MA 614 – Homework 28
Due Friday, April 22

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. For two bounded posets P and Q (that is, each poset has a $\hat{0}$ and $\hat{1}$), define the diamond product of P and Q to be $P \diamond Q := [(P - \hat{0}) \times (Q - \hat{0})] \cup \hat{0}$. Show that $\mu(P \diamond Q) = -\mu(P) \cdot \mu(Q)$. (For a poset R with $\hat{0}$ and $\hat{1}$, $\mu(R)$ is defined to be $\mu(\hat{0}, \hat{1})$.)

2. For the Boolean algebra B_n, prove that

$$\sum_{\hat{0} \leq a \leq b \leq c \leq d \leq e \leq \hat{1}} (\mu(\hat{0}, a))^4 (\mu(a, b))^3 (\mu(b, c))^6 (\mu(c, d)) (\mu(d, e))^{10} (\mu(e, \hat{1}))^2 = 2^n \, .$$

3. Let P be the poset on the set $\mathbf{Z}^2 = \{(a, b) : a, b \in \mathbf{Z}\}$ with the order relation

$$(a, b) \leq (c, d) \text{ if } b \leq d \text{ and } |c - a| \leq d - b \, .$$

   Find an expression for the Möbius function $\mu((0, 0), (m, n))$ and prove that it holds for all m, n.

4. Let P_k be the poset on the set $\hat{0} \cup [\{0, 1, 2, 3, \ldots\} \times \{1, 2, \ldots, k\}]$ for $k \geq 2$, where the cover relation is defined by the three relations

   (a) $\hat{0} \lessdot (0, j)$ for $1 \leq j \leq k$,

   (b) $(n, j) \lessdot (n + 1, j)$ for $1 \leq j \leq k$, and

   (c) $(n, j) \lessdot (n + 1, j + 1)$ for $1 \leq j \leq k$,

   where we let $k + 1 = 1$ in the second coordinate. Find an expression for the Möbius function $\mu(\hat{0}, (n, j))$ in the poset P_k.