# MA 614 – Homework 25
# Due Friday, April 1

Your answers should be detailed explanations in quality mathematical English.
NOTE: THIS HOMEWORK MAY BE HANDWRITTEN.

This problem deals with a sieving process in linear algebra that is important for combinatorics, algebra, and topology. Let

$$0 \to V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} W \to 0 \tag{1}$$

be an *exact sequence* of finite-dimensional vector spaces over some field, i.e. the $\partial_j$'s are linear transformations satisfying $\operatorname{im} \partial_{i+1} = \ker \partial_i$, where $\partial_n$ is injective and $\partial_0$ is surjective.

1. Show that

$$\dim W = \sum_{i=0}^{n} (-1)^i \dim V_i \, . \tag{2}$$

   HINT: Induct on $n$.

2. Show that for all $j$ such that $0 \leq j \leq n$,

$$\operatorname{rank} \partial_j = \sum_{i=j}^{n} (-1)^{i-j} \dim V_i \, , \tag{3}$$

   so in particular the quantity on the right-hand side is non-negative.

3. Suppose that we are given only that the sequence of spaces and maps (1) is a *complex*, i.e. $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$ for all $i$. Show that if equation (3) holds for all $j$ such that $0 \leq j \leq n$, then the complex (1) is exact.

   NOTE: A consequence of this is that for complexes of vector spaces, exactness is characterized by a family of sieving formulas using the dimensions of the spaces and ranks of the transformations.