MA 614 – Homework 7
Due Monday, Jan 31

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

NOTE: The following problems are taken from Problems 19 and 25 in Richard Stanley's Enumerative Combinatorics Volume 2. These exercises are available for free as a 27-page pdf file at http://www-math.mit.edu/∼rstan/ec/. Collected on Stanley's website are 190 different combinatorial interpretations of the Catalan numbers, and it is both fun and humbling to look through the list and establish bijections between some of these families. Here is a chance to try your hand at some of them.

RECALL: The reading established that the number of lattice paths from $(0,0)$ to (n,n) with steps from $(1,0)$ and $(0,1)$ that never rise above the line $x = y$ is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Establish enough bijections between the following sets to show that they are all counted by C_n. You do not need to give bijections between all pairs; you only need to find enough bijections to show they can all be put in bijection with each other (thus you will need at least three bijections).

1. Lattice paths from $(0,0)$ to (n,n) with steps from $(1,0)$ and $(0,1)$ that never rise above the line $x = y$. (Stanley 19(h))
2. The set of ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points. (Stanley 19(o))
3. Sequences $1 \le a_1 \le a_2 \le \cdots \le a_n$ of integers with $a_i \le i$. (Stanley 19(t))
4. Permutations $w_1 w_2 \cdots w_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e. there does not exist $i < j < k$ with $a_i > a_j > a_k$). These are called 321-*avoiding* permutations, as they avoid subsequences that decrease like 321 does. (Stanley 19(ee))

Just for fun, here are two challenge values that are also equal to the Catalan numbers (these problems are not required and you don't have to turn them in).

1. From Lie theory: Dimension of the space of invariants of $SL(2,\mathbb{C})$ acting on the $2n$-th tensor power $T^{2n}(V)$ of its defining representation. (Stanley 25(b))
2. From algebraic geometry: Dimension of the primitive intersection homology with real coefficients of the toric variety associated with the cube $[0,1]^n$. (Stanley 25(d))