MA 614 – Homework 14
Due Friday, April 24

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. For two bounded posets P and Q (that is, each poset has a $\hat{0}$ and $\hat{1}$), define the diamond product of P and Q to be $P \diamond Q := [(P - \hat{0}) \times (Q - \hat{0})] \cup \hat{0}$. Show that $\mu(P \diamond Q) = -\mu(P) \cdot \mu(Q)$. (For a poset R with $\hat{0}$ and $\hat{1}$, $\mu(R)$ is defined to be $\mu(\hat{0}, \hat{1})$.)

2. Let P be the poset on the set $\mathbf{Z}^2 = \{(a, b) : a, b \in \mathbf{Z}\}$ with the order relation

$$(a, b) \leq (c, d) \text{ if } b \leq d \text{ and } |c - a| \leq d - b\,.$$

   Find an expression for the Möbius function $\mu((0, 0), (m, n))$ and prove that it holds for all m, n.

3. Let P be a finite poset. A map $f : P \to P$ on a poset P is called a *closure operator* if for all $s, t \in P$ we have $t \leq f(t)$, $f(f(t)) = f(t)$, and $s \leq t \implies f(s) \leq f(t)$. The fixed points of f, i.e. those $s \in P$ such that $f(s) = s$, are called *closed* elements. The elements of $f(P)$, i.e. the image of f, form an induced subposet of P called the *quotient* of P relative to the closure operator f.

   (a) Prove that an element $s \in P$ is closed if and only if $s \in f(P)$.

   (b) Show that for all $s, t \in P$,

$$\sum_{\substack{u \in P \\ f(u) = f(t)}} \mu(s, u) = \begin{cases} \mu_{f(P)}(f(s), f(t)) & \text{if } s = f(s) \\ 0 & \text{if } s < f(s) \end{cases}$$