MA 614 – Homework 27
Due Friday, April 15

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. Let n be an integer greater than or equal to 3. Determine the number of rank n distributive lattices such that each rank between 1 and $n-1$ (inclusive) has exactly two elements.

2. Let $\mathcal{C}$ be the set of all compositions of all positive integers. Define a partial ordering on $\mathcal{C}$ by letting τ cover $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ if τ can be obtained from σ either by adding 1 to a part, or adding 1 to a part and then splitting this part into two parts. More precisely, for some j we have either

$$\tau = (\sigma_1, \sigma_2, \ldots, \sigma_{j-1}, \sigma_j + 1, \sigma_{j+1}, \ldots, \sigma_k)$$

or

$$\tau = (\sigma_1, \sigma_2, \ldots, \sigma_{j-1}, h, \sigma_j + 1 - h, \sigma_{j+1}, \ldots, \sigma_k)$$

for some $1 \leq h \leq \sigma_j$.

   (a) For each $\sigma \in \mathcal{C}$, find a relationship between the number of saturated chains from the composition 1 (being the bottom element of $\mathcal{C}$) to σ and permutations with specified descent sets.

   (b) For fixed n, what is the total number of saturated chains that begin at 1 and end at a composition of n?

3. Let P be a poset with elements $t_1, t_2, \ldots, t_p$, which we regard as indeterminates. Define a $p \times p$ matrix A by

$$A_{i,j} := \begin{cases} 0 & if\ t_i < t_j \\ 1 & otherwise \end{cases}$$

Define the diagonal matrix $D = \operatorname{diag}(t_1, \ldots, t_p)$ (that is, the diagonal matrix with (i,i)-th entry t_i), and let I denote the $p \times p$ identity matrix. Show that

$$\det(I + DA) = \sum_{C} \prod_{t_i \in C} t_i$$

where C ranges over all chains in P.

HINT: Recall that the coefficient of x^j in $\det(I + xDA)$ is the sum of the principal $j \times j$ minors of DA.