

Day 1: MATH

- Goal of this subject is to count, i.e., for a finite set S or sequence of finite sets $S_1, S_2, \dots$, find $|S|$ or $|S_1|, |S_2|, \dots$.

Remark: This problem has a trivial solution.

Thm (Cauchy Theorem): $|S| = \sum_{x \in S} 1$.

- So, the question isn't "can it be done," but rather "what does it mean to count something well, or effectively, or efficiently, or elegantly, etc?"
- This is the question Stanley answers in items #1-5 of EC1 reading @ beginning of Chapter 1. To elaborate:

So, but

Defⁿ: Given a sequence $(a_n)_{n \geq 0}$,

- (1) The ordinary generating function (ogf) is $\sum_{n \geq 0} a_n x^n$.
- (2) The exponential generating function (egf) is $\sum_{n \geq 0} a_n \frac{x^n}{n!}$.

Note: These functions exist in $\mathbb{C}[[x]]$, the ring of formal power series. We don't worry about convergence, the powers of x are really placeholders & helpful notation for algebraic manipulations.

Three incredibly useful generating functions:

$$(i) \quad 1 + x + x^2 + \dots = \frac{1}{1-x} \quad (\text{geometric series, egf for } 1, 1, 1, \dots)$$

$$(ii) \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x \quad (\text{egf for } 1, 1, 1, \dots)$$

$$(iii) \quad 0 + \frac{x^1}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \ln\left(\frac{1}{1-x}\right) = -\ln(1-x) \\ (\text{egf for } a_n = (n-1)!)$$

Note: We can use analysis if the series is analytic, but we don't require this.

General Philosophy: To enumerate elements in a set well, we need to exploit underlying structure in the elements, leading to the study of:

Combinatorial
Objects

Partial orders
(posets)

Sequences

Recurrences

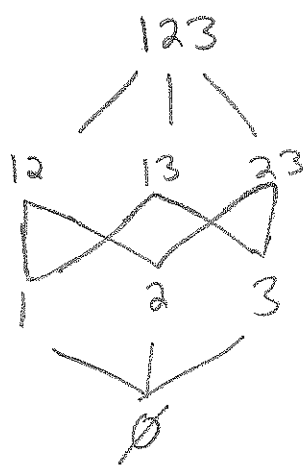
Refinements

Statistics

generating functions

To understand how these are related to each other is to understand enumeration at a basic level.

Example: Subsets of a 3-element set: There are 2^3 of them.



- partially ordered by inclusion
- Cayman $\Rightarrow 2^3 = \underbrace{1+1+1}_{\text{singles}} + \underbrace{1+1}_{\text{double}} + \underbrace{1}_{\text{triple}}$
- Statistic: For $A \subseteq [3] := \{1, 2, 3\}$, $|A|$ = size of A .
- Thus, 2^3 is refined to $(1, 3, 3, 1)$.

- ogf for $(1, 3, 3, 1)$ is $1 + 3x + 3x^2 + x^3$, which factors as $(1+x)^3$. So, we have a theorem:

$$\underbrace{(1+x)^3}_{\text{elegant, efficient}} = \sum_{\underbrace{A \subseteq [3]}_{\text{Refinement of Cayman}}} x^{|A|}$$

- Recurrences arise when extending this to subsets of arbitrary sets $[n] := \{1, 2, \dots, n\}$, $(2^n)_{n \geq 0}$

Example: Defⁿ: Given $n \in \mathbb{Z}_{>0}$, a composition of n is an ordered sum $a_1 + a_2 + \dots + a_k = n$, with $a_i \in \mathbb{Z}_{>0}$.

$$n=1: 1$$

$$n=2: 2, 1+1$$

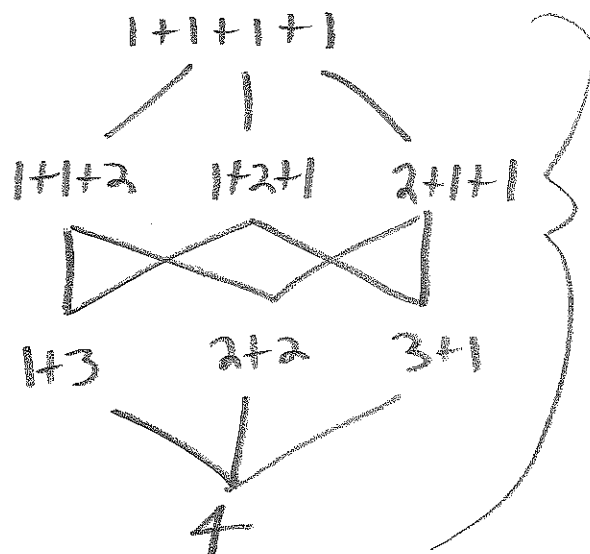
$$n=3: 3, 1+2, 2+1, 1+1+1$$

$$1 = 2^0$$

$$2 = 2^1$$

$$4 = 2^2$$

$n=4$:



This is very familiar!

Natural Q's: • Are there 2^{n-1} compositions of n ?

- Is there a bijection between these + subsets of $[n]$ that respects a statistic inducing $(1+x)^n$ as an ogf refinement for 2^n ?
 - Do nice recurrences for these sets have combinatorial interpretations that are respected by the bijection?
- e.g. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Remark 1: Answer to all Q's is Yes.

Remark 2: The sequence $(2^n)_{n \geq 0}$ has nice ogf + egf's,

• $\sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}$

• $\sum_{n \geq 0} 2^n \frac{x^n}{n!} = e^{2x}$

This is an example of how geometric series + exponentials are very fundamental.

Day 2: Pascal's Triangle, the binomial recurrence, and the binomial theorem.

The sequence $1, 2, 4, 8, 16, \dots$ is our starting point.

Notation: $[n] := \{1, \dots, n\}$, $2^S := \{A : A \subseteq S\}$, $|S|$ or $\#S$ is cardinality,
 $\uparrow$ "power set" of S

Prop 1: $|2^{[n]}| = 2^n$

Pf: $T \in 2^{[n]}$ is determined by making a choice for each $i \in [n]$ whether $i \in T$ or $i \notin T$. $\blacksquare$

Alternatively: For each subset of $[n-1]$, either choose to include n or not. Thus, $|2^{[n]}| = 2 \cdot |2^{[n-1]}|$. This is a recurrence, easy to solve.

Note: These proofs describe a combinatorial process of enumeration, but are not concretely combinatorial.

Best: Create a bijection $\Theta: 2^{[n]} \rightarrow \{0,1\}^n$, where

$$\Theta(A) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \text{ with } \varepsilon_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

This makes the proof of our Propⁿ explicit + clear.

Defⁿ: Call $\Theta(A)$ the characteristic vector of A .

Observe: $2^n = (1+1)^n = \overbrace{(1+1)(1+1)\dots(1+1)}^n$

$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{omit} & \text{incl.} & \text{omit} & \text{incl.} & \text{omit} & \text{incl.} \\ 1 & 1 & 2 & 2 & n & n \end{matrix}$

The Argument from pf of Propⁿ 1 is encoded in the arithmetic; we can turn it into algebra!

Ex: $(1+x_1)(1+x_2) = 1 + x_1 + x_2 + x_1x_2$

$(1+x_1)(1+x_2)(1+x_3) = 1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3$

subsets are encoded by monomials.

Propⁿ 2: $\prod_{i=1}^n (1+x_i) = \sum_{\{i_1, \dots, i_k\} \subseteq [n]} x_{i_1} x_{i_2} \dots x_{i_k}$

Pf: Exercise! $\square$

Defⁿ: Let $\binom{[n]}{k} := \{A : A \subseteq [n], |A|=k\}$ and $\binom{n}{k} := \left| \binom{[n]}{k} \right|$.
"n choose k"

The Binomial Thm:

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Pf: In propⁿ 2, evaluate each $x_i \equiv x$. $\square$

Remark: It is straightforward to extend this to

$\prod_{i=1}^n (y_i + x_i) = \sum_{A \subseteq [n]} \prod_{j \notin A} y_j \prod_{i \in A} x_i$, hence $(y+x)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

The recurrence $|2^{[n]}| = 2 \cdot |2^{[n-1]}|$ arises as $(1+x)^n = (1+x)(1+x)^{n-1}$.

This gives us a recurrence for $\binom{n}{k}$!

Pascal (aka Binomial) Recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof: We know $(1+x)^n = (1+x)(1+x)^{n-1}$, so

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x) \left(\binom{n-1}{0} + \binom{n-1}{1}x + \dots + \binom{n-1}{n-1}x^{n-1} \right)$$

$$= \binom{n-1}{0} + \left[\binom{n-1}{0} + \binom{n-1}{1} \right]x + \left[\binom{n-1}{1} + \binom{n-1}{2} \right]x^2 + \dots + \binom{n-1}{n-1}x^n.$$

Equating coefficients on both sides yields the result. ■

Note: The algebra confirms this is true; now, find a combinatorial proof!

2nd Proof: $\binom{n}{k} = \# \text{ of ways to pick } S \in \binom{[n]}{k}$

$$= \left[\# \text{ of ways to pick } S \in \binom{[n]}{k} \text{ with } n \in S \right] +$$

$$\left[\# \text{ of ways to pick } S \in \binom{[n]}{k} \text{ with } n \notin S \right]$$

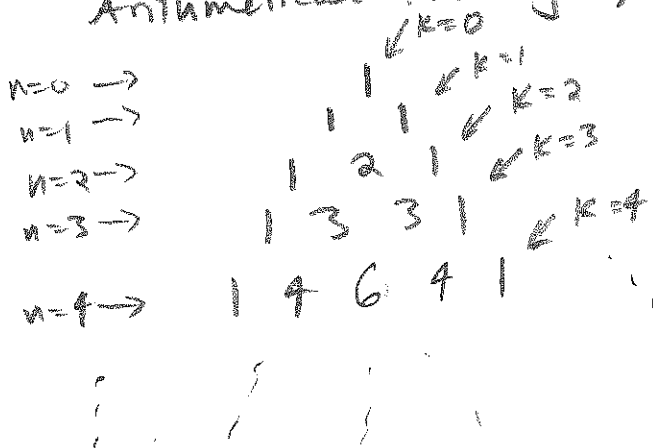
$$= \left[\# \text{ of } S \subseteq [n-1] \text{ w/ } |S| = k-1 \right] +$$

$$\left[\# \text{ of } S \subseteq [n-1] \text{ w/ } |S| = k \right]$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}.$$

■

∴ This recurrence gives rise to Pascal's Triangle (aka the Arithmetical triangle).



Final point: A formula exists for $\binom{n}{k}$, but we have not yet used it! Our goal is to use combinatorial arguments as much as possible, using algebra + formulas to investigate + discover new things.

Thm: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \leq k \leq n$, $k, n \in \mathbb{Z}$.

Pf: READ pg 16 in EC1. This proof is the model for a similar result about the Gaussian polynomial $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$. $\square$

Remark: Evaluations of $x+y$ in binomial theorem give interesting identities.

• $0 = (1-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k$.

• Differentiate $(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$ to get

$$n(1+x)^{n-1} = \sum_{k=0}^n \binom{n}{k} \cdot k \cdot x^{k-1}.$$

Set $x=1$, get

$$n \cdot 2^{n-1} = \sum_{k=0}^n \binom{n}{k} \cdot k. \quad \left. \vphantom{\sum_{k=0}^n} \right\} \leftarrow \text{HW\#2 includes finding a combinatorial proof of this.}$$

Day 3: Binomial Coefficients, integer compositions, & multisets.

Recall on Day 1 we discussed comps of 4 and $2^{[3]}$ and their similarities. Let's make this formal.

(I) We say a k-composition of n is a comp of n w/ k parts.

Ex: $1+2+1+4=8$ is a 4-comp of 8.

Thm: There is a bijection between k -comps of n and $\binom{[n-1]}{k-1}$.

Thus, there are $\binom{n-1}{k-1}$ k -comps of n .

Sketch of Pf: k -comp

$$\alpha = (a_1, a_2, \dots, a_k)$$

$$\text{s.t. } \sum a_i = n$$

records partial sums of α

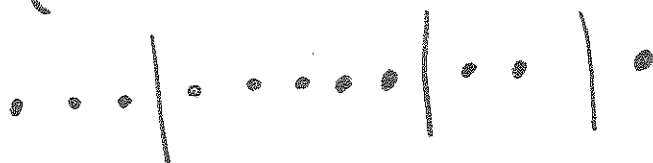
$$S_\alpha = \{a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\dots+a_{k-1}\} \subseteq [n-1]$$

Ex: $(3, 5, 2, 1) \longleftrightarrow \{3, 8, 10\} \subseteq [10]$.

4-comp of 11

3-subset

Picture ("stars + bars", or "balls into boxes", or "dots + bars")



dots: $3 + 5 + 2 + 1$

bars: $\begin{matrix} 3 & 8 & 10 \\ \text{dots to left} & \text{dots to left} & \text{dots to left} \end{matrix}$

Use 11 dots, 3 bars to break the dots into a composition.

Propⁿ: There are 2^{n-1} comps of n .

Pf: $2^{n-1} = \sum_{k=1}^n \binom{n-1}{k-1}$, by binomial thm.

of parts in composition

② Defⁿ: A weak composition of n is $\sum_{i=1}^k c_i = n$ w/ $c_i \in \mathbb{Z}_{\geq 0}$.

Ex: $1+0+2+1+0+0+4=8$ has 7 parts as a weak comp.

Q: How many weak k-comps of n are there? (Remember, $\binom{n-1}{k-1}$ k-comps of n ...)

Propⁿ: There are $\binom{n+k-1}{k-1}$ weak k-comps of n .

Pf: If $\sum_{i=1}^k c_i$ is a weak comp of n , then

$\sum_{i=1}^k (c_i + 1)$ is a comp of $n+k$. In fact,

all k-comps of $n+k$ arise in this way! We have a bijection. Hence, $\binom{n+k-1}{k-1}$ is an answer. $\square$

Ex: $1+0+2+0+0 \rightarrow 2+1+3+1+1=8$
 $= 3$

↖ subtract 1
 from each
 to go backward.

A Related Propⁿ: The # of solutions to $\sum_{i=1}^k c_i \leq n$ with $c_i \in \mathbb{Z}_{\geq 0}$ is $\binom{n+k}{k}$.

Pf: Introduce a slack variable $y = n - \sum_{i=1}^k c_i$, so we count solutions to $y + \sum_{i=1}^k c_i = n$ w/ $c_i, y \in \mathbb{Z}_{\geq 0}$.

By previous propⁿ, this is $\binom{n+k}{k}$. $\square$

III

Defⁿ: A multi-set is a set w/ some elements repeating a finite # of times.

Ex: $\{1, 1, 2, 5, 5, 5\} = \{1^2, 2^1, 5^3\}$

← multiplicities can be written as exponents.

← multiplicity 2
← mult 1
← mult 3

Defⁿ: $\left(\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right)\right)$ is the # of k-elt multisets of $[n]$.

Ex: $\left(\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)\right) = 6$, since $\left(\left(\begin{smallmatrix} [3] \\ 2 \end{smallmatrix}\right)\right) = \{11, 22, 33, 12, 13, 23\}$.

Propⁿ: $\left(\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right)\right) = \binom{n+k-1}{k}$.

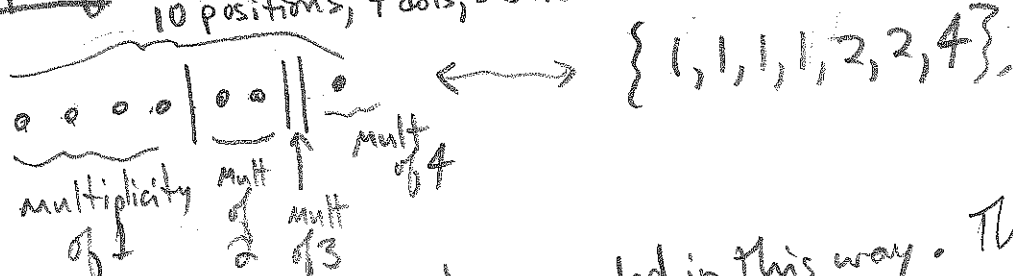
Remark: The reading contains 3 proofs of this.

1st Proof: Use weak comps. For each $A \in \left(\left(\begin{smallmatrix} [n] \\ k \end{smallmatrix}\right)\right)$, let $c_i := \#$ of times i appears in A .

Ex: for $\{1, 1, 2, 5, 5, 5\} \in \left(\left(\begin{smallmatrix} [5] \\ 6 \end{smallmatrix}\right)\right)$, $C = (2, 1, 0, 0, 3)$.

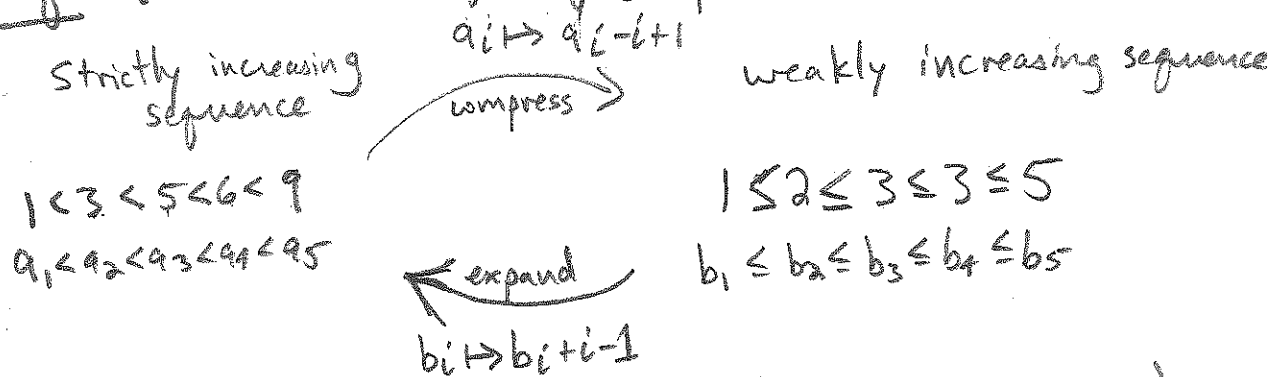
This gives a bijection btwn weak n-comps of k and k -multisets of $[n]$, hence the result. $\square$

2nd proof: Use "stars + bars": k dots (aka stars) and $n-1$ bars, 10 positions, 7 dots, 3 bars.



Every multiset can be encoded in this way. Thus, for $(n-1+k)$ total positions we choose k of them to be dots, giving $\binom{n-1+k}{k}$. $\square$

3rd proof: Use the technique of compressing sequences.



Using this, we obtain a bijection $\left(\begin{matrix} [n] \\ k \end{matrix} \right) \longleftrightarrow \left(\begin{matrix} [n+k-1] \\ k \end{matrix} \right)$

from which the result follows. $\blacksquare$

Ex: For $\left(\begin{matrix} [4] \\ 7 \end{matrix} \right)$ and $\left(\begin{matrix} 10 \\ 7 \end{matrix} \right)$, we have

$\{1, 1, 1, 1, 2, 2, 4\} \longleftrightarrow \{1, 2, 3, 4, 6, 7, 10\}$

multisets can be written in weakly increasing order

subsets can be written in strictly increasing order.

SUMMARY:

- $\left(\begin{matrix} n-1 \\ k-1 \end{matrix} \right)$ counts # of k -comps of n and stars + bars w/ n dots, k bars, no consecutive pair of bars.
- $\left(\begin{matrix} n+k-1 \\ k-1 \end{matrix} \right)$ counts weak k -comps of n and stars + bars w/ n dots, $k-1$ bars in arbitrary position.
- $\left(\begin{matrix} n+k \\ k \end{matrix} \right)$ counts ^{# of} k -comps of all elements of $\{0, 1, 2, \dots, n\}$.
- $\left(\begin{matrix} n+k-1 \\ k \end{matrix} \right)$ counts k -multisets of $[n]$, and weak k -comps of k , and stars + bars w/ $n-1$ bars + k dots.

Day 4: Generating functions:

There are two standard ways to produce a generating fn.

① Start w/ $a_0, a_1, a_2, a_3, \dots$, form egf or ecf.

② Start w/ a set + sum over the elements using a statistic.

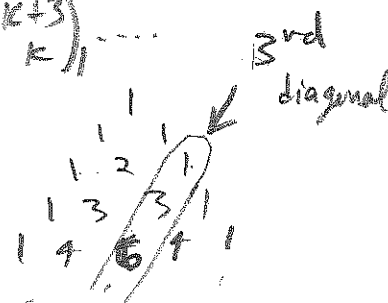
Defⁿ: A statistic on a set X is a function $\phi: X \rightarrow \mathbb{Z}$.

Ex: ① $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}, 0, 0, 0, \dots \Rightarrow$

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

② For $A \in 2^{[n]}$, use $|A|$ as statistic. Then $\sum_{A \in 2^{[n]}} x^{|A|} = (1+x)^n$.

Ex: Fix $k \geq 0$. The sequence $\binom{k}{k}, \binom{k+1}{k}, \binom{k+2}{k}, \binom{k+3}{k}, \dots$ is a diagonal in Pascal's triangle. Previous example was a row of this.



$$\text{Pf: } \sum_{n \geq 0} \binom{k+n}{k} x^n = \sum_{n \geq 0} \left(\sum_{\substack{c_1 + c_2 + \dots + c_{k+1} = n \\ c_i \in \mathbb{Z}_{\geq 0}}} x^n \right) = \sum_{n \geq 0} \sum_{\substack{c_1 + \dots + c_{k+1} = n \\ c_i \in \mathbb{Z}_{\geq 0}}} x^{c_1} x^{c_2} \dots x^{c_{k+1}}$$

$$= \underbrace{\left(1 + x + x^2 + \dots \right) \left(1 + x + x^2 + \dots \right) \dots \left(1 + x + x^2 + \dots \right)}_{k+1 \text{ copies}}$$

$$= \left(\frac{1}{1-x} \right)^{k+1} = \frac{1}{(1-x)^{k+1}}.$$

Another good Example: Let's re-prove that there are 2^{n-1} comps of n w/ parts in $\mathbb{Z}_{\geq 1}$. Set $a_n := \# \text{ comps of } n$.

