MA 614 – Homework 9
Due Friday, Mar 13

Your answers should be detailed explanations in quality mathematical English. You must type your homework in LaTeX.

1. For $n \geq 1$, prove that
$$\prod_{k=0}^{n-1}(1+q^k x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \,.$$
NOTE: This shows that one useful q-analogue of $f_n(x) = (1+x)^n$ is
$$f_n(x,q) = \prod_{k=0}^{n-1}(1+q^k x)\,.$$

2. Let x and y be variables satisfying the commutation relation $qxy = yx$, where q commutes with x and y. Show that
$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}\,.$$

3. Let a_n be the number of ways of doing the following: Partition the elements of $[n]$ into nonempty blocks, then partition the blocks themselves into nonempty "superblocks."
   (a) Determine the egf for a_n.
   (b) Calculate a_3 and confirm this value by explicitly enumerating all possibilities.

4. Find the egf and an expression for the number of permutations of $[n]$ having no 2-cycles.

5. Suppose we have a room with n children which are distinguishable. The children gather into circles by holding hands, and one child stands in the center of each circle. A circle may consist of as few as one child clasping his or her hands around the other child at the center of the circle. Let h_n be the number of ways this can be done. Find the egf for h_n.