt{x}} \frac{t^2}{t^4+1}\,dt$

21. $F(x) = \int_{\sin x}^1 \sqrt{1+t^2}\,dt$

Worksheet # 25: The Fundamental Theorem of Calculus

1. (a) State both parts of the Fundamental Theorem of Calculus using complete sentences.

   (b) Consider the function $f(x)$ on $[1,\infty)$ defined by $f(x) = \displaystyle\int_1^x \sqrt{t^5 - 1}\, dt$. Find the derivative of f. Explain why the function f is increasing.

   (c) Find the derivative of the function $g(x) = \int_1^{x^3} \sqrt{t^5 - 1}\, dt$ on $(1,\infty)$.

2. Use Part 2 of the Fundamental Theorem of Calculus to evaluate the following integrals or explain why the theorem does not apply:

   (a) $\displaystyle\int_{-2}^{5} 6x\, dx$

   (b) $\displaystyle\int_{-2}^{7} \frac{1}{x^5}\, dx$

   (c) $\displaystyle\int_{-1}^{1} e^{u+1}\, du$

   (d) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sin(2x)}{\sin(x)}\, dx$

3. Find the following integrals.

   (a) $\displaystyle\int_1^4 x^2 + x\, dx$

   (b) $\displaystyle\int_1^e \frac{dx}{x}$

   (c) $\displaystyle\int_0^{999\pi} \sin(x)\, dx$

   (d) $\displaystyle\int_0^1 e^x + x^e\, dx$

Worksheet # 26: Net Change and Substitution

1. Find each of the following indefinite integrals.

   (a) $\int 7x - 2\,dx$

   (b) $\int \frac{1}{x^{78}}\,dx$

   (c) $\int e^{u+2}\,du$

2. A population of rabbits at time t increases at a rate of $40 - 12t + 3t^2$ rabbits per year where t is measured in years. Find the population after 8 years if there are 10 rabbits at $t = 0$.

3. Suppose the velocity of a particle traveling along the x-axis is given by $v(t) = 3t^2 + 8t + 15$ m/s at time t seconds. The particle is initially located 5 meters left of the origin. How far does the particle travel from $t = 2$ seconds to $t = 3$ seconds? After 3 seconds, where is the particle with respect to the origin?

4. Suppose an object traveling in a straight line has a velocity function given by $v(t) = t^2 - 8t + 15$ km/hr. Find the displacement and distance traveled by the object from $t = 2$ to $t = 4$ hours.

5. (a) An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^t$ liters per minute. How much oil leaks out during the first hour?

   (b) Is this model realistic? In other words, is it realistic to use this function $r(t)$ to model the leak rate in this situation? Why or why not?

6. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

   (a) $\int \frac{4}{(1+2x)^3}\,dx$

   (b) $\int x^2\sqrt{x^3+1}\,dx$

   (c) $\int \cos^4(\theta)\sin(\theta)\,d\theta$

   (d) $\int \sec^3(x)\tan(x)\,dx$

   (e) $\int e^x \sin\left(e^x\right)\,dx$

   (f) $\int \frac{2x+3}{x^2+3x}\,dx$

7. Evaluate the following definite integrals, and indicate any substitutions that you use:

   (a) $\int_0^7 \sqrt{4+3x}\,dx$

   (b) $\int_0^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right)\,dx$

   (c) $\int_0^4 \frac{x}{\sqrt{1+2x^2}}\,dx$

   (d) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

   (e) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

8. If f is continuous on $(-\infty, \infty)$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx\,.$$

   For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

9. Assume f is a continuous function.

(a) If $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x \cdot f(x^2)\,dx$.

(b) If $\int_0^u f(x)\,dx = 1 + e^{u^2}$ for all real numbers u, find $\int_0^2 f(2x)\,dx$.

10. Do you remember our technique from worksheet #2 of writing $b^x = e^{x\ln(b)}$? Use this to find the indefinite integral $\int b^x\,dx$.

11. Which integral should be evaluated using substitution? Evaluate both integrals:

(a) $\int \frac{9\,dx}{1+x^2}$

(b) $\int \frac{x\,dx}{1+9x^2}$

12. Find a so that if $x = au$, then $\sqrt{16+x^2} = 4\sqrt{1+u^2}$.

13. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

(a) $\int \frac{dx}{x^2+3}$

(b) $\int \frac{\cos(\ln(t))\,dt}{t}$

(c) $\int \frac{x\,dx}{\sqrt{7-x^2}}$

(d) $\int \frac{dt}{4t^2+9}$

(e) $\int \frac{\ln(\arccos(x))\,dx}{\arccos(x)\sqrt{1-x^2}}$

(f) $\int \frac{dt}{|t|\sqrt{12t^2-3}}$

(g) $\int \frac{dx}{(4x-1)\ln(8x-2)}$

(h) $\int e^{9-2x}\,dx$

Worksheet # 27: Linear Approximation

1. What is the relation between the linearization of a function $f(x)$ at $x = a$ and the tangent line to the graph of the function $f(x)$ at $x = a$ on the graph?

2. (a) Use the linearization of $\sqrt{x}$ at $a = 16$ to estimate $\sqrt{18}$.

   (b) Find a decimal approximation to $\sqrt{18}$ using a calculator.

   (c) Compute both the absolute error and the percentage error if we use the linearization to approximate $\sqrt{18}$.