Then

$$\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \sum_{\substack{c_1 + \dots + c_k = n \\ \text{for some } k}} x^n = \sum_{n \geq 0} \left(\sum_{k \geq 0} \sum_{\substack{c_1 + \dots + c_k = n \\ k \text{ fixed}}} x^{c_1} x^{c_2} \dots x^{c_k} \right)$$

$$= \sum_{n \geq 0} \left(\frac{x}{1-x} \right)^k = \frac{1}{1 - \frac{x}{1-x}} = \frac{1-x}{1-2x} = (1-x)(1 + 2x + 2^2x^2 + 2^3x^3 + \dots)$$

$$= 1 + (2-1)x + (2^2-2)x^2 + (2^3-2^2)x^3 + \dots = 1 + 1x + 2x^2 + 2^2x^3 + 2^3x^4 + \dots$$

So, $a_n = 2^{n-1}$.

- Remarks:
- Remember, we are not concerned w/ convergence.
 - Writing things like $\frac{1}{1-x} = 1 + x + x^2 + \dots$ algebraically makes sense because it holds true analytically (also because of $\mathbb{C}[[x]]$ having field of fractions $\mathbb{C}((x))$).
 - Pgs 3-8 in EC1 have several important defⁿs. Make sure you read it carefully over time, especially pgs on inverses, pg 8 on composition and formal derivatives.

The Binomial Series: For any $F(x) \in \mathbb{C}[[x]]$ s.t. $F(0)=0$, for any $\lambda \in \mathbb{C}$,
define

$$(1+F(x))^\lambda := \sum_{n \geq 0} \binom{\lambda}{n} (F(x))^n$$

$$= \sum_{n \geq 0} \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!} (F(x))^n.$$

NOTE: Pf is on HW.

Ex: $(1+x)^n$ we have seen.

Ex: $\frac{1}{(1-x)^{k+1}} = (1+(-x))^{-k-1} = \sum_{n \geq 0} \frac{(-k-1)(-k-2)\cdots(-k-n)}{n!} (-x)^n =$

$$\sum_{n \geq 0} \frac{(n+k)(n+k-1)\cdots(k+1)}{n!} x^n = \sum_{n \geq 0} \frac{(n+k)!}{n! k!} x^n = \sum_{n \geq 0} \binom{k+n}{k} x^n.$$

NOTE: We just proved that! Cool!

WARNING: Watch out for negative exponents, they can cause trouble.

Ex: $x^{-2} + x^{-1} + x^0 + x^1 + x^2 + \cdots$

$$= x^{-2} \cdot (x^0 + x^1 + x^2 + \cdots) = x^{-2} \cdot \frac{1}{1-x} = \frac{1}{x^2(1-x)}.$$

This is okay.

Ex: $\cdots + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + \cdots = \sum_{n \in \mathbb{Z}} x^n = \sum_{n < 0} x^n + \sum_{n \geq 0} x^n$

$$= \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots \right) + (1 + x + x^2 + \cdots) = \frac{\frac{1}{x}}{1 - \frac{1}{x}} + \frac{1}{1-x} = \frac{1}{x-1} + \frac{1}{1-x}$$

$$= \frac{-1}{1-x} + \frac{1}{1-x} = 0. \quad \text{What?!? That's awesome. Watch out.}$$

Observation: To shift a sequence, derive + integrate!

Ex: To make $a_0, a_1, a_2, a_3, \dots$ into $a_1, a_2, a_3, a_4, \dots$ use an eqn. + take a derivative.

$$\frac{d}{dx} \left(a_0 + a_1 \frac{x^1}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \dots \right) \\ = a_1 + a_2 \frac{x^1}{1!} + a_3 \frac{x^2}{2!} + a_4 \frac{x^3}{3!} + \dots$$

Observation: To multiply a_i by i , apply $x \cdot \frac{d}{dx}$ to the eqn.

$$x \cdot \frac{d}{dx} \left(a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots \right) = \\ x \cdot \left(0 + 1 \cdot a_1 x^0 + 2 \cdot a_2 x^1 + 3 \cdot a_3 x^2 + 4 \cdot a_4 x^3 + \dots \right) = \\ 0 \cdot a_0 + 1 \cdot a_1 x^1 + 2 \cdot a_2 x^2 + 3 \cdot a_3 x^3 + 4 \cdot a_4 x^4 + \dots$$

Ex: Applying $x \cdot \frac{d}{dx}$ to $\frac{x}{1-x} = x + x^2 + x^3 + \dots$ m times yields

$$\underbrace{x \cdot \frac{d}{dx} \cdot x \cdot \frac{d}{dx} \dots x \cdot \frac{d}{dx}}_{m \text{ times}} \left(\frac{x}{1-x} \right) = 1^m x + 2^m x^2 + 3^m x^3 + 4^m x^4 + \dots \\ = \sum_{j \geq 1} j^m x^j.$$

Note: Euler studied this in relation to $J(z)$, and it will come up again when we study descents for permutations.

Day 5: Fibonacci Numbers and Fibonacci Polynomials

- Question: How many compositions of n are there using only 1s and 2s as parts?

	#	set
<u>Ex</u> : $n=1$:	1	1
$n=2$:	2	1+1, 2
$n=3$:	3	1+1+1, 2+1, 1+2
$n=4$:	5	1+1+1+1, 2+1+1, 1+2+1, 1+1+2, 2+2
$n=5$:	8	1+1+1+1+1, 2+1+1+1, 1+2+1+1, 1+1+2+1, 2+2+1, 1+1+1+2, 2+1+2, 1+2+2

Observe: Every comp ends (right-hand side) w/a 1 or a 2.

- If ending w/ 1, get a comp of $n-1$ w/ other parts.
- If ending w/ 2, get a comp of $n-2$ w/ other parts.

Hence:

Propⁿ:

The number of comps of n using 1s + 2s, denoted F_n , satisfies $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 2$.

Defⁿ: The Fibonacci numbers are $F_0 = 1, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. (Warning: $F_0 = 0, F_1 = 1$ is often used)

- Let's study the ogf for the sequence F_n :
- $$\sum_{n \geq 0} F_n x^n = \sum_{n \geq 0} \sum_{k \geq 0} \sum_{\substack{c_1 + \dots + c_k = n \\ c_i \in \{1, 2\}}} x^n = \sum_{k \geq 0} \sum_{\substack{c_1 + \dots + c_k \\ c_i \in \{1, 2\}}} x^{c_1 + \dots + c_k} =$$
- $\uparrow$ total value of comp $\uparrow$ # of parts $\uparrow$ swap order of summation

$$\sum_{k \geq 0} \left[\underbrace{(x+x^2)}_{x^{c_1}} \underbrace{(x+x^2)}_{x^{c_2}} \dots \underbrace{(x+x^2)}_{x^{c_k}} \right] = \sum_{k \geq 0} (x+x^2)^k = \frac{1}{1-x-x^2}$$

• Observe: This ogf encodes the Fibonacci recurrence!

$$\sum_{n \geq 0} F_n x^n = \frac{1}{1-x-x^2} \Leftrightarrow (1-x-x^2) \left(\sum_{n \geq 0} F_n x^n \right) = 1 \Leftrightarrow$$

$$\begin{cases} 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \dots + 0 \cdot x^n + \dots \\ = F_0 + (F_1 - F_0)x + (F_2 - F_1 - F_0)x^2 + (F_3 - F_2 - F_1)x^3 + \dots + (F_n - F_{n-1} - F_{n-2})x^n + \dots \end{cases}$$

Comparing coefficients tells us the recurrence.
We will return to this observation soon.

So, we know $1, 1, 2, 3, 5, 8, 13, \dots$ satisfies:

$$F_n = F_{n-1} + F_{n-2}$$

$$\text{and } \sum_{n \geq 0} F_n x^n = \frac{1}{1-x-x^2}$$

Our key observation was to find a disjoint union:

$$\text{Comp}_n := \left\{ \begin{array}{l} \text{compositions of } n \\ \text{using } 1\text{'s \& } 2\text{'s} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{comps of } n-1 \\ \text{using } 1\text{'s \& } 2\text{'s} \end{array} \right\} \cup \left\{ \begin{array}{l} \text{comps of } n-2 \\ \text{using } 1\text{'s \& } 2\text{'s} \end{array} \right\}$$

$\xleftarrow{\text{add 1 on right to get a comp of } n}$ $\xleftarrow{\text{add 2 on right to get a comp of } n}$

Let's introduce a statistic related to this:

$$\phi: \text{Comp}_n \rightarrow \mathbb{Z}, \quad \phi(c_1 + \dots + c_k) = |\{i: c_i = 1\}|$$

= number of 1's in comp.

Defⁿ: The n^{th} Fibonacci polynomial is

$$F_n(x) := \sum_{c \in \text{Comp}_n} x^{\phi(c)}$$

Two Questions:

- (i) Is $F_n(x)$ related to the Fib. recurrence?
- (ii) What are the coefficients of $F_n(x)$?

NOTE: $F_n(1) = \sum_{c \in \text{Comp}_n} 1 = F_n$. Cayman Theorem!

Answer (i):

$$\text{Prop}^n: F_n(x) = x \cdot F_{n-1}(x) + F_{n-2}(x).$$

$$\text{Pf: } F_n(x) = \sum_{c \in \text{Comp}_n} x^{\#\{i: c_i=1\}} = \sum_{b \in \text{Comp}_{n-1}} x^{\#\{i: b_i=1\}+1} +$$

$$\sum_{a \in \text{Comp}_{n-2}} x^{\#\{i: a_i=1\}} = x \cdot F_{n-1}(x) + F_{n-2}(x).$$

Answer (ii):

$$\text{Prop}^n: F_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1}$$

Sketch of Pf: Use the recursion and induction.
Assuming true for $F_{n-1}(x) + F_{n-2}(x)$, we have

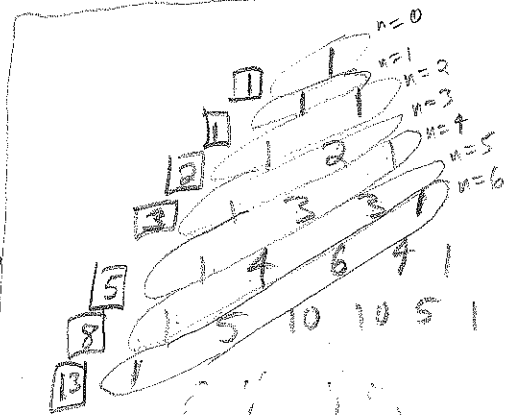
$$F_n(x) = x F_{n-1}(x) + F_{n-2}(x) =$$

$$x \left[\binom{n-2}{0} x^{n-2} + \binom{n-3}{1} x^{n-4} + \binom{n-4}{2} x^{n-6} + \dots \right] + \left[\binom{n-3}{0} x^{n-3} + \binom{n-4}{1} x^{n-5} + \binom{n-5}{2} x^{n-7} + \dots \right]$$

$$= \binom{n-2}{0} x^{n-1} + \left[\binom{n-3}{1} + \binom{n-3}{0} \right] x^{n-3} + \left[\binom{n-4}{2} + \binom{n-4}{1} \right] x^{n-5} + \dots$$

$$= \binom{n-2}{0} x^{n-1} + \binom{n-2}{1} x^{n-3} + \binom{n-3}{2} x^{n-5} + \dots$$

- Notes:
- (a) You have to carefully check what happens w/ linear + constant terms to make the proof rigorous.
 - (b) You can give a direct combinatorial argument as well, that the # of comps of n using 1s + 2s with $n-2j-1$ 1s is $\binom{n-j-1}{j}$.



"Shallow diagonals" in Pascal's triangle are coeffs of $F_n(x)$.

Our final Question: Can we extend $\sum_{n \geq 0} F_n x^n = \frac{1}{1-x-x^2}$ to $F_n(t)$? ↑
switch variable.

Yes!

Propⁿ: $1 + tx + (t^2+1)x^2 + (t^3+2t)x^3 + (t^4+3t^2+1)x^4 + \dots$
 $= \sum_{n \geq 0} F_n(t) x^n = \frac{1}{1-tx-x^2}$

Pf (I): We see that

$$(1-tx-x^2) \left(\sum_{n \geq 0} F_n(t) x^n \right) = F_0(t) + [F_1(t) - tF_0(t)]x + [F_2(t) - tF_1(t) - F_0(t)]x^2 + [F_3(t) - tF_2(t) - F_1(t)]x^3 + \dots = 1$$

Pf (II): $\sum_{n \geq 0} F_n(t) x^n = \sum_{k \geq 0} (tx + x^2)^k = \frac{1}{1-tx-x^2}$ □

comp's w/
k parts using 1's
+ 2's, a "t" for
each 1.

Day 6: Recurrence Relations + ogfs

Recall: $\sum_{n \geq 0} F_n x^n = \frac{1}{1-x-x^2}$. By quadratic formula, we have

$$1-x-x^2 = (1-\phi x)(1-\bar{\phi} x) \text{ where } \phi = \frac{1+\sqrt{5}}{2}, \bar{\phi} = 1-\phi = \frac{1-\sqrt{5}}{2}.$$

Here, we can use partial fractions!

$$\sum_{n \geq 0} F_n x^n = \frac{1}{1-x-x^2} = \frac{1}{(1-\phi x)(1-\bar{\phi} x)} \xrightarrow{\text{assume partial fractions}} \frac{\alpha}{1-\phi x} + \frac{\beta}{1-\bar{\phi} x} \xrightarrow{\text{expand geometric series}}$$

$$\sum_{n \geq 0} \alpha \cdot \phi^n x^n + \sum_{n \geq 0} \beta \cdot \bar{\phi}^n x^n = \sum_{n \geq 0} (\alpha \phi^n + \beta \bar{\phi}^n) x^n.$$

So, $F_n = \alpha \phi^n + \beta \bar{\phi}^n$ for some α, β . Using $n=0, 1$, we get

$$\left. \begin{array}{l} 2 \text{ eqns,} \\ 2 \text{ unknowns.} \end{array} \right\} \begin{cases} F_0 = 1 = \alpha + \beta, & F_1 = 1 = \alpha \phi + \beta \bar{\phi} \\ & = \alpha \left(\frac{1+\sqrt{5}}{2} \right) + \beta \left(\frac{1-\sqrt{5}}{2} \right). \end{cases}$$

Solving gives $\alpha = \frac{\phi}{\sqrt{5}}, \beta = -\frac{\bar{\phi}}{\sqrt{5}}.$

Hence, we have proved:

Thm (Binet's formula): $F_n = \frac{\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}}{\sqrt{5}}.$

Three key properties of F_n :

① Recurrence: $F_n - F_{n-1} - F_{n-2} = 0$

② Rational ogf: $\frac{1}{1-x-x^2}$

③ Explicit formula in terms of roots: $\frac{1}{\sqrt{5}} (\phi^{n+1} - \bar{\phi}^{n+1})$ of $1-x-x^2$

NOTE: Stanley uses $F_0=0, F_1=1$, hence his formula having $\frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$ for F_n .

Theorem: The following are equivalent, where for all we are given $\alpha_1, \alpha_2, \dots, \alpha_d$ a fixed sequence of complex numbers with $d \geq 1, \alpha_d \neq 0$, and consider a sequence $(a_n)_{n \geq 0}$:

(i) $\sum_{n \geq 0} a_n x^n = \frac{P(x)}{Q(x)}$ where

- $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d$
- $P(x)$ is a polynomial of degree $\leq d$.

(ii) For all $n \geq 0$, (a_n) satisfies the linear recurrence

$$a_{n+d} + \alpha_1 a_{n+d-1} + \alpha_2 a_{n+d-2} + \dots + \alpha_d a_n = 0$$

(iii) Let $1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d = \prod_{i=1}^k (1 - \gamma_i x)^{d_i}$, with

the γ_i 's distinct + non-zero. Then

$$a_n = \sum_{i=1}^k P_i(n) \gamma_i^n, \text{ where } P_1(n), P_2(n), \dots, P_k(n) \text{ are each a polynomial in } n \text{ of degree less than } d_1, d_2, d_3, \dots, d_k, \text{ respectively.}$$

See §4.1 in Stanley for a proof

Another example: For any $l \in \mathbb{Z}_{\geq 2}$, the l -sequence is

$$a_0 = 1, a_1 = l, a_n - l a_{n-1} + a_{n-2} = 0, \text{ (as in (ii) above).}$$

Ex: $l=2$: 1, 2, 3, 4, 5, ...

$l=3$: 1, 3, 8, 21, 55, ...

← every other Fib. #

The o.g.f. for an l -sequence is $\frac{1}{1 - lx + x^2}$, (as in (i) above).

By using partial fractions on $1 - lx + x^2 = (1 - \gamma_1 x)(1 - \gamma_2 x)$, we can find an explicit formula for any of these sequences given by powers of $\gamma_1 + \gamma_2$.

Group Work: For $a_0=0$, $a_1=1$, $a_n = 2a_{n-1} + 3a_{n-2}$,

Show that

$$\textcircled{1} \sum_{n \geq 0} a_n x^n = \frac{x}{1-2x-3x^2}$$

$$\textcircled{2} a_n = \frac{1}{4}((-1)^{n+1} + 3^n)$$

Group Work: For the 3-sequence 1, 3, 8, 21, 55, given by $a_n = 3a_{n-1} - a_{n-2}$, find the ogf and an explicit formula for a_n . Is this formula related to Binet's formula for Fibonacci numbers?

Day 7: Combinatorial interpretations for recurrence relations

Suppose we have a recurrence relation

$$a_n + \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \dots + \alpha_d a_{n-d} = 0$$

with corresponding polynomial

$$1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d.$$

We can always relate this to compositions.

Method 1: Consider compositions using only $\{1, 2, \dots, d\}$.

Then the ogf is $\sum_{k \geq 0} (x + x^2 + \dots + x^d)^k = \frac{1}{1 - x - x^2 - \dots - x^d}.$

what if we have 2 types of 1s, + 3 types of 2s?

Then we get $\sum_{k \geq 0} (x + x + x^2 + x^2 + x^2)^k = \sum_{k \geq 0} (2x + 3x^2)^k = \frac{1}{1 - 2x - 3x^2}.$

of types.
shows up here!

Principle: The ogf for compositions of n

using α_1 "colors" of 1, α_2 "colors" of 2, ..., α_d "colors" of d , is

$$\sum_{k \geq 0} (\alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d)^k = \frac{1}{1 - \alpha_1 x - \dots - \alpha_d x^d}.$$

Method 2: What if we have positive coeff's in denominator? eg.

$$\frac{1}{1 - 2x + x^2} = \frac{1}{1 - (2x - x^2)} = \sum_{k \geq 0} (2x - x^2)^k.$$

how to interpret this?

Here, we think of the composition as being weighted.
So, this example is the ogf for comps of n
using only 1s + 2s w/ 2 colors for 1, where each
comp is given the weight $(-1)^{\# \text{ 2s in comp.}}$

Principle: Since

$$\sum_{n \geq 0} b_n x^n = \frac{1}{1 + \beta_1 x + \beta_2 x^2 + \dots + \beta_d x^d} = \sum_{k \geq 0} (-\beta_1 x - \beta_2 x^2 - \dots - \beta_d x^d)^k, \text{ we}$$

see that $b_n = \sum_{\substack{c_1 + c_2 + \dots + c_d \\ \text{comps of } n \text{ using} \\ 1, 2, \dots, d \text{ as parts}}} (-\beta_1)^{\#1s} (-\beta_2)^{\#2s} \dots (-\beta_d)^{\#ds}$.

Ex: Recall that for $F_n(t)$, our Fib. polynomials, we had $\sum_{n \geq 0} F_n(t) x^n = \frac{1}{1 - tx - x^2} = \sum_{k \geq 0} (tx + x^2)^k$.

So, $F_n(t) = \sum_{\substack{\text{Comps of } n \\ \text{w/ } 1s \text{ \& } 2s}} t^{\#1s}$, exactly as we previously defined.

IMPORTANT TOOL: Recall the binomial theorem:

$$(1 + F(x))^1 = \sum_{n \geq 0} \binom{1}{n} [F(x)]^n.$$

Ex: Find an explicit formula for a_n satisfying

$$a_n - 9a_{n-1} + 27a_{n-2} - 27a_{n-3} = 0, \quad a_0 = 1.$$

The ogf is $\frac{1}{1 - 9x + 27x^2 - 27x^3} = \frac{1}{(1 - 3x)^3}$. Now, use bin.thm!

$$(1 + (-3x))^{-3} = \sum_{n \geq 0} \binom{-3}{n} (-3x)^n = \sum_{n \geq 0} \binom{-3}{n} (-3)^n x^n = \sum_{n \geq 0} a_n x^n.$$

$$\begin{aligned} \text{Thus, } a_n &= \binom{-3}{n} (-3)^n = \frac{(-3)(-3-1)\dots(-3-n+1)}{n!} (-1)^n 3^n \\ &= \frac{3 \cdot 4 \cdot 5 \cdot \dots \cdot (n+2)}{n!} 3^n = \frac{(n+1)(n+2) \cdot 3^n}{2}. \end{aligned}$$

Note: Recurrences can be non-linear.

Ex: $a_{n+1} = a_n + n \cdot a_{n-1}$, $n \geq 0$, $a_0 = 1$.

↑
not a constant coefficient!

