MA 614 – Homework 11
Due Wednesday, February 9

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. A *Schröder path* is a Delannoy path from $(0,0)$ to (n,n) that never rises above the line $x=y$. The *Schröder number* sc_n is the number of Schröder paths to (n,n).

   (a) Prove that $sc_{n+1} = sc_n + \sum_{i=0}^{n} sc_i sc_{n-i}$, with $sc_0 = 1$.

   (b) Prove that
   $$\sum_{n\geq 0} sc_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}\,.$$

   NOTE: We now have four classes of lattice paths with interesting recurrences and generating functions: the binomial coefficients count lattice paths with north and east steps, while the Catalan numbers count the ones that don't pass above the line $x=y$; the Delannoy numbers count lattice paths with north, east, and north-east steps, while the Schröder numbers count those that don't pass above the line $x=y$.

2. In this problem we consider Delannoy paths equipped with an interesting weighting system. Let P be a Delannoy path from $(0,0)$ to (m,n). Define the weight of P to be $w(P) = (-1)^k$, where k is the number of steps of type $(1,1)$ in P. Let $a_{m,n} = \sum_P w(P)$, where the sum is taken over all Delannoy paths to (m,n).

   (a) What does $a_{m,n}$ count?

   (b) Find the ogf $\sum_{m\geq 0}\sum_{n\geq 0} a_{m,n} x^m y^n$.

   (c) Use the ogf to find an explicit formula for $a_{m,n}$.