3. For each of the following, use a linear approximation to the change in the function and a convenient nearby point to estimate the value:

   (a) $(3.01)^3$

   (b) $\sqrt{17}$

   (c) $8.06^{2/3}$

4. Suppose we want to paint a sphere of radius 200 cm with a coat of paint 0.1 mm thick. Use a linear approximation to approximate the amount of paint we need to do the job.

5. Let $f(x) = \sqrt{16 + x}$. First, find the linearization to $f(x)$ at $x = 0$, then use the linearization to estimate $\sqrt{15.75}$. Present your solution as a rational number.

6. Find the linearization $L(x)$ to the function $f(x) = \sqrt{1 - 2x}$ at $x = -4$.

7. Find the linearization $L(x)$ to the function $f(x) = \sqrt[3]{x + 4}$ at $x = 4$, then use the linearization to estimate $\sqrt[3]{8.25}$.

8. Your physics professor tells you that you can replace $\sin(\theta)$ with θ when θ is close to zero. Explain why this is reasonable.

9. Suppose we measure the radius of a sphere as 10 cm with an accuracy of $\pm$ 0.2 cm. Use linear approximations to estimate the maximum error in:

   (a) the computed surface area.

   (b) the computed volume.

10. Suppose that $y = y(x)$ is a differentiable function which is defined near $x = 2$, satisfies $y(2) = -1$ and

$$x^2 + 3xy^2 + y^3 = 9.$$

Use the linear approximation to the change in y to approximate the value of $y(1.91)$.

Worksheet # 28: Review for Final

An Interesting Fact: Taylor polynomials were investigated in the 1600's in Europe. However, formulas equivalent to the Taylor polynomials for $\sin(x)$, $\cos(x)$, and $\arctan(x)$ had been previously discovered by Madhava of Sangamagrama in the Kerala region of India, two hundred years before the work of Taylor, Newton, Gregory, and others European mathematicians. Trigonometric functions were crucial ingredients for the study of astronomy, and many historians believe that the need to construct accurate astronomical tables led Madhava to develop this theory. While it is possible that Madhava's work was brought to European scholars by Jesuits traveling in India, thus influencing European mathematicians, there is not definitive evidence either for or against this transmission of knowledge to Europe.

1. Compute the following limits.

   (a) $\lim_{t\to 9} \frac{9-t}{3-\sqrt{t}}$

   (b) $\lim_{\theta\to 0} \frac{\sin(3\theta)}{10\theta}$

   (c) $\lim_{x\to\infty} \frac{e^x}{x^2}$

   (d) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

2. (a) State the limit definition of the continuity of a function f at $x = a$.

   (b) State the limit definition of the derivative of a function f at $x = a$.

   (c) Given $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4-3x & \text{if } x \geq 1 \end{cases}$. Is the function continuous at $x = 1$? Is the function differentiable at $x = 1$? Use the definition of the derivative. Graph the function to check your answer.

3. Provide the most general antiderivative of the following functions:

   (a) $f(x) = x^4 + x^2 + x + 1000$

   (b) $g(x) = (3x-2)^{20}$

   (c) $h(x) = \frac{\sin(\ln(x))}{x}$

4. Use implicit differentiation to find $\frac{dy}{dx}$, and compute the slope of the tangent line at (1,2) for the following curves:

   (a) $x^2 + xy + y^2 + 9x = 16$

   (b) $x^2 + 2xy - y^2 + x = 2$

5. An rock is thrown up the in the air and returns to the ground 4 seconds later. What is the initial velocity? What is the maximum height of the rock? Assume that the rock's motion is determined by the acceleration of gravity, 9.8 meters/second^2.

6. A conical tank with radius 5 meters and height 10 meters is being filled with water at a rate of 3 cubic meters per minute. How fast is the water level increasing when the water's depth is 3 meters?

7. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of the container is twice its width. Material for the base costs \$10 per square meter while material for the sides costs \$6 per square meter. Find the cost of materials for the least expensive possible container.

8. (a) State the Mean Value Theorem.

   (b) If $3 \leq f'(x) \leq 5$ for all x, find the maximum possible value for $f(8) - f(2)$.

9. Use linearization to approximate $\cos(\frac{11\pi}{60})$

   (a) Write down $L(x)$ at an appropriate point $x = a$ for a suitable function $f(x)$.

   (b) Use part(a) to find an approximation for $\cos(\frac{11\pi}{60})$

   (c) Find the absolute error in your approximation.

10. Find the value(s) c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} (cx)^3 & \text{if } x < 2 \\ \ln(x^c) & \text{if } x \geq 2 \end{cases}$$

11. (a) Find y' if $x^3 + y^3 = 6xy$.

    (b) Find the equation of the tangent line at $(3, 3)$.

12. Show that the function $f(x) = 3x^5 - 20x^3 + 60x$ has no absolute maximum or minimum.

13. Compute the following definite integrals:

    (a) $\displaystyle\int_{-1}^{1} e^{u+1}\,du$

    (b) $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\,dx$

    (c) $\displaystyle\int_{1}^{9} \frac{x-1}{\sqrt{x}}\,dx$

    (d) $\displaystyle\int_{0}^{10} |x - 5|\,dx$

    Hint: For some of the integrals, you will need to interpret the integral as an area and use facts from geometry to compute the integral.

14. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$