These are often best dealt with using exponential generating functions.

Ex: $\sum_{n \geq 0} a_n \frac{x^n}{n!} = e^{(x + \frac{x^2}{2})}$. Apply $\frac{d}{dx}$ to both sides, getting...

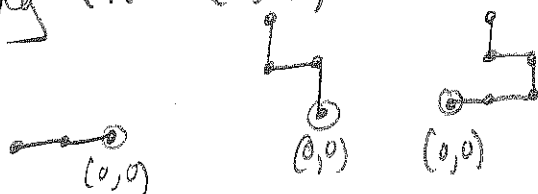
$\frac{d}{dx} \Rightarrow \sum_{n \geq 0} a_n \frac{x^{n-1}}{(n-1)!} = (1+x) e^{(x + \frac{x^2}{2})} = (1+x) \left[\sum_{n \geq 0} a_n \frac{x^n}{n!} \right]$

equating these gives $a_{n+1} = a_n + n \cdot a_{n-1}$.

NOTE: Compositions are only one combinatorial model for linear recurrence sequences.

Example: Let $f(n)$ be the number of paths w/ n steps starting from $(0,0)$, each step of the form $(-1,0), (1,0), (0,1)$, that never intersect themselves.

What is f_n ?



Remark: After a $(0,1)$ -step, we start fresh, since we can never go "back down" to a previous height. We can only self-intersect by doing $(1,0)$ and $(-1,0)$ after each other (in either order).

(next page)

We can directly count that $f(0)=1$, $f(1)=3$.

For $n \geq 2$, we can see that every path of length n ends with one of the following five sequences:

- ① $\uparrow (0,1)$ } $f(n-1)$ paths end this way.
 - ② $\rightarrow (1,0) + (1,0) \leftarrow$ associate to $(n-1)$ -paths ending in $(1,0)$
 - ③ $\leftarrow (-1,0) + (-1,0) \leftarrow$ associate to $(n-1)$ -paths ending in $(-1,0)$
 - ④ $\rightarrow (0,1) + (1,0) \leftarrow$ associate to $(n-1)$ -paths ending in $(0,1)$
 - ⑤ $\leftarrow (0,1) + (-1,0) \leftarrow$ } $f(n-2)$ paths end this way.
- $f(n-1)$ such paths

Thus, $f(n) = 2f(n-1) + f(n-2)$.

Since $f(0)=1$, $f(1)=3$, we get

$$\sum_{n \geq 0} f(n)x^n = \frac{1+x}{1-2x-x^2}.$$

See Example 4.1.3 in Stanley for an exact formula.

A Natural Question (I don't know the answer):

We can interpret $\frac{1+x}{1-2x-x^2} = \left[\frac{1}{1-2x-x^2} + \frac{x}{1-2x-x^2} \right]$ in terms of counting compositions using 2 types of 1's and 1 type of 2.

Is there a nice bijection from compositions counted by this ogf to the Paths described above?

Day 8: Partition identities

Defⁿ: A partition of $n \in \mathbb{Z}_{\geq 1}$ is a sequence $(\lambda_1, \lambda_2, \dots)$ s.t.
 $\sum_i \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$. Write $(\lambda_1, \dots, \lambda_k)$ for the non-zero parts. Write $\lambda \vdash n$ to denote this, and $|\lambda| = n$.

Notes: ① Partitions are compositions w/out order.

② Warning: These are tricky devils...

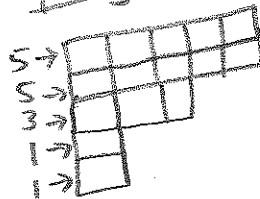
Observation: For compositions, dots & bars were a helpful schematic representation. For a partition, we use a Ferrers diagram or a Young diagram:

Ex: $1 + 1 + 3 + 5 + 5 = 15$

Ferrers

5 $\rightarrow$
5 $\rightarrow$
3 $\rightarrow$. . .
1 $\rightarrow$.
1 $\rightarrow$.

Young



Note: In advanced combinatorics, filling Young diagrams w/ numbers in boxes is common.

Q: We know there are 2^{n-1} comps of n . How many partitions of n ?

A: No one really has a good idea.

Defⁿ: Let $p(n)$ denote the number of partitions of n .

- $P_k(n) = \#$ partitions $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$ with $\lambda_k \geq 1$, i.e. w/ exactly k parts.
- $P_{\leq k}(n) = \#$ partitions of n w/ $\leq k$ parts.

Thm (Hardy + Ramanujan, 1918): $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\left[\pi \sqrt{\frac{2n}{3}}\right]}$ as $n \rightarrow \infty$.

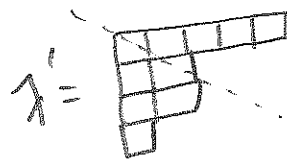
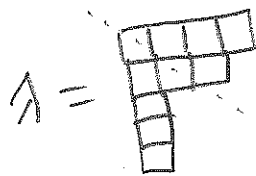
See the wikipedia page for more info on this + subsequent asymptotic series for $p(n)$. ①

$$\text{Prop}^n: \sum_{n \geq 0} p_{\leq k}(n) q^n = \frac{1}{(1-q)(1-q^2) \cdots (1-q^k)} = \prod_{j=1}^k \frac{1}{1-q^j}.$$

Remark: The fundamental tool for this proof is to observe an involution on partitions of n , where an involution is an order 2 permutation.

Defⁿ: Given $\lambda \vdash n$, define λ' , the conjugate of λ , to be the partition with $\lambda'_i = \lambda_{i+1}$ parts of size i .
Alternatively, geometrically "flip" the Young diagram of λ along its diagonal to get λ' .

Ex: $\lambda = (4, 3, 1, 1, 1)$ $\lambda' = (5, 2, 2, 1)$



flip to get

Note: If λ has $\leq k$ parts, then λ' has all parts $\leq k$.

$$\text{Proof of Prop}^n: \sum_{n \geq 0} p_{\leq k}(n) q^n = \sum_{n \geq 0} \left[\sum_{\substack{\lambda \vdash n \\ \lambda' w/ \text{largest part} \leq k}} q^{|\lambda|} \right] =$$

$$\sum_{n \geq 0} \left[\sum_{\substack{\lambda' \vdash n \\ \lambda' w/ \text{largest part} \leq k}} q^{|\lambda|} \right] = \sum_{n \geq 0} \sum_{\substack{m_1 + 2m_2 + 3m_3 + \cdots + km_k = n \\ m_i \geq 0}} q^{m_1 + 2m_2 + \cdots + km_k} \quad \begin{matrix} \uparrow \\ \text{factor geometric series} \end{matrix}$$

$\uparrow$ "multiplicity of i in λ' "

$$\left(\sum_{m_1 \geq 0} q^{m_1} \right) \left(\sum_{m_2 \geq 0} q^{2m_2} \right) \cdots \left(\sum_{m_k \geq 0} q^{km_k} \right) = \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdots \frac{1}{1-q^k}.$$

Corollary: (Euler's Partition Theorem)

$$\sum_{n \geq 0} p(n) q^n = \prod_{j \geq 1} \frac{1}{1 - q^j}$$

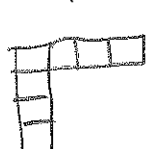
PF: Let $k \rightarrow \infty$ in the proposition; In terms of formal power series,

$$\lim_{k \rightarrow \infty} \left[\sum_{n \geq 0} p_{\leq k}(n) q^n \right] = \sum_{n \geq 0} p(n) q^n$$

Since for all $k \geq n$, the coefficient of q^n is stable. $\square$

Let's investigate the conjugation involution a bit more.

Q: Are any $\lambda = \lambda'$?

A: Yes! e.g.  ,  ,  , etc.

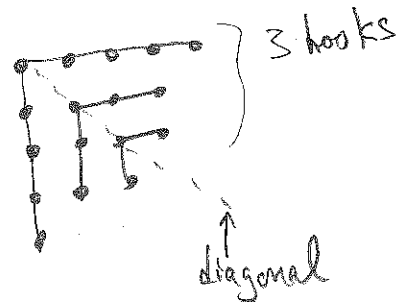
Defⁿ: Let $\text{Fix}(n)$ be the set of all self-conjugate partitions of n , i.e. $\lambda \vdash n$ with $\lambda = \lambda'$, i.e. the "Fixed" points of our involution.

Theorem: $|\text{Fix}(n)|$ is equal to the number of partitions of n w/ distinct, odd parts. Equivalently,

$$\sum_{n \geq 0} |\text{Fix}(n)| q^n = \prod_{i \geq 1} (1 + q^{2i-1}) = (1 + q)(1 + q^3)(1 + q^5) \dots$$

Sketch of PF: Create a bijection between $\text{Fix}(n)$ and $\{\mu \vdash n : \mu \text{ has distinct odd parts}\}$.
Let $\lambda \in \text{Fix}(n)$. The set of "diagonal hooks" of λ form a partition $\mu \vdash n$ with distinct odd parts. $\square$

Ex: $\lambda = (5, 4, 4, 3, 1)$



$\mapsto \mu = (9, 5, 3)$

Another well-known Theorem:

Euler's odd/distinct Theorem: Let $d(n) := \#$ of partitions of n into distinct parts,
 $P_{\text{odd}}(n) := \#$ of partitions of n into odd parts.

Then $\sum_{n \geq 0} d(n) q^n = \sum_{n \geq 0} P_{\text{odd}}(n) q^n$, i.e. $d(n) = P_{\text{odd}}(n)$.

Pf: $\sum_{n \geq 0} d(n) q^n = (1+q)(1+q^2)(1+q^3)(1+q^4) \dots$
 $= (1+q) \frac{(1-q)}{(1-q)} (1+q^2) \frac{(1-q^2)}{(1-q^2)} (1+q^3) \frac{(1-q^3)}{(1-q^3)} \dots$
 $= \frac{(1-q^2)}{(1-q)} \cdot \frac{(1-q^4)}{(1-q^2)} \cdot \frac{(1-q^6)}{(1-q^3)} \cdot \frac{(1-q^8)}{(1-q^4)} \dots$
cancel $\rightarrow = \frac{1}{1-q} \cdot \frac{1}{1-q^3} \cdot \frac{1}{1-q^5} \dots$
 $= \sum_{n \geq 0} P_{\text{odd}}(n) q^n$ □

Q: Is there an interesting interpretation of $F(q) = \frac{1}{1-q} \cdot \frac{1}{1-q^3} \cdot \dots \cdot \frac{1}{1-q^{2k-1}}$? i.e. a finite version?

A: Yes, found in 1997! By M. Bousquet-Mélou + K. Eriksson.

Thm: Coefficient of q^n in $F(q)$ is $\#$ of "lecture hall" partitions, i.e. non-negative integer solutions to

$$0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \frac{\lambda_3}{3} \leq \dots \leq \frac{\lambda_k}{k}$$

with $\sum_{i=1}^k \lambda_i = n$.

(this is the same k as in $F(q)$).

Days 9-10: The Twelfold Way

GOAL: Count the number of functions $f: N \rightarrow X$, where
 $|N| = n \in \mathbb{Z}_{\geq 1}$ and $|X| = x \in \mathbb{Z}_{\geq 1}$, subject to:

- two constraints on N ,
- two constraints on X , and
- three constraints on f .

This yields $2 \cdot 2 \cdot 3 = 12$ total situations.

Constraints: ① Elements of X and N can be distinguishable or indistinguishable.
② f can be arbitrary, injective, or surjective.

NOTE: Refer to page 73 of EC1 for the 12-fold way table.

Some required definitions & notation:

(i) $(x)_n := x(x-1)(x-2) \cdots (x-n+1)$. Thus, $\binom{x}{n} = \frac{(x)_n}{n!}$.

(ii) A set partition of $[n]$ is a collection $\pi = \{B_1, \dots, B_k\}$ of nonempty, pairwise disjoint subsets of $[n]$ s.t. $\bigcup_{i=1}^k B_i = [n]$.

Note: Since the B_i 's are unordered, we see that $|B_1|, |B_2|, \dots, |B_k|$ corresponds to an integer partition of n .

(iii) The number of set partitions of $[n]$ with k blocks is denoted $S(n, k)$, called the Stirling numbers of the second kind.
Ex: $\{\{1, 2, 8\}, \{3, 4, 7\}, \{5, 6\}\} \vdash [8]$, with 3 blocks.

(iv) The total number of set partitions of $[n]$ is called the n th Bell Number, denoted $B(n)$.

Propⁿ: $B(n) = \sum_{k=1}^n S(n, k)$.

Pf: Immediate. $\square$

We will work through 1, 2, & 3 in Stanley's table together.
Go through 4-12 in groups.

Day II: The Symmetric Group

There are two equivalent ways to define the set of permutations of $[n]$.

Defⁿ 1: A permutation of $[n]$ is a linear ordering $\pi = a_1 a_2 a_3 \dots a_n$ of the elements of $[n]$.

Defⁿ 2: A permutation of $[n]$ is a bijection $\sigma: [n] \rightarrow [n]$.

Ex: $\pi = 213465$, $\sigma: \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 & 6 & 5 \end{array}$

Note: These are not identical defⁿ's, they are equivalent. They extend in different ways.

Ex: • For a multiset S , a permutation is a linear ordering of the elements of S .

• For $\mathbb{R}$, a permutation is a bijection $\sigma: \mathbb{R} \rightarrow \mathbb{R}$.

Defⁿ: For a multiset S , let $\mathcal{G}_S$ denote the set of permutations of S . If $S = [n]$, write $\mathcal{G}_n$, called the symmetric group.

Propⁿ: $|\mathcal{G}_n| = n!$

Proof: In a linear order $a_1 a_2 \dots a_n$, choose a_1 in n ways, a_2 in $n-1$ ways, a_3 in $n-2$ ways, etc. $\square$

Observations: ① $\mathcal{G}_n$ is a group under composition of functions. If $\pi, \omega \in \mathcal{G}_n$, then $\pi, \omega: [n] \rightarrow [n]$, so we write $\pi\omega$ for $\pi \circ \omega$ as functions.

② For $\pi, \omega \in \mathcal{G}_n$, $\pi\omega \in \mathcal{G}_n$.

Ex: $\pi = 4132$, $w = 2143$ in G_4 . Write $\pi + w$ in "two line" notation:

$$\pi = \begin{array}{c} 1 \ 2 \ 3 \ 4 \\ \downarrow \\ 4 \ 1 \ 3 \ 2 \\ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \end{array}, \quad w = \begin{array}{c} 1 \ 2 \ 3 \ 4 \\ \downarrow \\ 2 \ 1 \ 4 \ 3 \\ w_1 \ w_2 \ w_3 \ w_4 \end{array} \quad \pi \circ w = \begin{array}{c} 1 \ 2 \ 3 \ 4 \\ \downarrow \\ 2 \ 1 \ 4 \ 3 \\ \downarrow w \\ 1 \ 4 \ 2 \ 3 \uparrow \pi \end{array}$$

Exercise: $w \circ \pi = 3241 \neq 1423 = \pi \circ w$, so G_n not abelian.

(2) Permutations in G_n can be represented by permutation matrices. If $w \in G_n$, P_w is $n \times n$ matrix with

$$(P_w)_{ij} := \begin{cases} 1 & \text{if } w_i = j \\ 0 & \text{else.} \end{cases}$$

If $w: [n] \rightarrow [n]$, then $P_w: \mathbb{F}^n \rightarrow \mathbb{F}^n$, where $\mathbb{F}$ is any field.

Ex: π, w as above: $P_\pi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$

$$P_\pi \cdot P_w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P_{3241} = P_{w \circ \pi}.$$

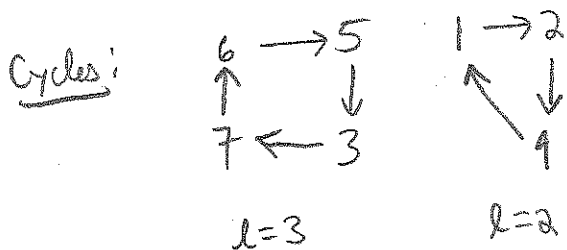
(3) Permutations in G_n can be written as a product of disjoint cycles.

Defⁿ: Let $w \in G_n$ and $x \in [n]$. Then we get a sequence

$$x \rightarrow w(x) \rightarrow w^2(x) \rightarrow w^3(x) \rightarrow \dots \rightarrow w^l(x) \text{ for some } l.$$

Since w is a bijection, we eventually return to x . We call this a cycle of length l and write $(x \ w(x) \ w^2(x) \ \dots \ w^{l-1}(x))$.

Ex: $w = 2471356 \in G_7$.



$$\text{So, } w = \underbrace{(6537)}_{\substack{\uparrow \\ \text{cyclic order does} \\ \text{not matter.}}} \underbrace{(241)}$$

Propⁿ: If $\pi \in G_n$, then $\pi = A_1 A_2 \dots A_k$ where the A_i 's are cycles whose entries are disjoint.

Proof: Exercise. Generalize the previous example. $\square$

Let's investigate cycle structure more closely.

Defⁿ: If $\pi = A_1 A_2 \dots A_k$ is the cycle decomposition of $\pi \in G_n$, the type of π , $\text{type}(\pi)$, is the sequence $(c_1, c_2, \dots, c_n)$ where $c_i = \# \{A_j : |A_j| = i\}$.

Ex: $\text{Type}(2471356) = (0, 0, 1, 1, 0, 0, 0)$.

• $\text{Type}(21436587) = (0, 4, 0, 0, 0, 0, 0)$

• $\text{Type}(234561) = (0, 0, 0, 0, 0, 1)$.

Propⁿ: Let G_n^c denote the set of permutations in G_n of type $c = (c_1, \dots, c_n)$.

Then
$$|G_n^c| = \frac{|G_n|}{\prod_{i=1}^n i^{c_i} c_i!} = \frac{n!}{1^{c_1} c_1! \cdot 2^{c_2} c_2! \cdot \dots \cdot n^{c_n} c_n!}.$$

Idea of Proof: Consider a sufficiently complicated example: $c = (2, 3, 0, 0, 1, 0, \dots, 0)$ for G_{13} . An element of G_{13}^c is of the form

$$(*) = (a_1)(a_2)(a_3 a_4)(a_5 a_6)(a_7 a_8)(a_9 a_{10} a_{11} a_{12} a_{13}),$$

when the disjoint cycles are arranged in increasing length.

I have a map $\Phi: G_{13} \rightarrow G_{13}^c$ that takes

$a_1, a_2, \dots, a_{13}$ to $(*)$ by inserting parentheses.

For example, $\pi = [13, 5, 2, 1, 7, 9, 8, 6, 4, 3, 11, 12, 10]$ has

$$\Phi(\pi) = (13)(5)(2,1)(7,9)(8,6)(4,3,11,12,10). \quad (\text{I'm using commas due to double-digits}).$$

Now, $\Phi(\pi)$ is the same permutation as:

$$\begin{array}{ccccccc} & \text{rotate} & & \text{rotate} & & & \\ & \downarrow & & \downarrow & & & \\ (5)(13) & (9,7) & (2,1) & (11,12,10,4,3) & & & \\ \uparrow \nearrow & \uparrow \nearrow & & & & & \\ \text{swap} & \text{swap} & & & & & \end{array}$$

$$\text{Thus, } \Phi(\pi) = \Phi(\sigma) \text{ for } \sigma = [5, 13, 9, 7, 2, 1, 11, 12, 10, 4, 3].$$

For any $\pi \in G_{13}$, the number of $\sigma \in G_{13}$ satisfying $\Phi(\pi) = \Phi(\sigma)$ is

$$(1 \cdot 1) \cdot 2 \cdot (2 \cdot 2 \cdot 2) \cdot 3! \cdot (5) \cdot 1$$

Annotations for the product terms:

- $(1 \cdot 1)$: no cycling for a_1, a_2
- 2 : $c_1! = 2$ possible orders for 1-cycles
- $(2 \cdot 2 \cdot 2)$: cycling for each of the 2-cycles
- $3!$: $c_2!$ possible orders for 2-cycles
- (5) : cycling for the 5-cycle
- 1 : $c_5! = 1$ 5-cycle

This matches the formula $1^{c_1} \cdot c_1! \cdot 2^{c_2} \cdot c_2! \cdot \dots \cdot 5^{c_5} \cdot c_5!$

Thus, we have a surjective map $\Phi: G_{13} \rightarrow G_{13}^c$ where

$$|\Phi^{-1}(w)| = \prod_{i=1}^{13} \binom{c_i}{i} c_i! \text{ for every } w \in G_{13}^c. \text{ Thus,}$$

$$|G_{13}| = |G_{13}^c| \cdot \prod_{i=1}^{13} \binom{c_i}{i} c_i!, \text{ as desired.} \quad \square$$

Day 12: Cycle Structure and Stirling Numbers

Defⁿ: Let $c([n], k)$ denote the set of permutations that are a product of exactly k disjoint cycles.

Ex: $c([n], 1)$ is the set of n -cycles in S_n .

There are $(n-1)!$ such cycles.

Pf: Apply our previous formula;

get $\frac{n!}{n^1 \cdot 1!} = (n-1)!$ ▀

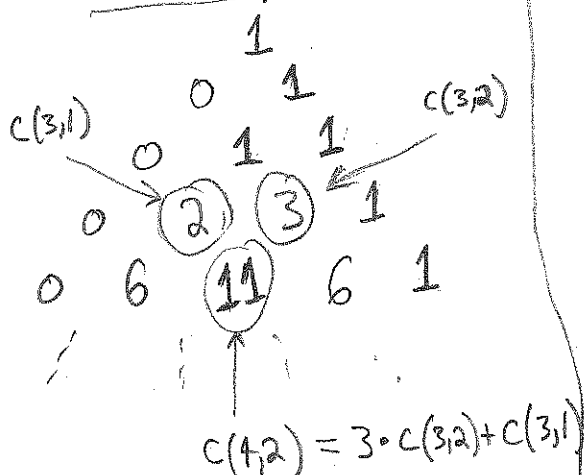
Ex: $c([n], n)$ contains only the identity permutation, as all cycles are 1-cycles.

