Name:

Date:

Start and End times:

MA 614, Enumerative Combinatorics
Exam 1 – Take-Home Exam

This is a take-home exam.

Complete five of the following six problems. Clearly indicate which five problems you want to have graded.

Each problem is worth twenty points. You are allowed two hours to complete it and are not allowed to use your book or notes. You may not use separate blocks of time totalling two hours; you are to take this exam in a single two-hour block of time. You should return this exam to the instructor by Monday, March 7, at the beginning of class.

You are not to discuss any aspect of this exam with anyone other than the instructor until after class on Monday, March 7.

Problem 1: Provide a bijective proof of the following identity, where p is a positive integer:

$$\sum_{\substack{i+j+k=p\\ i,j,k\geq 1}} \binom{p}{i}\binom{p}{j}\binom{p}{k} = \binom{3p}{p} - 3\binom{2p}{p} + 3\,.$$

Problem 2: Let C_n be all compositions of n into odd parts. For an odd integer k define its weight to be $w(k) = 2^{(k-1)/2}$ and for a composition $c = (c_1, \dots, c_r)$ define its weight to be the product $w(c) = w(c_1) \cdot w(c_2) \cdots w(c_r)$. Let a_n be the sum

$$a_n := \sum_{c \in C_n} w(c)\,.$$

Find the generating function $\sum_{n \geq 0} a_n x^n$ and use this to find an explicit formula for a_n.

Problem 3: Let x and y be variables satisfying the commutation relation $qxy = yx$, where q commutes with x and y. Using the lattice path interpretation of the q-binomial coefficients, give a combinatorial proof that

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k} .$$

Problem 4: Let $S(n, k)$ denote the Stirling numbers of the second kind and $s(n, k) = (-1)^{n-k} c(n, k)$ denote the Stirling numbers of the first kind. Prove that for any two positive integers m and n,

$$\sum_{k \geq 0} S(m, k) s(k, n) = \delta_{m,n}$$

Problem 5: Provide a combinatorial proof that

$$\prod_{i\geq 1}(1+xq^i) = \sum_{k\geq 0} \frac{x^k q^{\binom{k+1}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

Problem 6: A *Motzkin path* is a lattice path from $(0,0)$ to $(n,0)$, using only the steps $(1,1)$, $(1,-1)$, and $(1,0)$, that does not drop below the x-axis. The *Motzkin number* $M(n)$ is the number of Motzkin paths to $(n,0)$. Find the ordinary generating function of the sequence $M(n)$ for $n \geq 0$.