MA 614 – Homework 7
Due Friday, Feb 27

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. Prove that the following are enumerated by the Catalan numbers, either by finding a bijection with a known Catalan object or by showing that the objects satisfy the Catalan recurrence.

   (a) The set of ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points.

   (b) Sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ of integers with $a_i \leq i$.

2. A *Delannoy path* is a lattice path from $(0,0)$ to (m,n) using only steps of the form $(1,0)$,$(0,1)$, and $(1,1)$. The *Delannoy numbers* $d_{m,n}$ are the number of Delannoy paths to (m,n).

   (a) Find the ogf $\sum_{m\geq 0}\sum_{n\geq 0} d_{m,n}x^m y^n$.
   (HINT: Find and use a simple multivariable recursion.)

   (b) Prove that
   $$\frac{x^j}{(1-x)^{j+1}} = \sum_{n\geq 0}\binom{n}{j}x^n .$$

   (c) Prove using generating functions that
   $$d_{m,n} = \sum_{j\geq 0} 2^j \binom{m}{j}\binom{n}{j} .$$
   Suggestion: Multiply both sides by $x^m y^n$, sum over m and n, and work toward the ogf you obtained in part 2a.

3. In this problem we consider Delannoy paths equipped with an interesting weighting system. Let P be a Delannoy path from $(0,0)$ to (m,n). Define the weight of P to be $w(P) = (-1)^k$, where k is the number of steps of type $(1,1)$ in P. Let $a_{m,n} = \sum_P w(P)$, where the sum is taken over all Delannoy paths to (m,n).

   (a) What does $a_{m,n}$ count?

   (b) Find the ogf $\sum_{m\geq 0}\sum_{n\geq 0} a_{m,n}x^m y^n$.

   (c) Use the ogf to find an explicit formula for $a_{m,n}$.

4. A *Schröder path* is a Delannoy path from $(0,0)$ to (n,n) that never rises above the line $x = y$. The *Schröder number* sc_n is the number of Schröder paths to (n,n).

   (a) Prove that $sc_{n+1} = sc_n + \sum_{i=0}^{n} sc_i sc_{n-i}$, with $sc_0 = 1$.

   (b) Prove that
   $$\sum_{n\geq 0} sc_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x} .$$

NOTE: We now have four classes of lattice paths with interesting recurrences and generating functions: the binomial coefficients count lattice paths with north and east steps, while the Catalan numbers count the ones that don't pass above the line $x = y$; the Delannoy numbers count lattice paths with north, east, and north-east steps, while the Schröder numbers count those that don't pass above the line $x = y$.