Defⁿ: $c(n, k) := |c([n], k)|$ is a Stirling number of the first kind (signless). $s(n, k) := (-1)^{n-k} c(n, k)$ is a Stirling number of the first kind (signed).

Theorem: Using initial conditions $c(n, k) = 0$ if $n < k$ or $k = 0$, except $c(0, 0) = 1$, we have

$$c(n, k) = (n-1)c(n-1, k) + c(n-1, k-1).$$

Pascal-type triangle:



Pf: Start w/ $\pi = \pi_1 \pi_2 \dots \pi_n \in c([n], k)$.

We have 2 cases:

Case ① $\pi_n = n$. Then $\pi' = \pi_1 \pi_2 \dots \pi_{n-1}$ is in $c([n-1], k-1)$, since (n) is a 1-cycle in π . Every such π arises uniquely in this way.

Case 2: If $\pi_n \neq n$, then n appears in a j -cycle with $j > 1$.
 If $\pi = A_1 A_2 \dots A_k$ is the cycle expression for π ,
 remove n from its cycle to obtain $\pi' \in C([n-1], k)$.
 There are $n-1$ elements in $C([n], k)$ yielding π'
 in this manner.

Thus, $C(n, k) = (n-1)C(n-1, k) + C(n-1, k-1)$. ■

Remark: We have seen two other "triangles" w/ recurrences:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad S(n, k) = kS(n-1, k) + S(n-1, k-1)$$

$$\begin{array}{cccc} & & 1 & \\ & 1 & & 1 \\ & 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\begin{array}{cccc} & & & 1 \\ & & 0 & 1 \\ & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Q: Are the $C(n, k)$'s similar/related to these?

A: Yes!

Propⁿ: For $n \geq 1$, $\sum_{k=0}^n C(n, k) t^k = t(t+1)(t+2) \dots (t+n-1)$.

Remark: Compare to $\sum_{k=0}^n \binom{n}{k} t^k = (1+t)^n$; hence an similarity.

Pf 1: Use the recurrence and induction. Base Case is
 Straight forward, $C(1, 0) + C(1, 1)t = 0 + t = t$. Thus,

$$(t+n-1)(t+n-2) \dots (t+1)t = (t+n-1) \sum_{k=0}^{n-1} C(n-1, k) t^k$$

Induct

$$\begin{aligned} \sum_{k=1}^n C(n-1, k-1) t^k + \sum_{k=0}^{n-1} (n-1) C(n-1, k) t^k &= \sum_{k=0}^n \left[(n-1) C(n-1, k) + C(n-1, k-1) \right] t^k \\ &= \sum_{k=0}^n C(n, k) t^k. \quad \blacksquare \end{aligned}$$

To connect to $S(n, k)$, we use $s(n, k)$, the signed Stirling numbers.

Thm (Propⁿ 1.9.1 in Stanley EC1): For all $m, n \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{k \geq 0} S(m, k) s(k, n) = \sum_{k \geq 0} S(m, k) (-1)^{k-n} c(k, n) = \delta_{mn},$$

where $\delta_{mn} = 1$ if $m=n$ and 0 else.

Note: This has an interpretation via matrix multiplication, see pages 76-77 in Stanley EC1 for an explanation.

Idea of proof: Consider the bases for $\mathbb{C}[X]$, the set of all polynomials over $\mathbb{C}$, given by

$B_1 = \{1, x, x^2, x^3, \dots\}$ and $B_2 = \{1, (x)_1, (x)_2, (x)_3, \dots\}$.
Then $S(n, k)$ are the entries for the change of basis matrix from B_2 to B_1 , and $s(n, k)$ are the entries for the change of basis matrix from B_1 to B_2 , i.e.

$$x^n = \sum_{k=0}^n S(n, k) (x)_k \quad \text{and}$$

$$(x)(x-1) \dots (x-n+1) = (x)_n = \sum_{k=0}^n s(n, k) x^k = \sum_{k=0}^n (-1)^{n-k} c(n, k) x^k. \quad \blacksquare$$

Observation: By the 12-fold way, t^k is the number of functions from $[k]$ to $[t]$. Thus, for any $t \in \mathbb{Z}_{\geq 1}$,

we see that

$$\sum_{k=0}^n c(n, k) t^k$$

enumerates all possible pairs (w, f) where $w \in \mathcal{C}([n], k)$ and $f: [k] \rightarrow [t]$. Now, if for each $w \in \mathcal{C}([n], k)$ we number the cycles in w by $1, \dots, k$, we see f is a function from the cycles of w to $[t]$.

Similarly, $t(t+1)\dots(t+n-1)$ counts the number of sequences $(a_1, a_2, \dots, a_n)$ where $0 \leq a_i \leq t+n-1-i$, since there are the sequences in

$$\underbrace{\{0, 1, 2, \dots, t+n-2\}}_{\substack{t+n-1 \\ \text{elts}}} \times \underbrace{\{0, 1, 2, \dots, t+n-3\}}_{\substack{t+n-2 \\ \text{elts}}} \times \dots \times \underbrace{\{0, 1, \dots, t-1\}}_{t \text{ elts.}}$$

Thus, $\sum_{k=0}^n C(n, k) t^k = t(t+1)\dots(t+n-1)$ will be proved if we show that for all $t \in \mathbb{Z}_{\geq 1}$, there is a bijection between

$$\{(w, f) : w \in \mathcal{G}_n, f : [\#_{\text{cycles}} w] \rightarrow [t]\} \text{ and } \{0, 1, \dots, t+n-2\} \times \dots \times \{0, 1, \dots, t-1\},$$

since then the two polynomials will agree on infinitely many values.

Definition of bijection: Start w/ $(a_1, a_2, \dots, a_n)$ as above. We will produce $w = \underbrace{C_k C_{k-1} \dots C_1}_{\text{disjoint cycles}}$ and $f : \{C_1, \dots, C_k\} \rightarrow [t]$.

First Step: place n into C_1 , set $f(C_1) = n+1$.

* Assume $n, n-1, \dots, n-i+1$ have been inserted to cycles of w .

Inductive Step: We have two cases:

i) If $0 \leq a_{n-i} \leq t-1$, Start new cycle C_j with $n-i$, set $f(C_j) = a_{n-i} + 1$. (Note: C_j is to the "left" of previous cycles listed)

ii) For some $0 \leq k \leq i-1$, $a_{n-i} = t+k$. Then insert $n-i$ into an existing cycle so that it isn't the left-most element of any cycle, and it appears to the right of $k+1$ previously inserted numbers.

Ex: $(3, 3, 1, 2) \in \{0, 1, 2, 3, 4, 5\} \times \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3\} \times \{0, 1, 2\}$, where $n=4, t=3$.

$$(4) \mapsto (3)(4) \mapsto (32)(4) \mapsto (312)(4)$$

$n=4$ $a_{n-1}=1 \leq a=t-1$ $a_{n-2}=3=t+0$ $a_{n-3}=3=t+0$
 $f(C_1)=2+1=3$ $f(C_2)=1+1=2$ $k=0$ $k=0$

Day 13: Inversions

Let's look carefully at one-line notation for awhile.

Defⁿ: For $\pi \in G_n$, an inversion in π is a pair (π_i, π_j) where $i < j$ and $\pi_i > \pi_j$. The number of inversions in π is denoted $\text{inv}(\pi)$.

Ex: G_3 : $\text{inv}(123) = 0$ $\text{inv}(\widehat{21}3) = 1$ $\text{inv}(3\widehat{1}2) = 2$
 $\text{inv}(13\widehat{2}) = 1$ $\text{inv}(\widehat{23}1) = 2$ $\text{inv}(\widehat{32}1) = 3$

Consider the o.g.f. for inv over G_3 : $\sum_{\pi \in G_3} q^{\text{inv}(\pi)} = 1 + 2q + 2q^2 + q^3$
 $= (1+q)(1+q+q^2)$.

factors into finite geometric series!

Notation: (q -analogues of n & $n!$)

$$[n]_q := \frac{1-q^n}{1-q} = 1+q+q^2+\dots+q^{n-1}$$

$$[n]_q! := [n]_q [n-1]_q \dots [1]_q$$

Remark: Stanley uses bold (n) for our $[n]_q$.

Theorem (MacMahon): For $n \geq 1$, $\sum_{\pi \in G_n} q^{\text{inv}(\pi)} = [n]_q! = \prod_{j=1}^n (1+q+\dots+q^{j-1})$.

To prove this theorem, we will construct a bijection

$$I: G_n \rightarrow \uparrow_n \text{ where } \uparrow_n = \{(a_1, \dots, a_n) : 0 \leq a_i \leq n-i\}$$

$$= [0, n-1] \times [0, n-2] \times \dots \times [0, 0],$$

such that $\text{inv}(\pi) = a_1 + \dots + a_n$ when $I(\pi) = (a_1, \dots, a_n)$.

Pf of Thm: Assume we have a bijection of this type. Then

$$\sum_{\pi \in G_n} q^{\text{inv}(\pi)} = \sum_{(a_1, \dots, a_n) \in \uparrow_n} q^{a_1 + \dots + a_n} = \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \dots \sum_{a_n=0}^0 q^{a_1 + \dots + a_n} =$$

$$\left(\sum_{a_1=0}^{n-1} q^{a_1} \right) \dots \left(\sum_{a_{n-1}=0}^1 q^{a_{n-1}} \right) \left(\sum_{a_n=0}^0 q^{a_n} \right) = \prod_{j=1}^n \frac{(1-q^j)}{1-q} = [n]_q!$$

Thus, we now need to construct our bijection I . We will actually construct I^{-1} . Note: $I(\pi)$ is called the inversion table for π .

Given $(a_1, \dots, a_n) \in T_n$, inductively assume that $n, n-1, \dots, n-i+1$ have been inserted into π . Next, insert $n-i$ so that there are a_{n-i} elements in the word (so far) to its left.

Ex: $a = (0, 3, 3, 3, 2, 1, 0) \in T_7$
 $\uparrow$
 always = 0!

construction

7 $\leftarrow a_7 = 0$, so start w/ 7.

76 $\leftarrow$ If a_6 were 0, then 67

765 $\leftarrow$ If a_5 were 1, then 756.

7654

76534 $\leftarrow$ Since $a_3 = 3$, need 3 to the left.

765234

$\rightarrow$ 1765234 $\leftarrow$ since $a_1 = 0$, 1 at far left.

$I^{-1}(a)$.

Claim: I^{-1} is injective.

Pf: If $a \neq b$, then some $a_i \neq b_i$. Thus, $n-i$ is placed in a different order relative to $n, n-1, \dots, n-i+1$ for $I^{-1}(a)$ and $I^{-1}(b)$.

Hence, $I^{-1}(a) \neq I^{-1}(b)$. $\blacksquare$

By definition, $|T_n| = n!$. Thus, we have a bijection.

To see why $\text{inv}(I^{-1}(a)) = a_1 + \dots + a_n$, consider $I(\pi)$. Reversing I^{-1} shows us that $I(\pi) = (a_1, \dots, a_n)$ where a_i is equal to the number of entries to the left of i in $\pi_1 \dots \pi_n$ that are greater than i .

Another Example:

$\pi = 21354 \Rightarrow I(\pi) = (1, 0, 0, 1, 0)$.

$I^{-1}(I(\pi)) =$

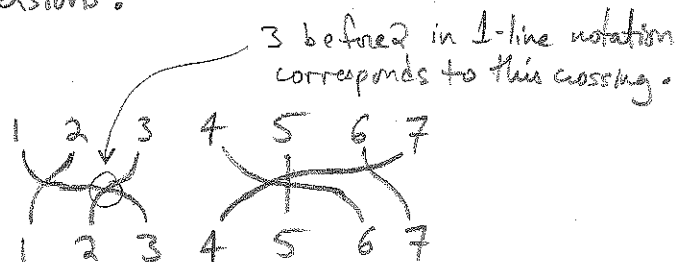
$\begin{pmatrix} 5 \\ 54 \\ 354 \\ 2354 \\ 21354 \end{pmatrix}$

$\leftarrow \pi$, as desired.

Two useful ways to visualize inversions:

(i) Use a "wire diagram".

Ex: $w = 3126574 \in \mathbb{G}_7$. $w \downarrow$



$\text{inv}(w) = 6$; each "crossing" in the diagram is an inversion! So, count crossings.

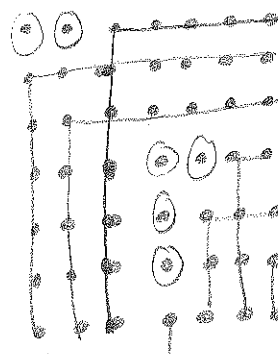
Propⁿ: For any $w \in \mathbb{G}_n$, let w^{-1} denote the inverse of w . We have that $\text{inv}(w) = \text{inv}(w^{-1})$ for all $w \in \mathbb{G}_n$.

Pf: The wiring diagram for w (read top to bottom) is the wiring diagram for w^{-1} (read bottom to top). Thus, the number of crossings is the same in w & w^{-1} . $\square$

(ii) Use the (matrix) diagram for w . Construct D_w as follows, if $w \in \mathbb{G}_n$. Start w/ an $n \times n$ grid of dots. If $w_i = j$, then from the j^{th} column in the i^{th} row, draw lines to the right & down. Circle the remaining dots.

Ex: $w = 3126574$.

$D_w =$



Remark: Each circled dot corresponds to an inversion. Before, we saw that $\text{inv}(w) = 6$, which we see here as well.

Propⁿ: For $w \in \mathbb{G}_n$, if $I(w) = (a_1, a_2, \dots, a_n)$, then a_j is the number of circled dots in column j .

Pf: If $w_i = j$, then j appears in the 1st position in an inversion precisely when there is a circled dot in D_w to the left of the $(i,j)^{\text{th}}$ dot. j appears in the 2nd position in an inversion when a circled dot appears above the $(i,j)^{\text{th}}$ dot. The propⁿ follows from this. $\square$

Day 14: Descents

A special type of inversion occurs when $\pi_i > \pi_{i+1}$; we call these inversions descents.

Defⁿ: Given $\pi \in G_n$, a descent in π is a position $i \in \{1, 2, \dots, n-1\}$ such that $\pi_i > \pi_{i+1}$. We define $\text{Des}(\pi) = \{i : i \text{ is a descent of } \pi\}$ and $\text{des}(\pi) = |\text{Des}(\pi)|$. Note that $\text{Des}(\pi) \subseteq [n-1]$ for $\pi \in G_n$.

Ex: $\pi = \widehat{3} \widehat{1} 2 4 6 \widehat{5} \in G_6$ has $\text{Des}(\pi) = \{1, 5\}$, $\text{des}(\pi) = 2$.

Defⁿ: The d^{th} Eulerian polynomial is

$$A_d(x) := \sum_{\pi \in G_d} x^{1+\text{des}(\pi)} = \sum_{k=1}^d A_{d,k} x^k, \text{ the ogf for } 1+\text{des}(\pi).$$

The numbers $A(d,k)$ are called Eulerian numbers.

Ex: $A_0(x) = 1$, $A_1(x) = x$, $A_2(x) = x + x^2$, $A_3(x) = x + 4x^2 + x^3$, etc.

Notes: Eulerian polynomials have many remarkable properties!

① They have palindromic coefficients, eg. $A_4(x) = x + 11x^2 + 11x^3 + x^4$.

② Every root of $A_d(x)$ is a real number, i.e. $A_d(x)$ is real-rooted.

③ The Eulerian numbers are unimodal, i.e.

$$A_{d,1} \leq A_{d,2} \leq \dots \leq A_{d,\lfloor d/2 \rfloor} \geq \dots \geq A_{d,d-1} \geq A_{d,d}.$$

④ The Eulerian polynomials are γ -non-negative, i.e. for $d \geq 1$ there exist non-negative integers $\gamma_{d,i}$, $i = 1, \dots, \lfloor d/2 \rfloor$, such that

$$A_d(x) = \sum_{i=1}^{\lfloor d/2 \rfloor} \gamma_{d,i} \cdot x^i \cdot (1+x)^{d+1-2i}.$$

Ex: $A_4(x) = x(1+x)^3 + 8x^2(1+x)$.

Note: γ -non-negative $\Rightarrow$ unimodal.

Q: What did Euler have to do with them?

A: Euler didn't know about descent sets & permutations; he discovered $A_d(x)$ as numerators of power series he was interested in, related to work of his on the Riemann zeta function.

Thm: For $d \geq 0$, $\sum_{m \geq 0} m^d x^m = \frac{A_d(x)}{(1-x)^{d+1}}$.

Ex: $\sum_{m \geq 0} m x^m = \frac{x}{(1-x)^2}$
 $\sum_{m \geq 0} m^2 x^m = \frac{x + 4x^2 + x^3}{(1-x)^4}$

Remark 1: Note that $(1-x)^{d+1}$ is a polynomial w/ constant 1, hence we have a rational agt. for the sequence m^d , hence we have recursions! See EC1, §4.3 for a general statement.

Remark 2: $\sum_{m \geq 0} m^d x^m$ is $x \frac{d}{dx}$ applied to $\frac{1}{1-x}$ d times.

Pf of Thm: Induct on d . The base case is straight forward. For the induction step, use Remark 2:

$$\sum_{m \geq 0} m^{d+1} x^m = x \frac{d}{dx} \sum_{m \geq 0} m^d x^m = x \frac{d}{dx} \frac{A_d(x)}{(1-x)^{d+1}} =$$

$$\frac{x(1-x)A_d'(x) + (d+1)x A_d(x)}{(1-x)^{d+2}}.$$

Thus, we need to show that

$$A_{d+1}(x) = x(1-x)A_d'(x) + (d+1)x A_d(x),$$

which coefficient-by-coefficient is equal to the recurrence

$$A_{d+1,k} = k \cdot A_{d,k} + (d-k+2)A_{d,k-1}.$$

To prove the recurrence, observe that for $\pi \in \mathcal{G}_{d+1}$, with $k-1$ descents, we obtain π uniquely in one of two ways:

(i) Given $w \in \mathcal{G}_d$ with $k-1$ descents, insert $d+1$ after w_i for some $i \in \text{Des}(w)$, or at the end of w . There are $k \cdot A_{d,k}$ π 's arising in this way.

(ii) Given a permutation $w \in \mathcal{G}_d$ with $k-2$ descents, insert $d+1$ after w_i if $i \notin \text{Des}(w)$, or at the beginning of w . There are $(d-k+2)A_{d,k-1}$ π 's arising in this manner.



A neat property of Eulerian polynomials is that $A_d(x)$ has a nice egf..

Thm: Set $A_0(x) := 1$. Then $\sum_{d \geq 0} A_d(x) \frac{t^d}{d!} = \frac{1-x}{1-xe^{(1-x)t}}$.

Pf: Multiply both sides of $\sum_{m \geq 0} m^d x^m = \frac{A_d(x)}{(1-x)^{d+1}}$ by $\frac{t^d}{d!}$, and

sum over all d to obtain:

$$\sum_{d \geq 0} \frac{A_d(x)}{(1-x)^{d+1}} \frac{t^d}{d!} = \sum_{d \geq 0} \left[\sum_{m \geq 0} m^d x^m \right] \frac{t^d}{d!} = \sum_{m \geq 0} \left[\sum_{d \geq 0} \frac{(mt)^d}{d!} \right] x^m =$$

$$\sum_{m \geq 0} e^{mt} x^m = \sum_{m \geq 0} (e^t x)^m = \frac{1}{1-xe^t}.$$

Next, multiply both sides by $1-x$ and substitute $(1-x)t$ for t , to obtain the desired identity.

Properties of permutations such as inversions + descents were first seriously studied in the late 1800's + early 1900's by Major Percy MacMahon, an officer in the British army. He investigated the following statistic, which is now named after him.

Defⁿ: For $\pi \in G_n$, the major index of π is

$$\text{maj}(\pi) := \sum_{j \in \text{Des}(\pi)} j.$$

Ex: $\pi = 4\hat{1}3\hat{2}$ has $\text{Des}(\pi) = \{1, 3\}$ so $\text{maj}(\pi) = 1+3 = 4$

Without proof, let's state some awesome theorems related to maj .

$$\text{Thm (MacMahon)}: \sum_{\pi \in G_n} q^{\text{maj}(\pi)} = \sum_{w \in G_n} q^{\text{inv}(\pi)} = [n]_q!.$$

Remark: Consider $\pi = 51234$, which has $\text{maj}(\pi) = 1$, but $\text{inv}(\pi) = 4$. The above theorem says that while maj & inv can differ for a specific permutation, they are equidistributed over G_n .

Thm (Carlitz formally, MacMahon informally)

$$\sum_{m \geq 0} [m+1]_q^n X^m = \frac{\sum_{\pi \in G_n} X^{\text{des}(\pi)} q^{\text{maj}(\pi)}}{\prod_{j=0}^n (1 - Xq^j)}.$$

Remark: This is a " q -analogue" of our generating function identity involving Eulerian polynomials. Note what the analogues are:

$$(m+1)^n \rightarrow [m+1]_q^n = (1 + q + \dots + q^m)^n.$$

$$\sum_{\pi \in G_n} X^{\text{des}(\pi)} \rightarrow \sum_{\pi \in G_n} X^{\text{des}(\pi)} q^{\text{maj}(\pi)}.$$

$$(1-X)^{n+1} \rightarrow (1-X)(1-Xq)(1-Xq^2) \dots (1-Xq^n).$$

Setting $q=1$ in the above identity, and multiplying both sides by X , we recover an original (single-variable) identity.

