MA 614 – Homework 11
Due Friday, Apr 3

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. Prove that $\binom{n}{k}$, $k \geq 0$ and n fixed, is log-concave using simple algebraic calculations.

2. Prove the following identities by finding a sign-reversing involution on an appropriate set.

   (a) $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

   (b) $\displaystyle\sum_{\substack{i+j+k+l=n \\ i,j,k,l\geq 0}}(-1)^{i+k}\binom{n}{i,j,k,l} = 0$

3. Use a sign-reversing involution to prove that $\begin{bmatrix}2n \\ 2k\end{bmatrix}_q$ evaluated at $q=-1$ is equal to $\binom{n}{k}$.

4. This problem deals with a sieving process in linear algebra that is important for combinatorics, algebra, and topology. Let

$$0 \to V_n \xrightarrow{\partial_n} V_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} W \to 0 \tag{1}$$

   be an *exact sequence* of finite-dimensional vector spaces over some field, which is defined to mean that the maps ∂_j are linear transformations satisfying $\operatorname{im}\partial_{i+1} = \ker\partial_i$, where ∂_n is injective and ∂_0 is surjective.

   (a) Show that

$$\dim W = \sum_{i=0}^{n}(-1)^i \dim V_i\,. \tag{2}$$

   (b) Show that for all j such that $0 \leq j \leq n$,

$$\operatorname{rank}\partial_j = \sum_{i=j}^{n}(-1)^{i-j}\dim V_i\,, \tag{3}$$

   so in particular the quantity on the right-hand side is non-negative.

   (c) Suppose that we are given only that the sequence of spaces and maps (1) is a *complex*, i.e. $\operatorname{im}\partial_{i+1} \subseteq \ker\partial_i$ for all i. Show that if equation (3) holds for all j such that $0 \leq j \leq n$, then the complex (1) is exact.

   NOTE: A consequence of this is that for complexes of vector spaces, exactness is characterized by a family of sieving formulas using the dimensions of the spaces and ranks of the transformations.

5. Find a simple formula for the number of maximal chains in the partition lattice Π_n.