MA 614 – Homework 10
Due Monday, February 7

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. Consider the sequence defined by $a_n = \sum_c 2^{c_1-1} \cdot 2^{c_2-1} \cdots 2^{c_k-1}$ where $c = (c_1, c_2, \ldots, c_k)$ ranges over all compositions of n.

   (a) Find the generating function $\sum_{n\geq 0} a_n x^n$.

   (b) Use the generating function to find an explicit formula for a_n.

2. (a) For all $n \geq 1$, provide a bijection between the sets A_n and B_n, defined as follows. A_n is the set of all compositions $c_1 + \cdots + c_k$ of n in which each c_i comes in 2^{c_i-1} "colors." B_n is the set of all sequences $d_1 d_2 \cdots d_{n-1}$ where each $d_i \in \{0, 1, 2\}$.

   (b) Explain why this bijection gives a second proof of your explicit formula for a_n in Problem 1.

3. A *Delannoy path* is a lattice path from $(0, 0)$ to (m, n) using only steps of the form $(1, 0)$,$(0, 1)$, and $(1, 1)$. The *Delannoy numbers* $d_{m,n}$ are the number of Delannoy paths to (m, n).

   (a) Find the ogf $\sum_{m\geq 0} \sum_{n\geq 0} d_{m,n} x^m y^n$.
   Suggestion: Find and use a simple multivariable recursion.

   (b) Prove that
   $$\frac{x^j}{(1-x)^{j+1}} = \sum_{n\geq 0} \binom{n}{j} x^n \,.$$

   (c) Prove using generating functions that
   $$d_{m,n} = \sum_{j\geq 0} 2^j \binom{m}{j} \binom{n}{j} \,.$$

   Suggestion: Multiply both sides by $x^m y^n$, sum over m and n, and work toward the ogf you obtained in part 3a.