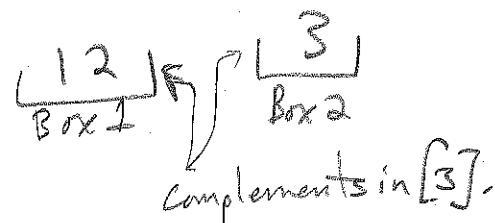
Day 15: Multinomial Coefficients + lattice paths

Our focus today is on two new ways to interpret/generalize binomial coefficients.

① Multinomial Coefficients

Ex: $\binom{m+n}{k}$ is the # of ways of selecting k objects from $[m+n]$ to go in one box and the remaining $(m+n-k)$ objects to go in another box. Since k and $m+n-k$ are clearly interchangeable, we see $\binom{m+n}{k} = \binom{m+n}{m+n-k}$, the known symmetry of Pascal.

Ex: $\binom{2+1}{2} = \binom{2+1}{1}$



$$\{1, 2, 1, 3, 2, 3\} \longleftrightarrow \{3, 2, 1\}$$

Notation: Write $\binom{m+n}{k, m+n-k}$ for $\binom{m+n}{k}$ to emphasize this. note the complement relationship.

Defⁿ: The number of ways of placing the elements of $[n]$ into k labeled categories $C_1, C_2, \dots, C_k$ with a_i elements in category C_i is denoted $\binom{n}{a_1, a_2, a_3, \dots, a_k}$, called a multinomial coefficient.

Remark: When considering $\binom{n}{k} = \binom{n}{k, n-k}$, think of selecting k elts to go in category 1 & the complement in Category 2.

Three important (and relatively simple) results

Thm 1: $\binom{n}{a_1, \dots, a_k} = \prod_{j=1}^k \binom{n-a_1-\dots-a_{j-1}}{a_j} = \binom{n}{a_1} \binom{n-a_1}{a_2} \dots \binom{n-a_1-\dots-a_{k-1}}{a_k}$

Pf: Pick a_1 elts for C_1 , then a_2 of the remaining for C_2 , etc.



Thm 2 (Multinomial Theorem):

$$(x_1 + \dots + x_m)^n = \sum_{\substack{a_1 + \dots + a_m = n \\ a_i \geq 0}} \binom{n}{a_1, \dots, a_m} x_1^{a_1} x_2^{a_2} \dots x_m^{a_m}.$$

Sketch of pf: $(x_1 + x_2 + x_3)^4 = \underbrace{(x_1 + x_2 + x_3)}_1 \cdot \underbrace{(x_1 + x_2 + x_3)}_2 \cdot \underbrace{(x_1 + x_2 + x_3)}_3 \cdot \underbrace{(x_1 + x_2 + x_3)}_4$

elts of $[4] \rightarrow$

While distributing, picking x_j from first term assigns 1 to C_j , picking x_i from second term assigns 2 to C_i , etc.

For example, $(x_2) \cdot (x_3) \cdot (x_1) \cdot (x_1)$ means $\boxed{34} \boxed{1} \boxed{2}$

1 2 3 4 C_1 C_2 C_3

Thm 3: Let $M = \{\gamma_1^{a_1}, \gamma_2^{a_2}, \dots, \gamma_m^{a_m}\}$ be a multiset w/ elements $\gamma_1, \dots, \gamma_m$ of multiplicities $a_1, \dots, a_m$. If $\sum_{i=1}^m a_i = n$, i.e. $|M| = n$, then $|\mathcal{G}_M| = \binom{n}{a_1, \dots, a_m}$.

pf: If γ_i appears in position j of $\sigma \in \mathcal{G}_M$, then put the element $j \in [n]$ in C_i .

Ex: $M = \{1, 1, 1, 2, 2, 3, 3\}$, $\sigma = 2131132$ corresponds to

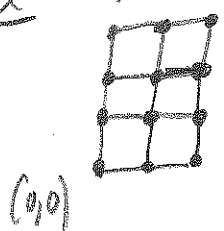
$$\boxed{2, 4, 5} \boxed{1, 7} \boxed{3, 6}$$

C_1 C_2 C_3

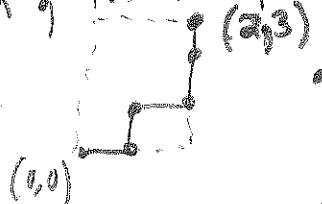
Ⓐ Lattice Paths

Thm: $\binom{m+n}{m} = \binom{m+n}{m, n}$ is the number of lattice paths w/ steps $(0,1)$ and $(1,0)$ from $(0,0)$ to (m,n) .

Ex: $(m,n) = (2,3)$.



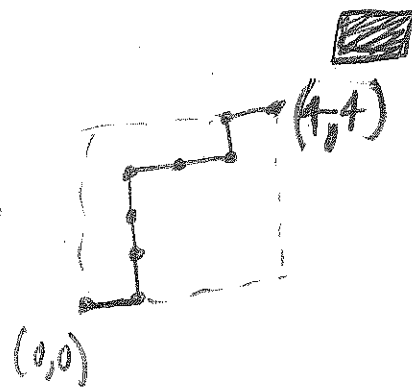
We need to take two steps "East" and three "North", for example ENENN corresponds to



Observation: Every lattice path of this type to (m,n) can be written as a word with m E's and n N's. Thus, we are counting permutations of $\{E^m, N^n\}$.

Pf of Thm: These lattice paths are in bijection with the elements of $\{E^m, N^n\}$. Thus, the number of them is $\binom{m+n}{m, n}$.

Ex: ENNNEENE $\leftrightarrow$ $E \in \{E^4, N^4\}$



This can be generalized to higher dimensions. For example, taking steps $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, how many paths are there to $(3,4,7)$? Write S_1 , S_2 , and S_3 for our three types of steps, and we see that any lattice path is a permutation of $\{S_1^3, S_2^4, S_3^7\}$.

(over)

Thus, the number of such paths is $\binom{3+4+7}{3,4,7} = \binom{14}{3,4,7}$.

Thm: The number of lattice paths in $\mathbb{Z}^d$ from $(0, \dots, 0)$ to $(a_1, \dots, a_d)$ w/ steps $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$ is given by

$$\binom{a_1 + a_2 + \dots + a_d}{a_1, a_2, \dots, a_d}.$$

pf: Exercise. 

Day 16: Catalan Numbers

We next consider a family of restricted lattice paths.

Defⁿ: A Dyck path ^{of length $2n$} is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) using steps $(1,0)$ and $(0,1)$ whose edges lie below the line $x=y$.

Thm: The number of Dyck paths of length $2n$ is

$$\frac{1}{n+1} \binom{2n}{n}.$$

Pf: See André's reflection trick on the Wikipedia page for Catalan numbers. 

Defⁿ: The sequence $\frac{1}{n+1} \binom{2n}{n}$ for $n \geq 0$ is called the sequence of Catalan numbers, where we write $C_n = \frac{1}{n+1} \binom{2n}{n}$.

Thm: The set of Dyck paths of length $2n$ is in bijection with the set of Dyck words, i.e. those elements in $\tilde{G}_{\{x,y\}^n}$ such that no initial segment of the permutation has more X's than Y's.

Pf: Exercise. 

Thm: For $n \geq 0$, $C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$. 

Pf: Exercise. 

Day 17: Catalan Numbers, Generating Functions, & Narayana Numbers

Let's investigate Catalan numbers from a generating function perspective. Define $C(x) := \sum_{n \geq 0} C_n x^n$.

Observation: The recurrence $C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$ is equivalent to the functional equation

$$C(x) = 1 + x[C(x)]^2.$$

From this, we can use the quadratic formula to solve for $C(x)$, and we obtain:

Thm: $C(x) = \frac{1 - \sqrt{1-4x}}{2x} = \frac{2}{1 + \sqrt{1-4x}}.$

Use this form to get a power series about 0 w/ positive radius of convergence.

Recall: We used Dyck paths + André's reflection method to prove that $C_n = \frac{1}{n+1} \binom{2n}{n}$. Let's use generating functions next.

Computation: Expand $\sqrt{1-4x} = (1-4x)^{1/2} = \sum_{n \geq 0} \binom{1/2}{n} (-4x)^n = \sum_{n \geq 0} \frac{(\frac{1}{2})(\frac{1}{2}-1) \cdots (\frac{1}{2}-n+1)}{n!} (-4)^n x^n$

$$= 1 + \sum_{n \geq 1} \binom{1/2}{n} \frac{(1)(-1)(-3) \cdots (-n+3)}{n!} (-1)^n 4^n x^n = 1 + \sum_{n \geq 1} 2^n \frac{(n-3)(n-5) \cdots (3)(1)(-1)}{n!} x^n$$

cancel 4's & 2's,
use $(-1)^n$

$$= 1 - \sum_{n \geq 1} \binom{2}{n} 2^{n-1} \frac{(n-3)(n-5) \cdots (3)(1)}{(n-1)!} x^n = 1 - \sum_{n \geq 1} \binom{2}{n} \frac{[2(n-1)]!}{(n-1)!(n-1)!} x^n$$

$$= 1 - \sum_{n \geq 1} \frac{2}{n} \binom{2n-2}{n-1} x^n.$$

$$\text{Thus, } \frac{1 - \sqrt{1-4x}}{2x} = \frac{1 - \left(1 - \sum_{n \geq 1} \frac{2}{n} \binom{2n-2}{n-1} x^n\right)}{2x} = \sum_{n \geq 1} \frac{1}{n} \binom{2(n-1)}{n-1} x^{n-1} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^n.$$

Hence, we have $C_n = \frac{1}{n+1} \binom{2n}{n}.$

So far, we've seen:

- $C_n = \frac{1}{n+1} \binom{2n}{n}$, via Dyck path + via ogf's.
- $C_{n+1} = \sum_{i=0}^n C_i C_{n-i}$, and $C(x) = \frac{1 - \sqrt{1-4x}}{2x}$
- Dyck Words encoding Dyck paths.

Remark 1: Stanley has collected 214 combinatorial interpretations of C_n in a new book.

Remark 2: Catalan #'s first combinatorially investigated in 1751-1759 by Euler, Goldbach, + Segner (see Igor Pak's website for good history).
E. Catalan first found formula $\frac{1}{n+1} \binom{2n}{n}$ in 1838.

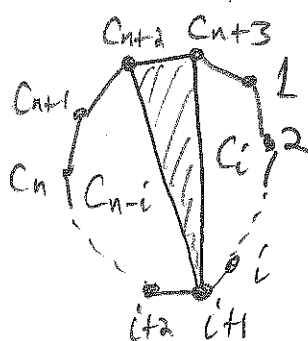
A few more combinatorial interpretations of C_n :

Ⓘ (Euler, Goldbach, Segner) C_n counts # of triangulations of a convex $(n+2)$ -gon (labeled vertices).

Ex: $C_3 = 5$, from 5  and 5 rotations.

$C_4 = 14$, eg. 6  6 , etc

Pf: We show that # of triangulations of an $(n+2)$ -gon satisfies the recurrence.



C_{n+1} should count triangulations of an $(n+3)$ -gon. Fix i , consider all triangulations where $(i-1, i, i+1)$ form a triangle and i is minimal for this property. There are $C_i \cdot C_{n-i}$ such triangulations, so $C_{n+1} = \sum_{i=0}^n C_i \cdot C_{n-i}$. 

Ⓜ 312-avoiding permutations in G_n , i.e. $\pi \in G_n$ s.t. there do not exist $i < j < k$ and $\pi_j < \pi_k < \pi_i$.

Sketch of Pf: Deleting "1" from any 312-avoiding π yields Catalan recurrence. 

Ⓐ C_n counts the number of 2-sided ideals in the ring of all $(n-1) \times (n-1)$ upper triangular matrices over a field.

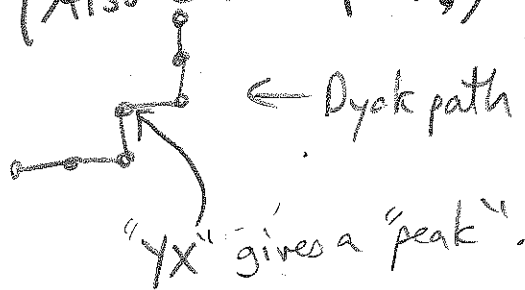
Cool Fact: o.g.f. for reciprocals of Catalan #'s is:

$$\sum_{n \geq 0} \frac{x^n}{C_n} = \frac{2(x+8)}{(4-x)^2} + \frac{24\sqrt{x} \sin^{-1}(\frac{1}{2}\sqrt{x})}{(4-x)^{5/2}}.$$

Hence, $\sum_{n \geq 0} \frac{1}{C_n} = 2 + \frac{4\sqrt{3}\pi}{27}.$

Recall: A Dyck path is a word $w \in \{x^n, y^n\}$ st. no initial term has more y 's than x 's. Consider x to be less than y ; Define a descent in w to be the x -position where " $y x$ " occurs in w . (Also called a "peak".)

Ex: $x x y x y y = w$
 $\text{Des}(w) = \{3\}$
 $\text{maj}(w) = 3$
 = sum of descent positions.



There is an analogue of the Eulerian polynomial in this setting.

Thm: Define the Narayana polynomials to be

$$NA_n(q) := \sum_{\substack{w \text{ Dyck} \\ \text{word in } \{x^n, y^n\}}} q^{\text{des}(w)+1} = \sum_{k=1}^n N(n,k) q^k.$$

Then $N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}$, called the (n,k) -Narayana number.

pf: Exercise. ■

Remark: Like Eulerian polynomials, $NA_n(q)$ are palindromic, real-rooted, & unimodal.

Thm: $\sum_{\substack{w \text{ Dyck} \\ \text{word in } G_{\{x, y\}^n}}} q^{\text{maj}(w)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q := \frac{1}{[n+1]_q} \frac{[2n]_q!}{[n]_q! [n]_q!}$

q -analogue of Catalan #'s.

Pf: Exercise. 

Remark: This motivates our study of q -analogues!

Thm: (Analogue of Carlitz identity for permutations) Let t be a variable.

$$\sum_{\substack{w \text{ Dyck} \\ \text{words in } G_{\{x, y\}^n}}} q^{\text{maj}(w) + (t-1) \text{des}(w)} = \sum_{i=0}^n \frac{1}{[n+1]_q} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} n \\ i+1 \end{bmatrix}_q q^{i^2 + it}$$

Close to q -analogue of Narayana #'s...

Note: This result and many related ones can be found in the exercises to chapter 6 of Stanley's EC Volume 2.

Q to Consider: By the first theorem above, $\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ is a polynomial w/ non-negative integer coefficients. This is quite surprising.....

Days 18 and 19: q -analogues

We have several interpretations of $\binom{n}{k}$:

- 1) # of k -subsets of $[n]$.
- 2) # of N-E lattice paths from $(0,0)$ to $(k, n-k)$.
- 3) # of elements in $G_{\{1^k, 2^{n-k}\}}$

We will extend all of these to polynomial refinements of $\binom{n}{k}$.

Defⁿ: For $n \in \mathbb{Z}_{\geq 1}$, and $a_1, a_2, \dots, a_m \in \mathbb{Z}_{\geq 1}$ such that $\sum_{i=1}^m a_i = n$, define

$$\left[\begin{matrix} n \\ a_1, a_2, \dots, a_m \end{matrix} \right]_q := \frac{[n]_q!}{[a_1]_q! [a_2]_q! \cdots [a_m]_q!}.$$

Note: We have seen these twice already.

i) When studying the o.g.f. for the descent statistic on Dyck words.

ii) When studying the o.g.f. for inv on G_n , i.e.

$$\sum_{\pi \in G_n} q^{\text{inv}(\pi)} = [n]_q! = \left[\begin{matrix} n \\ 1, 1, \dots, 1 \end{matrix} \right]_q.$$

Generalizing k -subsets:

Recall that for each prime power q , there is a finite field of order q , denoted $\mathbb{F}_q$. When q is prime, $\mathbb{F}_q \cong \mathbb{Z}/q\mathbb{Z}$. Our first theorem considers q as a prime power, not as a variable.

Thm: $\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \#$ of k -dim^s subspaces of $(\mathbb{F}_q)^n = \bigoplus_{i=1}^n \mathbb{F}_q$.

Pf: This proof is a generalization of Stanley's proof that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ found on page 16 in EC1.

We count ordered k -tuples of linearly independent vectors in $(\mathbb{F}_q)^n$ in two ways; denote the number of such k -tuples by N . Denote the number of k -subspaces of $(\mathbb{F}_q)^n$ by $G(n, k)$.

(i) $N = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{k-1})$ ← Pick first vector from $(\mathbb{F}_q)^n - \{\vec{0}\}$, the second from outside span of the first, etc.

(ii) $N = G(n, k) \cdot (q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})$ ← From each subspace, pick one nonzero vector, then another from outside the span of the first, etc.

Setting these equal yields

$$G(n, k) = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Generalizing inversions to multiset permutations:

We now consider q as a variable, and establish two properties of q -multinomials.

Propⁿ 1: $\begin{bmatrix} n \\ a_1, a_2, \dots, a_m \end{bmatrix}_q = \begin{bmatrix} n \\ a_1 \end{bmatrix}_q \begin{bmatrix} n - a_1 \\ a_2 \end{bmatrix}_q \begin{bmatrix} n - a_1 - a_2 \\ a_3 \end{bmatrix}_q \cdots \begin{bmatrix} n - a_1 - \dots - a_{m-1} \\ a_m \end{bmatrix}_q$

PF: Express the right-hand side using the definition, + cancel algebraically. 

Propⁿ 2: (q -Pascal relation) For $n=0$, set $\begin{bmatrix} 0 \\ k \end{bmatrix}_q = \delta_{0k}$. For $n \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \cdot q^{n-k} + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

PF: Exercise by algebra; expand, find common denominators, etc. 

Corollary: $\begin{bmatrix} n \\ a_1, a_2, \dots, a_m \end{bmatrix}_q$ is a polynomial in q with non-neg. integer coeffs.

PF: By Propⁿ 2, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is of this type (use induction). Using Propⁿ 1, multiplying such polynomials gives such a polynomial. 

Ex:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1 \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = 1$$

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q = 1 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1+q \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q = 1$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q = 1 \quad \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 1+q+q^2 \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = 1+q+q^2 \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q = 1$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$$

Defⁿ: For $w \in \mathcal{G}_{\{1^{a_1}, 2^{a_2}, \dots, m^{a_m}\}}$ an inversion is a 4-tuple (i, j, u_i, u_j) such that $i < j$ and $u_i > u_j$. Let $\text{inv}(w)$ denote the number of inversions in w .

Ex:

$\mathcal{G}_{\{1^2, 2^2\}}$	inv
1122	0
1212	1
1221	2
2112	2
2121	3
2211	4

Thus, $\sum_{w \in \mathcal{G}_{\{1^2, 2^2\}}} q^{\text{inv}(w)} = 1+q+2q^2+q^3+q^4$

$= \begin{bmatrix} 4 \\ 2, 2 \end{bmatrix}_q$
 multiplicities of
 elts in $\{1^2, 2^2\}$.

Thm: For $M = \{1^{a_1}, 2^{a_2}, \dots, m^{a_m}\}$ and $n = \sum_{i=1}^m a_i$,

$$\sum_{w \in \mathcal{G}_M} q^{\text{inv}(w)} = \begin{bmatrix} n \\ a_1, a_2, \dots, a_m \end{bmatrix}_q.$$

Remark: When $M = [n]$, we recover our previous result of this type.

Pf in special case $m=2$: (This is one step in the full proof given in EC1).
 For $n=0$, $q^0 = 1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q$. For $n \geq 1$, we go by induction, considering the last element of $\pi \in \mathcal{G}_{\{1^{a_1}, 2^{a_2}\}}$. (Compare this to Fibonacci's and comps using 1's & 2's.)

$$\sum_{\pi \in G_{\{1^{q_1}, 2^{q_2}\}}} q^{\text{inv}(\pi)} = \sum_{\substack{\pi \in G_{\{1^{q_1}, 2^{q_2}\}} \\ \pi_n = 1}} q^{\text{inv}(\pi)} + \sum_{\substack{\pi \in G_{\{1^{q_1}, 2^{q_2}\}} \\ \pi_n = 2}} q^{\text{inv}(\pi)}$$

$$= \sum_{\pi' \in G_{\{1^{q_1-1}, 2^{q_2}\}}} q^{\text{inv}(\pi') + q_2} + \sum_{\pi' \in G_{\{1^{q_1}, 2^{q_2-1}\}}} q^{\text{inv}(\pi')} = q^{q_2} \begin{bmatrix} n-1 \\ q_1-1, q_2 \end{bmatrix} + \begin{bmatrix} n-1 \\ q_1, q_2-1 \end{bmatrix} q = \textcircled{*}$$

by removing the last 1, we removed q_2 inversions

2 at end of π does not contribute any inversions

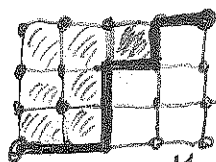
setting $q_1 = k, q_2 = n-k$, we see $\textcircled{*} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q + \begin{bmatrix} n-1 \\ k \end{bmatrix} q = \begin{bmatrix} n \\ k \end{bmatrix} q$.
 By q -Pascal. 

Remark: Removing the first element of π yields $\begin{bmatrix} n-1 \\ k \end{bmatrix} q + q^k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q$, and q -Pascal still applies.

Generalizing lattice paths:

To expand our lattice path interpretation of $\begin{pmatrix} n \\ k, n-k \end{pmatrix}$ to $\begin{bmatrix} n \\ k \end{bmatrix} q$, we will first develop a connection between lattice paths, partitions, and words.

Ex:



- Lattice path.
- Area between y -axis and the path is 7.

- The shaded area is Young diagram for $(3, 2, 2)$.
- This is a partition of 7.

- The N-E word for this lattice path is EENNENE.

Using $N < E$, we have an inversion if E precedes N .

- There are 7 inversions in this word.

Observation: There seems to be a correspondence among these three objects that preserves these statistics!

Thm: There is a bijection among the following three sets:

- ① Lattice paths w/ N-E steps from $(0,0)$ to $(k, n-k)$.
- ② Partitions with $\leq n-k$ parts, each part $\leq k$.
- ③ Elements of $G_{\{E^k, N^{n-k}\}}$.

Further, for a lattice path L , define $\text{Area}(L)$ to be area between L + y -axis.

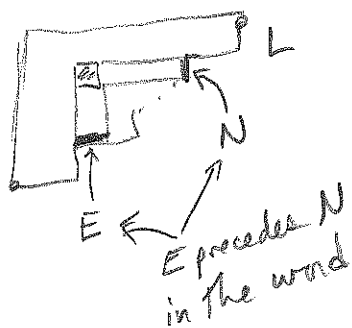
For a partition λ , write $|\lambda| = n$ when $\sum \lambda_i = n$. For a word in $G_{\{E^k, N^{n-k}\}}$

we consider inversions with respect to the order $N < E$.

This bijection preserves these three statistics.

Pf: We already have established the correspondence ① $\leftrightarrow$ ③.

To see that Area is equal to inv , note that every inversion in an N-E word corresponds to a box between the lattice path + the y -axis. The area of the path is the number of boxes, hence $\text{inv} = \text{Area}$.



To see the correspondence ① $\leftrightarrow$ ②, observe that a lattice path forms the boundary of a Young diagram for a partition λ , and $|\lambda|$ is the number of boxes between the path and the y -axis. □

Corollary:
$$\sum_{\substack{\text{Lattice paths} \\ L \text{ from } (0,0) \text{ to } (j,k)}} q^{\text{Area}(L)} = \begin{bmatrix} j+k \\ k \end{bmatrix}_q.$$

Corollary: Let $p(j,k,n)$ denote the number of partitions of n with at most k non-zero parts, with largest part at most j . Then

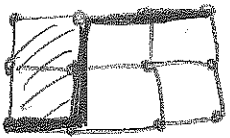
$$\sum_{n \geq 0} p(j,k,n) q^n = \begin{bmatrix} j+k \\ j \end{bmatrix}_q = \begin{bmatrix} j+k \\ k \end{bmatrix}_q.$$

Proofs: We know $\sum_{w \in G_{\{E^j, N^k\}}} q^{\text{inv}(w)} = \begin{bmatrix} j+k \\ k \end{bmatrix}_q$, and the corollaries follow from the bijection. □

We call $\begin{bmatrix} n \\ k \end{bmatrix}_q$ a q-binomial coefficient. These have a number of interesting properties as polynomials in q .

Propⁿ: $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has symmetric (aka. palindromic) coefficients.

PF: Note $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has degree $k(n-k)$. We will establish an involution on $\mathcal{G}_{\{E^k, N^{n-k}\}}$ taking a word with i inversions to one with $k(n-k)-i$ inversions. For a word $\pi = \pi_1 \dots \pi_n \in \mathcal{G}_{\{E^k, N^{n-k}\}}$ define the reverse of π to be $R(\pi) = \pi_n \pi_{n-1} \dots \pi_1$. It is straight forward to show $\text{inv}(R(\pi)) = k(n-k) - \text{inv}(\pi)$. 

Ex:  $\pi = \text{ENNEE}$. $\text{inv}(\pi) = 2$. $R(\pi) = \text{EENNE}$. $\text{inv}(R(\pi)) = 4$ 

Reversing the word takes the complementary area in the rectangle.

Thm: The coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are unimodal.

PF: **Hard!** We will omit it. 

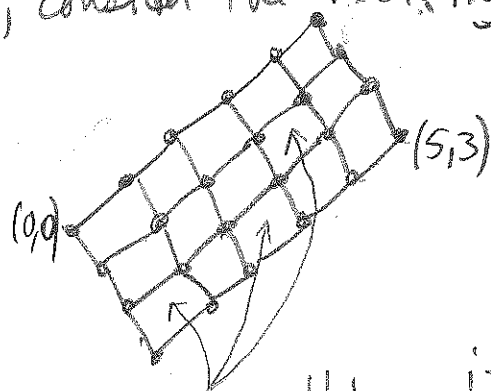
Remark: The unimodality of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ was first proved in 1913 by Elliot, using techniques from representation theory + algebraic geometry due to Sylvester and Cayley in the 1800's. The most recent proof of this is due to Pak & Panova, published in 2014 in Journal of Algebraic Combinatorics, and uses techniques from the representation Theory of G_n .

A poset-theoretic interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$:

Looking ahead, consider the rectangle from $(0,0)$ to (j,k) rotated as such:

Ex with

$(j,k) = (5,3)$.



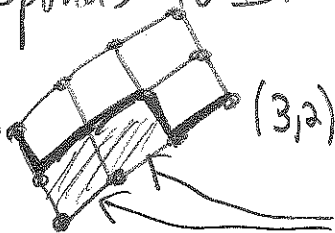
Boxes are related if one is in rectangle formed by another.

The boxes in this rectangle form a partially ordered set, or poset.

A lower order ideal I of this is a set of boxes such that if box $B \in I$, then all boxes below B are also in I .

Thm: There is a bijection between $G_{\{E^j; N^k\}}$ and the set of lower order ideals in this poset, where $\text{inv}(w) = |I|$ when w corresponds to I .

Ex: $(0,0)$



NEENE has $\text{inv} = 2$.

Two boxes in this lower order ideal.

$$\text{Cor: } \sum_{\substack{\text{Lower order} \\ \text{ideals } I \text{ of} \\ \text{poset corresponding} \\ \text{to } (0,0), (j,k)}} q^{|I|} = \begin{bmatrix} j+k \\ k \end{bmatrix}_q$$

Remark: This poset-theoretic view is the road to the fastest proof that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has unimodal coefficients, using representation theory of wreath products.

Exponential Generating Functions, Derangements, and Involutions

We begin with three quotes:

"OGFs are ideally suited for counting ordered structures like integer partitions, ordered trees, and words. But many combinatorial families are really sets, where the order is immaterial. For these, the natural concept is that of an exponential generating function."

-Doron Zeilberger

"If we can somehow associate a binomial poset with generating functions of the form $\sum f(n)x^n/B(n)$, then we will have... provided some justification of the heuristic principle that ordinary generating functions are associated with the non-negative integers, exponential generating functions with sets, Eulerian generating functions with vector spaces, and so on."

-Richard Stanley.

"A mathematical object is what it does."

-Timothy Gowers.

- When we multiply OGFs, what do they do?

$$\left(\sum_{n \geq 0} a_n x^n\right) \left(\sum_{m \geq 0} b_m x^m\right) = \sum_{n \geq 0} \left(\sum_{j=0}^n a_j b_{n-j}\right) x^n.$$

The resulting coefficient arises by concatenating objects of "type" $(a_n)_{n \geq 0}$ and "type" $(b_n)_{n \geq 0}$.

- What about EGFs?

$$\left(\sum_{n \geq 0} a_n \frac{x^n}{n!}\right) \left(\sum_{m \geq 0} b_m \frac{x^m}{m!}\right) = \sum_{n \geq 0} \left(\sum_{j=0}^n \frac{a_j}{j!} \frac{b_{n-j}}{(n-j)!}\right) x^n = \sum_{n \geq 0} \left(\sum_{j=0}^n \binom{n}{j} a_j b_{n-j}\right) \frac{x^n}{n!}$$

The factor of $\binom{n}{j}$ on $a_j b_{n-j}$ indicates we are no longer concatenating! Instead, we decompose an n -set into two parts in all possible ways, (over) ①

and for each way we assign a type $(a_n)_{n \geq 0}$ structure and a type $(b_n)_{n \geq 0}$ structure.

Let's look at two examples:

① Derangements

The Hat-Check Problem: Suppose n people give their hats to a hat-check person at a fancy event. How many ways can the hats be returned with everyone receiving the wrong hat?

This motivates the following:

Defⁿ: A derangement $\pi \in G_n$ is a permutation satisfying one of the following (equivalent) conditions:

① $\nexists \pi = C_1 C_2 \dots C_k$ as a disjoint cycle decomposition, then $|C_i| \geq 2$ for all i .

② $\nexists \pi = \pi_1 \pi_2 \dots \pi_n$ as a word, then $\pi_i \neq i$ for all i .

Ex: • $(13)(245)$, i.e. 34152, in G_5 .
• (12435) , i.e. 24531, in G_5 .

Derangements are "fixed-point free", i.e. have no fixed points.

Let D_n denote the number of derangements in G_n , and let

$$D(x) := \sum_{n \geq 0} D_n \frac{x^n}{n!}.$$

Theorem: ① $D(x) = e^{-x} \cdot \frac{1}{1-x} = \frac{e^{-x}}{1-x}.$

② $n! = \sum_{k=0}^n \binom{n}{k} D_{n-k}.$

③ $D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$

Pf: ③ follows easily from ①, since

$$\sum_{n \geq 0} D_n \frac{x^n}{n!} = D(x) = e^{-x} \cdot \frac{1}{1-x} = \left(\sum_{n \geq 0} (-1)^n \frac{x^n}{n!} \right) \left(\sum_{j \geq 0} x^j \right)$$

$$= \sum_{n \geq 0} \left[\sum_{j=0}^n \frac{(-1)^j}{j!} \cdot 1 \right] x^n = \sum_{n \geq 0} \left[n! \sum_{j=0}^n \frac{(-1)^j}{j!} \right] \frac{x^n}{n!}.$$

Remark: This is an important step; we are dealing with an egf, so we must re-introduce the $n!$ in the denominator. This is a frequent source of mistakes. End Remark.

Next, observe that ① and ② are equivalent:

$$D(x) = \frac{e^{-x}}{1-x} \text{ iff } e^x \cdot D(x) = \frac{1}{1-x} \text{ iff}$$

Remark: Here the $n!$ issue arises again!

$$\left(\sum_{n \geq 0} \frac{x^n}{n!} \right) \left(\sum_{m \geq 0} D_m \frac{x^m}{m!} \right) = \sum_{n \geq 0} n! \frac{x^n}{n!} \text{ iff}$$

$$\sum_{n \geq 0} \left[\sum_{k=0}^n \frac{1}{k!} \frac{D_{n-k}}{(n-k)!} \right] x^n = \sum_{n \geq 0} \left[\sum_{k=0}^n \binom{n}{k} D_{n-k} \right] \frac{x^n}{n!} = \sum_{n \geq 0} n! \frac{x^n}{n!} \text{ iff}$$

$$\sum_{k=0}^n \binom{n}{k} D_{n-k} = n! \text{ for all } n \geq 0.$$

So, we see that the product of egfs $e^x \cdot D(x) = \frac{1}{1-x}$ encodes this identity. Why is it true?

Every permutation $\pi \in \mathcal{G}_n$ consists of the identity permutation on the fixed points of π , and a derangement on the non-fixed points.



(over)

③

Thus, we see

$$\underbrace{n!}_{\substack{\text{\# elements} \\ \text{in } \mathbb{S}_n}} = \sum_{k=0}^n \underbrace{\binom{n}{k}}_{\substack{\text{\# identity} \\ \text{permutations on} \\ \text{a } k\text{-set of fixed} \\ \text{points}}} \cdot \underbrace{1}_{\substack{\text{\# of derangements on} \\ \text{the } (n-k)\text{-set of non-fixed} \\ \text{points}}} \cdot \underbrace{D_{n-k}}_{\substack{\text{\# of derangements on} \\ \text{the } (n-k)\text{-set of non-fixed} \\ \text{points}}}$$

II Involutions

Defⁿ: An involution $\pi \in \mathbb{S}_n$ is a permutation satisfying $\pi^2 = \text{id}$, equivalently satisfying that if $\pi = C_1 \cdots C_k$ is the disjoint cycle decomposition, then $|C_i| \leq 2$ for all i .

Remark: Derangements are $|C_i| \geq 2$ for all i , so these classes are natural to compare.

Let invol_n denote the number of involutions in $\mathbb{S}_n$.

Thm: $\sum_{n \geq 0} \text{invol}_n \frac{x^n}{n!} = e^x \cdot e^{\frac{1}{2}x^2} = e^{x + \frac{1}{2}x^2}$

Remark: The power $2m$ makes this harder.

Proof: Note that $e^x \cdot e^{\frac{1}{2}x^2} = \left(\sum_{n \geq 0} \frac{x^n}{n!} \right) \left(\sum_{m \geq 0} \frac{1}{2^m} \frac{x^{2m}}{m!} \right) =$

$$\sum_{n \geq 0} \left(\sum_{\substack{j! \cdot \frac{1}{2^m} \cdot \frac{1}{m!} \\ 1+1+\dots+2+2+\dots=n}} \right) x^n =$$

$$\sum_{n \geq 0} \left[\sum_{\substack{j! \cdot \frac{1}{2^m} \cdot \frac{1}{m!} \\ 1+1+\dots+2+2+\dots=n}} \frac{1}{j!} \cdot \frac{1}{2 \cdot 1 \cdot 2 \cdot \dots \cdot 2m} \cdot (j+2m)! \right] \frac{x^n}{n!} =$$

multiply and divide by the missing odd #'s.

note that $n = j + 2m$

$$\sum_{n \geq 0} \left[\sum_{\substack{j! \cdot \frac{1}{2^m} \cdot \frac{1}{m!} \\ 1+1+\dots+2+2+\dots=n}} \binom{j+2m}{j} \cdot 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1) \right] \frac{x^n}{n!}.$$

Note: Define $1 \cdot 3 \cdot 5 \cdots (2m-1) =: \underbrace{(2m-1)!!}_{\text{odd \#}}, \text{ and } (2m)!! = 0, \text{ double-factorial.}$

We thus need to show that

$$\text{invol}_n = \sum_{j=0}^n \binom{n}{j} (n-j-1)!!; \text{ which follows from:}$$

Picture:

2-cycles of involution. 1-cycles in an involution

$$\sum_{j=0}^n \binom{n}{j} \cdot \underbrace{1}_{\substack{\text{choose } j \text{ fixed} \\ \text{points}}} \cdot \underbrace{(n-j-1)!!}_{\substack{\text{one identity} \\ \text{permutation} \\ \text{on fixed} \\ \text{points}}}$$

This is the number of permutations consisting of exactly $\frac{n-j}{2}$ disjoint 2-cycles, as we will see shortly.

Remark: For both derangements and involutions, we had a product of egl's that led to an identity involving choosing a subset of $[n]$ and placing structure on the subset and its complement. Let's generalize this.

Let M denote a type of combinatorial structure on labeled finite sets, and let m_n denote the number of such structures on a labeled n -set. In each case, we have to decide on a value of m_0 , usually equal to 0 or 1.

Define $M(x) := \sum_{n \geq 0} m_n \frac{x^n}{n!}$ as the egl of type M .

Ex: • $M = \text{permutations}$, so $m_n = |S_n| = n!$ and $m_0 = 1$.

So, $\sum_{n \geq 0} m_n \frac{x^n}{n!} = \sum_{n \geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$.

• $M = \text{structure of a set}$, so $m_n = 1$, and $m_0 = 1$, so

$$\sum_{n \geq 0} m_n \frac{x^n}{n!} = \sum_{n \geq 0} \frac{x^n}{n!} = e^x.$$

• $P =$ set partitions of $[n]$ into exactly $\frac{n}{2}$ pairs. Then

$$P_{2j+1} = 0 \text{ and } P_{2j} = (2j-1) P_{2j-2} \Rightarrow P_{2j} = (2j-1)(2j-3) \dots (3)(1) = (2j-1)!!.$$

↑
Pick the element
to go with $2j$ in
a pair.

$$\text{Thus, } P(x) = \sum_{n \geq 0} \frac{(2n-1)!!}{(2n)!} x^{2n} = e^{\frac{1}{2}x^2}.$$

Pairwise-Compound Structure Principle:

To obtain the eg.f. for a combinatorial structure M formed as a pairwise-compound structure of two other structures of types A and B , where each structure in M decomposes uniquely into two structures of type $A+B$, respectively, we compute

$$m_n = \sum_{j=0}^n \binom{n}{j} a_j b_{n-j}, \text{ equivalently } M(x) = A(x)B(x).$$

Proof: The right-hand side records the unique decomposition,
since $\sum_{j=0}^n \binom{n}{j} a_j \cdot b_{n-j} = m_n.$

choose elements
from the n -set

↑
of type A
structures
on the j elements

↑
of type B
structures on
the $(n-j)$ other
elements.

Note: We have that $e^x \cdot D(x) = \frac{1}{1-x}$ and

$$\sum_{n \geq 0} \text{inv} \frac{x^n}{n!} = e^x \cdot e^{x^2/2}$$

both follow from this principle.

Block-partitioned Structure Principle and the Exponential formula

Observation: The egf for involn is $e^x \cdot e^{x^2/2} = e^{x + \frac{x^2}{2}}$,

Thus, this is both a product and a composition.

Recall Defⁿ: For two formal power series $f(x)$ and $g(x)$ with $g(0)=0$, we may form $f(g(x)) = \sum_{n \geq 0} f_n (g(x))^n = \sum_{n \geq 0} f_n \left[\sum_{j \geq 1} g_j x^j \right]^n$.

The coefficient of x^i in $f(g(x))$ is determined by the terms for $n=0$ to i , as $n > i$ contributes to higher-degree terms only.
 n factors

Example: $e^{x + x^2/2} = \sum_{n \geq 0} \frac{(x + \frac{x^2}{2})^n}{n!} = \sum_{n \geq 0} \frac{(x + \frac{x^2}{2}) \cdots (x + \frac{x^2}{2})}{n!}$.

We can interpret this as follows. If an involution π has n disjoint cycles, each of these is either a 1-cycle or a 2-cycle. If it is a 2-cycle, we multiply by $\frac{x^2}{2}$. If a 1-cycle, by x .

For example, $\frac{(x + \frac{x^2}{2})(x + \frac{x^2}{2})}{2!} \leftarrow$ multiplying out records all involutions w/ 2 ordered cycles.
 $2! \leftarrow$ eliminates overcounting from ordering.

$(2)(1)$ and $(1)(2) \leftarrow x \cdot x = \frac{2x^2}{2!}$ $\leftarrow$ 2 of the cycles $\leftarrow$ 2 negt

$(12)(3) \leftarrow \frac{x^2}{2} \cdot x = \frac{3x^3}{3!}$ and $\frac{3x^3}{3!} = x \cdot \frac{x^2}{2} \rightarrow$ $(3)(12)$
 $(23)(1) \leftarrow$ $(1)(23)$
 $(13)(2) \leftarrow$ $(2)(13)$

$(12)(34)$ and $(34)(12) \leftarrow \frac{x^2}{2} \cdot \frac{x^2}{2} = \frac{6x^4}{4!}$
 $(13)(24)$ and $(24)(13) \leftarrow$
 $(14)(23)$ and $(23)(14) \leftarrow$

In each case, dividing by $2! = 2$ eliminates the overcounting.

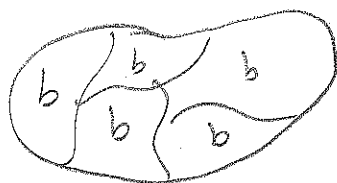
This motivates the following principle.

Block-partitioned Structure Principle:

Suppose M is a combinatorial structure on n -sets where each structure is uniquely described as a union of non-empty blocks of type B , where the blocks themselves are related by a type A structure, and $b_0 = 0$. Then

$$M(x) = A(B(x)) \quad \text{where as usual } M(x), A(x), \text{ \& } B(x) \text{ are egfs.}$$

Picture:



"B-structure on each block" = inside fn.
"A-structure on the set of blocks" = outside fn.

Sketch/Proof: $A(B(x)) = \sum_{k \geq 0} a_k \frac{(B(x))^k}{k!}$.

Step 1: Observe that $(B(x))^k$ is the egf for labeled set structures built as k ordered blocks of type B . Since $B(0) = 0$, each block is non-empty.

Step 2: Dividing $(B(x))^k$ by $k!$ removes the order, so $\frac{(B(x))^k}{k!}$ is the egf. for structures built with k nonempty blocks of type B .

Step 3: For each of these structures, there are a_k type A structures for the k blocks of type B , giving $a_k \frac{(B(x))^k}{k!}$ total structures w/ k blocks satisfying the principle.

Summing over all k yields $M(x) = \sum_{k \geq 0} a_k \frac{(B(x))^k}{k!} = A(B(x))$.

Corollary: (Exponential Formula): If the type A structure is trivial, i.e. $a_k = 1$ for all k , then $A(x) = e^x$, and

$$M(x) = \exp(B(x)) = e^{B(x)}.$$

Before going through examples, let's record some useful e.g.f.s (you must know all of these by heart).

• trivial structure: $e^x = \sum_{n \geq 0} 1 \cdot \frac{x^n}{n!}$

• all orderings possible: $\frac{1}{1-x} = \sum_{n \geq 0} n! \frac{x^n}{n!}$

• trivial on even size, none on odd: $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

• trivial on odd size, none on even: $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

• e.g.f. for # cycles of length n in $\mathbb{S}_n$, i.e. $(n-1)!$:

$$-\ln(1-x) = \ln\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

$$= \sum_{n \geq 1} (n-1)! \frac{x^n}{n!}$$

Applications of Block-Partitioned Principle + Exponential Formula:

① $B_n = n^{\text{th}}$ Bell number = # of set partitions of $[n]$.

Here, we break $[n]$ into non-empty, trivially structured blocks.

Hence, $B(x) = \sum_{n \geq 0} B_n \frac{x^n}{n!} = \underbrace{e}_{\text{trivial on blocks}} \underbrace{\frac{e^x - 1}{x}}_{\text{trivial non-empty in blocks}}$

② E.g.f. for set partitions of $[n]$ into blocks of even cardinality:

$$\underbrace{e}_{\text{trivial on blocks}} \underbrace{\frac{\cosh(x) - 1}{x}}_{\text{even-sized, trivial in blocks}}$$

If odd cardinality, we get

$$e^{\sinh(x)}$$

③ Eg.f. for the number of partitions of an n -set into k non-empty blocks:

$$\frac{1}{k!} (e^x - 1)^k = \sum_{n \geq 0} S(n, k) \frac{x^n}{n!}, \text{ as we have seen.}$$

Annotations:
 - $\frac{1}{k!}$: remove order on the blocks
 - $(e^x - 1)^k$: no order inside blocks, k blocks.

④ Eg.f. for the number of partitions of an n -set where the blocks have size 2, 3, or 5, and the size 5 blocks have 3 possible colors:

$$e^{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + 3 \frac{x^5}{5!}\right)}.$$

⑤ Eg.f. for the number of permutations of an n -set:

$$\exp\left(x + \frac{x^2}{2!} + 2! \frac{x^3}{3!} + 3! \frac{x^4}{4!} + \dots\right) = \exp(-\ln(1-x)) = \frac{1}{1-x}.$$

Annotations:
 - x : # of 1-cycles
 - $\frac{x^2}{2!}$: # of 2-cycles
 - $2! \frac{x^3}{3!}$: # of 3-cycles
 - $3! \frac{x^4}{4!}$: # of 4-cycles
 - "breaks the n -set into blocks to be disjoint cycles"
 - "orders each block as a cycle."
 - "This agrees with our earlier calculation."

⑥ Eg.f. for the number of permutations π of an n -set with only 1-, 2-, and 4-cycles in disjoint cycle decomposition:

$$e^{\left(x + \frac{x^2}{2!} + \frac{3! x^4}{4!}\right)}$$

Advanced Applications of E.g.f.'s:

Ex: Let h_n denote the number of ways for n people to form into nonempty lines, and then to arrange these in a circular order.



Since there are $j!$ ways to order j people, and $(j-1)!$ ways to circularly order $j \geq 1$ lines, we have that

$$\sum_{n \geq 0} h_n \frac{x^n}{n!} = 1 + \overbrace{\log \left(1 - \left[\frac{1}{1-x} \right] \right)^{-1}}^{\substack{\text{linearly order} \\ \text{cyclically order}}} = 1 + \log(1-2x)^{-1} - \log(1-x)^{-1}$$

$$= 1 + \sum_{n \geq 0} (2^n - 1)(n-1)! \frac{x^n}{n!}.$$

Thus, $h_n = (2^n - 1)(n-1)!$.

Exercise: Prove this formula combinatorially.

Ex: Let g_n denote the number of graphs for which every connected component is an (undirected) cycle. Since the number of undirected cycles of length n is $\frac{(n-1)!}{2}$, we have that the

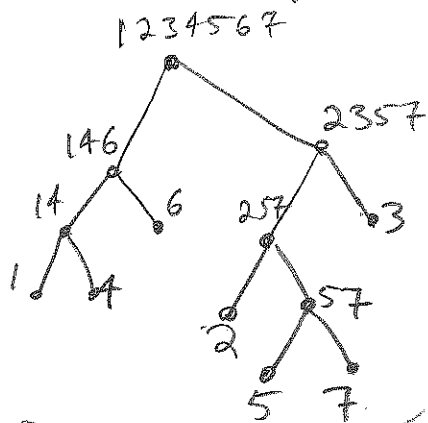
e.g.f. for undirected cycles (of length ≥ 3) is $\sum_{n=3}^{\infty} \frac{1}{2} \frac{(n-1)!}{2} \frac{x^n}{n!} = \frac{1}{2} \log \left(\frac{1}{1-x} \right) - \frac{x}{2} - \frac{x^2}{4}.$

$\nearrow 2$
2 orientations per cycle

By the exponential formula, we have

$$\sum_{n \geq 0} g_n \frac{x^n}{n!} = \exp \left(\frac{1}{2} \log \left(\frac{1}{1-x} \right) - \frac{x}{2} - \frac{x^2}{4} \right) = \frac{e^{(-\frac{x}{2} - \frac{x^2}{4})}}{\sqrt{1-x}}.$$

Ex: Let b_n denote the number of unordered, complete binary trees with n labeled endpoints. We can model these by successively splitting an n -set into pairs of non-empty, complementary blocks. For example, when $n=7$ we might have:



Note: Branches have no relative order, only labels.

Set $b_0 := 0$
Now, observe that every such tree can be split by removing the top element, which produces a unique decomposition of such a tree into two smaller (unordered) such trees. Setting $B(x) := \sum_{n \geq 1} b_n \frac{x^n}{n!}$

thus see that $B(x)$ can be written as a function of itself:

$$\frac{1}{2!} [B(x)]^2 = B(x) - x.$$

Note: This should remind you of Catalan numbers! Solve the quadratic!

$$\text{Thus, } B(x) = 1 - \sqrt{1-2x} = 1 - \sum_{n \geq 0} \binom{\frac{1}{2}}{n} (-2)^n x^n$$

$\uparrow$ Binomial Series

$\uparrow$ messy algebra, exercise

$$\sum_{n \geq 1} 1 \cdot 3 \cdot 5 \cdots (2n-3) \frac{x^n}{n!}.$$

$$\text{Thus, } b_n = (2n-3)!!.$$

Exercise: Prove this combinatorially.

Recurrences for Egt.'s, and permutations with restricted cycle types:

General idea: For egt.'s arising from the exponential formula, we get an expression of the form $M(x) = e^{B(x)}$.

Taking the derivative of both sides, we see

$$M'(x) = \sum_{n \geq 0} n \cdot m_n \frac{x^{n-1}}{n!} = \sum_{n \geq 0} m_n \frac{x^{n-1}}{(n-1)!} = B'(x) e^{B(x)} = B'(x) M(x).$$

For B 's that are polynomials, this gives us non-linear recurrences.

Example: Recall $I(x) = \sum_{n \geq 0} \text{invol}_n \frac{x^n}{n!} = e^{x + x^2/2}$.

$$\text{Thus } I'(x) = \sum_{n \geq 1} \text{invol}_n \frac{x^{n-1}}{(n-1)!} = (1+x) e^{x + x^2/2} = (1+x) I(x) =$$

$$(1+x) \sum_{n \geq 0} \text{invol}_n \frac{x^n}{n!} = \sum_{n \geq 0} \text{invol}_n \frac{x^n}{n!} + \sum_{n \geq 0} (n+1) \text{invol}_n \frac{x^{n+1}}{(n+1)!}.$$

This implies $\text{invol}_{n+1} = \text{invol}_n + n \cdot \text{invol}_{n-1}$. By egt.'s we know it's true, but let's give a ---

Combinatorial Proof: Set $\text{invol}_0 = 1$. Let $\pi \in G_{n+1}$ be an involution, + consider the element $n+1$. If it is in a 1-cycle, then (disjoint cycle decomp)

$$\pi = C_1 C_2 \dots C_k (n+1) \text{ and } C_1 \dots C_k \text{ is an involution in } G_n.$$

If it is in a 2-cycle, then

$$\pi = C_1 C_2 \dots C_k (i, n+1) \text{ with } C_1 C_2 \dots C_k \text{ an involution in } G_{n-1}$$

and we have n choices for i . The recurrence follows. $\square$

Remark: This recurrence can be found without egt.'s using a difference table. Define the operator (finite difference)

$$\Delta(a_n) = a_{n+1} - a_n \quad \text{for a sequence } (a_n)_{n \geq 0}.$$

Then we have:

n	0	1	2	3	4	5	6
invol _n	1	1	2	4	10	26	76
$\Delta(\text{invol}_n)$	0	1	1	2	6	16	50

One can guess the recurrence from this.

Example: Let T_n denote the number of elements of G_n only having cycle lengths 1, 2, and 3. Then

$$T(x) = \sum_{n \geq 0} T_n \frac{x^n}{n!} = e^{(x + \frac{x^2}{2} + \frac{x^3}{3})}. \text{ Hence,}$$

$$T'(x) = \sum_{n \geq 1} T_n \frac{x^{n-1}}{(n-1)!} = \left(1 + x + \frac{x^2}{2}\right) T(x) = \left(1 + x + \frac{x^2}{2}\right) \left(\sum_{n \geq 0} T_n \frac{x^n}{n!}\right);$$

hence

$$\sum_{n \geq 0} T_{n+1} \frac{x^n}{n!} = \sum_{n \geq 0} \left(T_n + nT_{n-1} + \frac{n(n-1)}{2} T_{n-2}\right) \frac{x^n}{n!}, \text{ and we have}$$

$$T_{n+1} = T_n + nT_{n-1} + \binom{n}{2} T_{n-2}.$$

Exercise: Prove this combinatorially, similarly to the case of invol_n.

Even if the exponent in $M(x) = e^{B(x)}$ is not a polynomial, we get interesting (infinite) recurrence formulas in this manner. These recurrences can shed light on subtle combinatorial structure in objects.

A deeper look at cycle structure:

Let $\pi \in G_n$ be a permutation, and suppose π has c_i cycles of length i .
Thus, $\sum_i i c_i = n$. Form a monomial, the cycle index of π , by

$$Z(\pi) := t_1^{c_1} t_2^{c_2} \dots t_n^{c_n}.$$

Define the (augmented) cycle index of G_n to be

$$\tilde{Z}(G_n) = \sum_{\pi \in G_n} Z(\pi), \text{ with } \tilde{Z}(G_0) := 1.$$

Example: $\tilde{Z}(G_1) = t_1.$

$$\tilde{Z}(G_2) = t_1^2 + t_2$$

$$\tilde{Z}(G_3) = t_1^3 + 3(t_1 t_2) + 2 t_3$$

$$\tilde{Z}(G_4) = t_1^4 + 6 t_1^2 t_2 + 8 t_1 t_3 + 3 t_2^2 + 6 t_4.$$

Note: If we set $t_i = 1$ for all i , $\tilde{Z}(G_n)|_{t_i=1 \forall i} = n!.$

The following is an algebraic extension of our formula $\frac{1}{1-x} = e^{\ln(\frac{1}{1-x})}.$

Thm: $\sum_{n \geq 0} \tilde{Z}(G_n) \frac{x^n}{n!} = \exp\left(\sum_{i \geq 1} t_i \frac{x^i}{i}\right).$

Let's give a proof from first principles, connecting to material earlier in the course.

Pf: Expand the right-hand side to get

$$\exp\left(\sum_{i \geq 1} t_i \frac{x^i}{i}\right) = \prod_{i \geq 1} \exp\left(t_i \frac{x^i}{i}\right) = \prod_{i \geq 1} \sum_{j \geq 0} t_i^j \frac{x^{ij}}{i^j j!}.$$

The coefficient of $t_1^{c_1} t_2^{c_2} \dots t_n^{c_n} x^n$ is therefore equal to zero unless $\sum_i i \cdot c_i = n$, in which case it is equal to

(over)

$$\frac{1}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!} = \frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!} \cdot \frac{1}{n!}$$

This gives the denominator we need for the e.g.f.

↑
We proved earlier in the semester that this is the number of permutations in S_n with cycle type $(c_1, c_2, \dots, c_n)$.

Thus, our theorem is proved. $\square$

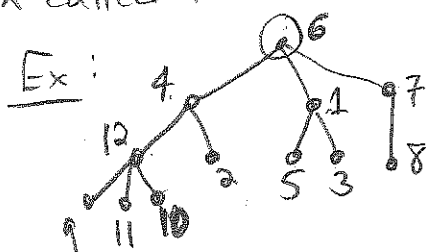
Point-Removal Structure Principle and Lagrange Inversion

Recall: The egf. for the number of unordered, complete binary trees with n labeled endpoints satisfied the functional equation

$$\frac{1}{2} [B(x)]^2 = B(x) - x.$$

There are certain types of enumerative problems that lead to more complicated functional equations on eg.f's, which we will now discuss.

Defⁿ: A tree T is a connected graph with no cycles. T is labeled if the vertices of T are labeled, and rooted if it has a distinguished vertex called the root.

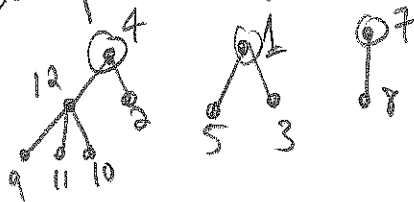


"6" is the root, which we denote by circling it.

Let $A(x) := \sum_{n \geq 1} a_n \frac{x^n}{n!}$, so $a_0 := 0$, denote the eg.f. for the number of rooted, labeled trees on n vertices.

Claim: $A(x) = x \cdot e^{A(x)}$.

Pf: For a rooted, labeled tree, removing the root yields a family of rooted, labeled trees. For example, removing "6" from the example above yields three rooted, labeled trees:



Thus, $A(x) = \underbrace{x}_{\text{add a new root}} \cdot \underbrace{e^{A(x)}}_{\substack{\text{By exponential formula, this forms} \\ \text{families of disjoint} \\ \text{labeled, rooted trees, i.e.} \\ \text{forests.}}}$



Remark: $A(x) = x \cdot e^{A(x)}$ if and only if $\frac{A(x)}{e^{A(x)}} = x$. Thus,

substituting $A(x)$ into $\frac{y}{e^y}$ yields x . We have a name for this.

Defⁿ: If $f(x) = \sum_{n \geq 1} a_n x^n$, we say $g(x)$ is a compositional inverse of f if $f(g(x)) = x = g(f(x))$, and we write $g(x) = f^{<-1>}(x)$.

Propⁿ: $f(x)$ has a compositional inverse if and only if $a_1 \neq 0$, in which case $f^{<-1>}(x)$ is unique. Also, $f(g(x)) = x$ if and only if $g(f(x)) = x$.

Pf: Exercise. This is very similar to the homework problem about multiplicative inverses earlier in the semester.

This leads us to the following.

Point-Removal Structure principle:

If a combinatorial structure M on labeled n -sets always has a unique distinguished element whose removal yields a collection of blocks of type M structures with distinguished elements, where the blocks get a type R structure, then the following three (equivalent) statements hold.

① $M(x) = x \cdot R(M(x))$

② $x = \frac{M(x)}{R(M(x))}$

③ $M^{<-1>}(x) = \frac{x}{R(x)}$

$M(x)$ and $R(x)$ are assumed to be e.g.f.s.

Pf: Exercise.

Big Question: How can I use a functional equation of this type to find a formula for the coefficients of $M(x)$?

Answer: Lagrange Inversion Formula!

②

What we want: A way to recover the coefficients of $M(x)$ from knowledge of $R(x)$ or $M^{(-1)}(x)$ in the equations

$$M(x) = x \cdot R(M(x)) \quad \text{or} \quad M^{(-1)}(x) = \frac{x}{R(x)}.$$

Remark: Suppose we know that $(e^x - 1)^{(-1)} = \log(1+x)$ and we know that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$; if we want to determine the coefficients of $e^x - 1$ as a power series, we don't want to have to solve for $R(x)$ in $\log(1+x) = \frac{x}{R(x)}$, which we would do

$$\text{by writing } R(x) = \frac{x}{\log(1+x)} = \frac{x}{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots} = \frac{1}{1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots} =$$

$$\sum_{k \geq 0} \left(\frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \frac{x^4}{5} + \dots \right)^k = \dots \text{ etc. On the other hand,}$$

if we know $A^{(-1)}(x) = \frac{x}{e^x}$, we know $R(x) = e^x$ and don't want to have to find the power series expansion for $A^{(-1)}(x)$ directly.

Recall Notation: $[x^n] \left(\sum_{k \geq 0} a_k x^k \right) = a_n$. For an e.g.f.,

$$[x^n] \left(\sum_{k \geq 0} a_k \frac{x^k}{k!} \right) = \frac{a_n}{n!}. \quad \text{Watch out for this!}$$

Lagrange Inversion Formula: Suppose $M(x) = \sum_{n \geq 1} m_n \frac{x^n}{n!}$ with $m_1 \neq 0$, and

that $M(x) = x \cdot R(M(x))$, so $M^{(-1)}(x) = \frac{x}{R(x)}$. Then

$$n [x^n] (M(x))^k = k \cdot [x^{n-k}] (R(x))^n = k \cdot [x^{-k}] \left(M^{(-1)}(x) \right)^{-n}.$$

Note: We will omit the proof, using Lagrange inversion as a "black box" tool. For proofs, see §5.4 of Stanley EC Volume 2.

Applications of Lagrange Inversion

Rooted, labeled Trees: We have $A(x) = xe^{A(x)}$, so $R(x) = e^x$ and

$M(x) = A(x) = \sum_{n \geq 0} \frac{A_n x^n}{n!}$. Use $k=1$ in Lagrange inversion:

$$n \cdot \frac{A_n}{n!} = n [x^n] (A(x)) \stackrel{\text{L.I.F.}}{=} [x^{n-1}] (e^x)^n = [x^{n-1}] e^{nx} =$$

$$[x^{n-1}] \left(\sum_{j \geq 0} \frac{(nx)^j}{j!} \right) = \frac{n^{n-1}}{(n-1)!}. \quad \text{Thus, } n \cdot \frac{A_n}{n!} = \frac{n^{n-1}}{(n-1)!} \text{ and we see}$$

that $A_n = n^{n-1}$ is the number of rooted, labeled trees on n vertices.

Corollary: (Cayley's Formula)

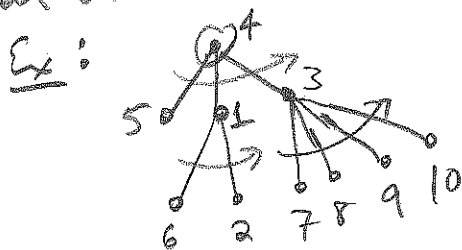
The number of labeled, connected trees on n vertices is n^{n-2} .

PF: For each labeled tree on n vertices, we have n choices for a root. Thus, if t_n is the number of labeled, connected trees on $[n]$, then

$$n \cdot t_n = n^{n-1}, \text{ hence } t_n = n^{n-2}. \quad \square$$

Note: In MA 514, this is proved combinatorially using Prüfer sequences.

Example: Let $F(x)$ be the egf. for labeled, rooted trees with a linear order on the children at each vertex.



Claim: $F(x) = x \cdot \frac{1}{1-F(x)}$, i.e.

$$R(x) = \frac{1}{1-x}.$$

PF: The block types are rooted, labeled trees, and the blocks are ordered using a permutation. The egf for # of permutations is $\frac{1}{1-x}$. $\square$

(over)

①

Thus, by Lagrange inversion, we have (using $k=1$)

$$n[x^n] F(x) = [x^{n-1}] \left(\frac{1}{1-x} \right)^n = [x^{n-1}] \sum_{k \geq 0} \binom{k+n-1}{n-1} x^k = \binom{2n-2}{n-1}.$$

Thus, if $F(x) = \sum_{n \geq 1} f_n \frac{x^n}{n!}$, we have

$$\frac{n}{n!} f_n = \binom{2n-2}{n-1} \Rightarrow f_n = (n-1)! \cdot \binom{2n-2}{n-1} = \frac{(2n-2)!}{(n-1)!}.$$

Example: Let A be a subset of $\{2, 3, 4, \dots\}$. Let $t_A(n)$ denote the number of ways of starting with an n -set S , then partitioning S into k blocks where $k \in A$, then partitioning each non-singleton block into k blocks where $k \in A$, etc, until only singletons remain. Set $t_A(0) := 0$.

For example, if $A = \{2\}$, we get complete leaf-labeled binary trees, which we saw before.

$$\text{Let } E_A(x) = \sum_{n \geq 1} t_A(n) \frac{x^n}{n!}.$$

Claim: $\sum_{k \in A} \frac{(E_A(x))^k}{k!} = E_A(x) - x$. Note: If $A = \{2\}$, we get $\frac{1}{2}(E_2(x))^2 = E_2(x) - x$, as in the binary tree case.

Pf: Exercise. $\square$

Thus, setting $G(x) = x - \sum_{k \in A} \frac{x^k}{k!}$, we see $x = G(E_A(x))$, so

$$E_A(x)^{<-1>} = x - \sum_{k \in A} \frac{x^k}{k!}. \quad \text{By Lagrange Inversion, we have } (k=1)$$

$$n[x^n] E_A(x) = [x^{-1}] \left(E_A(x)^{<-1>} \right)^{-n} = [x^{-1}] \left(x - \sum_{k \in A} \frac{x^k}{k!} \right)^{-n} =$$

(over)

$$[x^{-1}] \left((x^{-n}) \left(1 - \sum_{k \in A} \frac{x^{k-1}}{k!} \right)^{-n} \right) = [x^{n-1}] \left(1 - \sum_{k \in A} \frac{x^{k-1}}{k!} \right)^{-n}.$$

Consider the special case when $A = \{k\}$, a single element. Then

$$\left(1 - \frac{x^{k-1}}{k!} \right)^{-n} = \sum_{j \geq 0} \binom{n+j-1}{j} \frac{x^{j(k-1)}}{k!^j} = \sum_{j \geq 0} \binom{n+j-1}{j} \frac{(j(k-1))!}{k!^j} \frac{x^{j(k-1)}}{(j(k-1))!}.$$

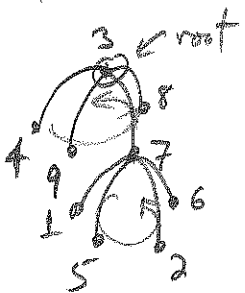
$$\text{Thus, } n[x^n] \left(\sum_{j \geq 0} t_k(j) \frac{x^j}{j!} \right) = [x^{n-1}] \left(\sum_{j \geq 0} \binom{n+j-1}{j} \frac{(j(k-1))!}{k!^j} \cdot \frac{x^{j(k-1)}}{(j(k-1))!} \right).$$

So, we see that $t_k(n) = 0$ unless $n = j(k-1) + 1$ for some $j \in \mathbb{Z}_{\geq 0}$, and

$$t_k(j(k-1)+1) = \binom{jk}{j} \frac{(j(k-1))!}{k!^j} = \frac{(jk)!}{j! k!^j}.$$

Example (The Spider Plant Problem): Find the number of rooted, labeled trees on n nodes such that each node is either a leaf or has exactly four children, arranged in a cycle.

Picture:



} Looks like a hanging plant!

Note: Cycles are ordered cyclically!

Let $T(x) = \sum_{n \geq 0} t_n \frac{x^n}{n!}$. Note that $t_n = 0$ if $n \not\equiv 1 \pmod{4}$, as a valid tree has $4k+1$ vertices. Let $C(x) = 1 + \frac{3! x^4}{4!}$ be the block structure for the e.g.f., so $C(x) = 1 + \frac{x^4}{4}$.

Claim: $T(x) = x C(T(x))$.

Pf: Exercise. $\square$

(over)

③

Applying Lagrange inversion, we see that (with $k=1$)

$$n \frac{t_n}{n!} = n [x^n] T(x) = [x^{n-1}] (C(x))^n = [x^{n-1}] \left(1 + \frac{x^4}{4}\right)^n =$$

$$[x^{n-1}] \left(\sum_{k=0}^n \binom{n}{k} \frac{x^{4k}}{4^k} \right). \text{ Thus, we have}$$

$$\frac{n t_n}{n!} = \frac{t_n}{(n-1)!} = [x^{n-1}] \left(\sum_{k=0}^n \binom{n}{k} \frac{x^{4k}}{4^k} \right).$$

when is $4k = n-1$? When $n = 4k+1$, as expected.

So, this confirms $t_n = 0$ if $n \not\equiv 1 \pmod{4}$.

If $n \equiv 1 \pmod{4}$, then

$$[x^{n-1}] \left(\sum_{k=0}^n \binom{n}{k} \frac{x^{4k}}{4^k} \right) = \binom{n}{\frac{n-1}{4}} \cdot \frac{1}{4^{\frac{n-1}{4}}}. \text{ As a result, we see}$$

$$t_n = (n-1)! \cdot \binom{n}{\frac{n-1}{4}} \cdot \frac{1}{4^{\frac{n-1}{4}}}.$$

□

Principle of Inclusion-Exclusion

Let's consider a different way to compute the number of derangements in G_n . For $T \subseteq [n]$, let $f_{\geq}(T)$ denote the number of permutations whose fixed point set contains T .

Observation: $f_{\geq}(T) = (n - |T|)!$, since we have $|T|$ fixed points in T and permute the elements of $[n] \setminus T$.

Let $f_{=}(T)$ be the number of permutations whose fixed point set is exactly T . Thus, $f_{=}(T) = D_{n-|T|}$.

Remark: We saw previously that $n! = \sum_{i=0}^n \binom{n}{i} D_{n-i}$.

Rewriting this, we have the equivalent expression

$$\underbrace{f_{\geq}(\emptyset)}_{\substack{\uparrow \\ \text{all permutations}}} = \sum_{\emptyset \subseteq T \subseteq [n]} \underbrace{f_{=}(T)}_{\substack{\uparrow \\ \text{fixed set exactly } T}}.$$

Question: Suppose we want to go in reverse, and write $D_n = f_{=}(\emptyset)$ as a sum of $f_{\geq}(T)$ over all $T \subseteq [n]$? Can we do this?

Recall: From eg.f.'s, we saw $D_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}$.

$$\text{So, } f_{=}(\emptyset) = D_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!} = \sum_{\emptyset \subseteq T \subseteq [n]} \binom{n}{i} (-1)^i (n-i)! = \sum_{\emptyset \subseteq T \subseteq [n]} (-1)^{|T|} f_{\geq}(T).$$

Notice: $f_{\geq}(\emptyset) = \sum_{\emptyset \subseteq T \subseteq [n]} f_{=}(T)$ and $f_{=}(\emptyset) = \sum_{\emptyset \subseteq T \subseteq [n]} (-1)^{|T|} f_{\geq}(T)$.

This is an example of a broader phenomenon.

Principle of Inclusion-Exclusion (P.I.E.)

Let A be a set of objects and S be a set of properties.
Let $T \subseteq S$ and denote by $f_{\geq}(T)$ the number of objects in A with at least the properties in T , and by $f_{=}(T)$ the number of objects in A with exactly the properties in T . Then:

$$(1) f_{\geq}(T) = \sum_{T \subseteq Y} f_{=}(Y).$$

$$(2) f_{=}(T) = \sum_{T \subseteq Y} (-1)^{|Y-T|} f_{\geq}(Y).$$

Proof: (1) is immediate, since every object in A satisfying at least T satisfies some $Y \supseteq T$ exactly for a unique Y .

For (2), let $A_{=T} \subseteq A$ be the elements of A satisfying exactly T , so $f_{=}(T) = |A_{=T}|$. Similarly, $A_{\geq T}$ are the elements satisfying at least T , and $f_{\geq}(T) = |A_{\geq T}|$.

Goal: Show:

$$f_{=}(T) = \sum_{a \in A_{=T}} 1 \stackrel{\text{focus on this equality, think "caveman".}}{=} \sum_{T \subseteq Y} (-1)^{|Y-T|} \sum_{a \in A_{\geq Y}} 1 = \sum_{T \subseteq Y} (-1)^{|Y-T|} f_{\geq}(Y).$$

3 cases: (i) If $a \in A \setminus A_{\geq T}$, then "a" contributes nothing to either side of $\circledast$.

(ii) If $a \in A_{=T}$, then "a" contributes exactly one to each side.

(iii) If $a \in A_{\geq T} \setminus A_{=T}$, then there is a unique X with $T \subsetneq X \subseteq S$ and $a \in A_X$. Then "a" contributes nothing to the left-hand side of $\circledast$, and to the right-hand side "a" contributes

$$\sum_{X \subseteq Y \subseteq S} (-1)^{|Y-X|} = (1-1)^{|S-X|} = 0.$$

Thus, $\circledast$ holds. 

Typical Application: The number of objects having none of the properties of S is given by $f_=(\emptyset) = \sum_{\emptyset \subseteq Y \subseteq S} (-1)^{|Y|} f_=(Y)$.

Example: In the 12-fold way homework, you proved that $S(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n$ using generating functions. Let's prove $k! S(n, k) = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$ using P.I.E., giving another proof.

Let $\mathcal{A} := \{ (A_1, \dots, A_k) : A_j \cap A_i = \emptyset \text{ if } i \neq j, A_j \subseteq [n] \text{ for all } j, \bigcup_{j=1}^k A_j = [n] \}$. Thus, $\mathcal{A}$ is all ordered set partitions of $[n]$ into k (possibly empty) blocks. Now, for $p \in [k]$, property " $\hat{p}$ " means $A_p = \emptyset$, and for $T \subseteq [k]$, we have $A_p = \emptyset$ whenever $p \in T$. Thus,

$$k! S(n, k) = f_=(\emptyset) = \sum_{\emptyset \subseteq Y \subseteq [k]} (-1)^{|Y|} f_=(Y).$$

Now, what is $f_=(Y)$? i.e., how many set partitions in $\mathcal{A}$ have at least the positions in Y empty? Every such partition corresponds uniquely to a function $[n] \rightarrow [k] \setminus Y$, so there are $(k-|Y|)^n$ such partitions. Since there are $\binom{k}{i}$ subsets of $[k]$ of size i , we have

$$k! S(n, k) = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n, \text{ as desired.} \quad \blacksquare$$

Remark: We will soon see that determinants are often involved in signed counting. Here is a general statement.

Propⁿ: (EC1, 2.2.6) Let $S = \{P_1, \dots, P_n\}$ be a set of properties, and let $T = \{P_{s_1}, \dots, P_{s_k}\} \subseteq S$, where $1 \leq s_1 \leq \dots \leq s_k \leq n$. Suppose that $f_{\pm}(T) = h(n) e(s_0, s_1) e(s_1, s_2) \dots e(s_{k-1}, s_k)$ for certain functions h on $\mathbb{Z}_{\geq 0}$ and e on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, where $s_0 := 0$, $s_{k+1} := n$, $e(i, i) := 1$, $e(i, j) := 0$ if $i > j$. Then

$$f_{\pm}(T) = h(n) \det \left[e(s_i, s_{j+1}) \right]_{i,j=0}^k$$

Pf: (omit). $\square$

Unimodal and Log-Concave Sequences

Defⁿ: A sequence $\{a_n\}_{n \geq 0}$ is unimodal if there is a k for which $a_0 \leq a_1 \leq \dots \leq a_k \geq \dots$, or if $\{a_n\}_{n \geq 0}$ is always increasing. The sequence is log-concave if $a_{k-1} \cdot a_{k+1} \leq a_k^2$ for all $k \geq 1$.

Remark: $a_{k-1} \cdot a_{k+1} \leq a_k^2$ if and only if (take natural log of both sides) $\frac{\log(a_{k-1}) + \log(a_{k+1}))}{2} \leq \log(a_k)$. Thus, the sequence is concave after the log is taken.

Remark: See exercise 1.50 in EC1 for a list of important properties of such sequences. In particular:

Theorem (Newton): If $\sum_{j=0}^n \binom{n}{j} a_j x^j$ has only real roots, then $\{a_j\}$ is log-concave. OMIT Proof.

Propⁿ: If $\{a_k\}$ is log-concave, it is unimodal.

Pf: If $a_i \leq a_{i+1}$ for all i , we are done. Otherwise, suppose a_i is the first i for which $a_i \geq a_{i+1}$. Then we have

$$a_i \cdot a_{i+2} \leq (a_{i+1})^2 \Rightarrow \frac{a_i}{a_{i+1}} \leq \frac{a_{i+1}}{a_{i+2}} \leq \frac{a_i}{a_{i+2}}, \text{ implying } a_{i+1} \geq a_{i+2}.$$

Now proceed inductively. ■

Theorem: The following sequences are log-concave for all n .

• $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$.

• $S(n,1), S(n,2), \dots, S(n,n)$.

• $A_{n,0}, A_{n,1}, \dots, A_{n,n-1}$

← Eulerian numbers, related to descents in S_n .

We will omit proofs for these two sequences.

We will see three proofs that $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$ is log-concave.

Proof 1: Use $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and do explicit algebraic calculations. This is on your homework.

Proof 2: Let's sketch a combinatorial proof. To do so, we will find explicit injections $\phi_k: \binom{[n]}{k-1} \times \binom{[n]}{k+1} \rightarrow \binom{[n]}{k} \times \binom{[n]}{k}$ to demonstrate $\binom{n}{k-1} \cdot \binom{n}{k+1} \leq \binom{n}{k}^2$.

For any $X \subseteq [n]$, let $X_j := X \cap [j]$. Given (A, B) in $\binom{[n]}{k-1} \times \binom{[n]}{k+1}$, let j be the largest integer such that

$|A_j| = |B_j| - 1$. Exercise: Prove such a j exists.

Then define $C := A_j \cup (B \setminus B_j)$, $D := B_j \cup (A \setminus A_j)$, and set

$\phi_k(A, B) := (C, D)$. This gives our desired injection.

Example: $n=7, k=3$. $A = \{1, 4\} \in \binom{[7]}{2}$, $B = \{2, 4, 6, 7\} \in \binom{[7]}{4}$.

For $j=6$, we have $|A_6| = |\{1, 4\}| = |\{2, 4, 6\}| - 1 = |B_6| - 1$.

Then we have $C = A_6 \cup (B \setminus B_6) = \{1, 4, 7\}$, $D = B_6 \cup (A \setminus A_6) = \{2, 4, 6\}$.

Thus, $\phi_3(\{1, 4\}, \{2, 4, 6, 7\}) = (\{1, 4, 7\}, \{2, 4, 6\})$.

Remark: It is not too hard to fill in the details by proving such a j exists, proving ϕ_k is well-defined, and proving ϕ_k is injective. However, it is very tedious. We will focus on another path.

Proof 3: We recast log-concavity in terms of determinants.

Observe that $a_{k-1} \cdot a_{k+1} \leq a_k^2$ if and only if

$$a_k^2 - a_{k-1} a_{k+1} \geq 0 \text{ if and only if } \det \begin{bmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{bmatrix} \geq 0.$$

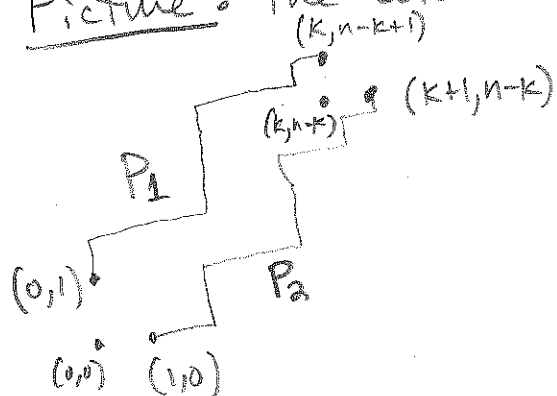
Thus, our goal is to prove that $\det \begin{bmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} \end{bmatrix} \geq 0.$

Recall that $\binom{n}{k}$ counts lattice paths to $(k, n-k)$. This leads to...

Theorem (Special Case of Gessel-Viennot Theorem):

Consider lattice paths using steps $(0,1)$ and $(1,0)$. The value of $\det \begin{bmatrix} \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-1} & \binom{n}{k} \end{bmatrix}$ is the number of pairs (P_1, P_2) of lattice paths that do not intersect, where P_1 goes from $(0,1)$ to $(k, n-k+1)$ and P_2 goes from $(1,0)$ to $(k+1, n-k)$.

Picture: The determinant counts non-intersecting pairs:



Remark: The third proof is complete once we prove this theorem. 

Our strategy to prove Gessel-Viennot is via sign-reversing involutions.

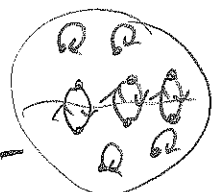
Sign-reversing Involutions

Defⁿ: A signed set is a set X with a function $\omega: X \rightarrow \{-1, 1\}$.

Let $X^+ = \omega^{-1}(1)$, and $X^- = \omega^{-1}(-1)$.

Example: $X = 2^{[n]}$ is a signed set with $\omega(T) = (-1)^{|T|}$ for $T \in 2^{[n]}$.

Defⁿ: An involution $\pi \in \mathfrak{S}_X$ for a signed set X is sign-reversing if for every 2-cycle $(a\ b)$ in π , $\omega(a) = -\omega(b)$.

X^+  $\} 2\text{-cycles interchange elements of } X^+ \text{ and } X^-.$

Proposition: Let π be a sign-reversing involution with fixed point set F . Then $\sum_{x \in X} \omega(x) = \sum_{x \in F} \omega(x)$.

Proof: In the left-hand side, the summands in 2-cycles cancel in pairs. $\blacksquare$

Example: Let n be odd, and for $T \in 2^{[n]}$, let $\pi(T) := [n] \setminus T$.

Then $\omega(\pi(T)) = (-1)^{n-|T|} = (-1) \cdot (-1)^{|T|} = (-1) \cdot \omega(T)$, so π is sign-reversing. Thus, since π has no fixed points,

$$\sum_{T \subseteq [n]} (-1)^{|T|} = 0.$$

Remark: There is a better proof of this, one that works for all n , both odd and even. You will find it on your homework. Hint: Your goal is to find a simple rule for making odd subsets even and vice-versa.

Corollary: For F, Π, X as in our previous proposition, suppose $F \subseteq X^+$. Then $\sum_{X \in F} w(X) = |F|$.

Remark: Section 2.6 in EC1 provides a proof of P.I.E. using sign-reversing involutions. This connection illustrates why P.I.E. and sign-reversing involutions are both "sieve" methods: they assign signs to objects, then sum the signs to cancel terms out.

Proof of Special Case of Gessel-Viennot:

Let $X = \left\{ (P_1, P_2) : \begin{array}{l} P_1: (0,1) \rightarrow (k, n-k+1) \text{ and } P_2: (1,0) \rightarrow (k+1, n-k) \\ \text{OR} \\ P_1: (0,1) \rightarrow (k+1, n-k) \text{ and } P_2: (1,0) \rightarrow (k, n-k+1) \end{array} \right\}$.

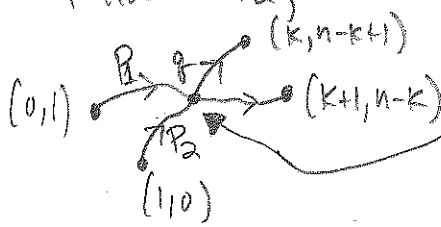
We first must make X into a signed set. Define:

$X^+ := \left\{ (P_1, P_2) : P_1: (0,1) \rightarrow (k, n-k+1) \text{ and } P_2: (1,0) \rightarrow (k+1, n-k) \right\} \leftarrow \left\{ \text{diagram of non-intersecting paths} \right\}$
 $X^- := \left\{ (P_1, P_2) : P_1: (0,1) \rightarrow (k+1, n-k) \text{ and } P_2: (1,0) \rightarrow (k, n-k+1) \right\} \leftarrow \left\{ \text{diagram of intersecting paths} \right\}$

Define π on X as follows:

- If P_1 and P_2 do not intersect, $\pi(P_1, P_2) = (P_1, P_2)$.
- If (P_1, P_2) do intersect, let q be the first intersection point on the paths (after starting from $(0,1)$ and $(1,0)$).

Set $\pi(P_1, P_2) = (Q_1, Q_2)$ where Q_1 follows P_1 to q then follows P_2 , and Q_2 follows P_2 to q then follows P_1 .



q can be viewed either as a crossing point or as a "tangential" intersection point. π exchanges these views.

Now, observe:

- ① If F is the fixed point set for π , then by definition $F \subseteq X^+$.
- ② All pairs in $X \setminus F$ must intersect.
- ③ The number of paths $P: (0,1) \rightarrow (k, n-k+1)$ is $\binom{n}{k}$.
- ④ The number of paths $P: (1,0) \rightarrow (k+1, n-k+1)$ is $\binom{n}{k}$.
- ⑤ The number of paths $P: (0,1) \rightarrow (k+1, n-k)$ is $\binom{n}{k+1}$.
- ⑥ The number of paths $P: (1,0) \rightarrow (k, n-k+1)$ is $\binom{n}{k+1}$.

Thus, the number of paths in X^+ is $\binom{n}{k} \cdot \binom{n}{k} = \binom{n}{k}^2$.

The number of paths in X^- is $\binom{n}{k+1} \binom{n}{k-1}$.

Thus, the number of non-intersecting paths is

$$|F| = \sum_{X \in X} w(X) = \binom{n}{k}^2 - \binom{n}{k+1} \binom{n}{k-1}.$$

$X^+ \qquad \qquad \qquad X^-$

Remarks: • The full Gessel-Viennot Theorem is more general, and applies to weighted paths in graphs.

• The full Gessel-Viennot theorem is used to prove theorems in the theory of symmetric functions; it can also serve to give a combinatorial proof of the Binet-Cauchy Theorem in matrix algebra.

• This theorem is recent, less than 50 years old; however, it has a funny history, with several "rediscoveries" along the way.

Posets

Why study posets?

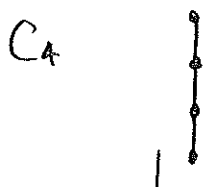
- Many combinatorial objects admit meaningful poset structures.
- Möbius inversion for posets generalizes
 - a) finite difference operators
 - b) P.I.E.
 - c) Möbius function of number theory.
- Binomial-type posets are used to understand various classes of generating functions.
- Posets provide deep connections with algebra & topology.

Defⁿ: A partially ordered set, aka poset, is a set P together with a binary relation denoted $\leq$ or $\leq_P$ satisfying:

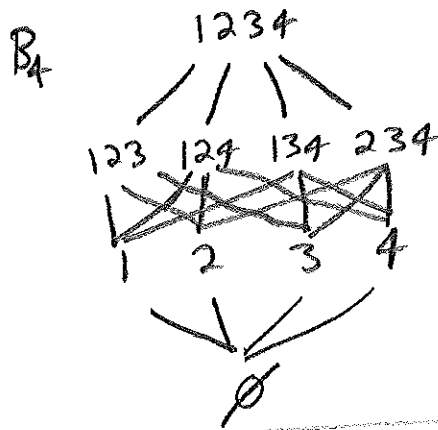
- For all $t \in P$, $t \leq t$ (reflexivity)
- If $s \leq t$ and $t \leq s$, then $s = t$. (antisymmetry)
- If $s \leq t$ and $t \leq u$, then $s \leq u$ (transitivity)

Note: a) If $s \neq t$ and $s \leq t$, write $s < t$. If $s \leq t$, s and t are comparable.
b) If s and t are not related, we say they are incomparable and write $s \parallel t$.

Examples: a) The set $[n]$ with the relations $1 < 2 < 3 < \dots < n$ is a poset. We refer to this poset as "bld" n , called the n -chain.



① For $n \in \mathbb{Z}_{>0}$, denote $B_n := 2^{[n]}$, with the order relation $S \leq T$ if $S \subseteq T$. B_n is called the Boolean algebra.



Defⁿ: A total order is a poset P satisfying the trichotomy law that for all $t, s \in P$, either $t \leq s$, $s \leq t$, or $s = t$.

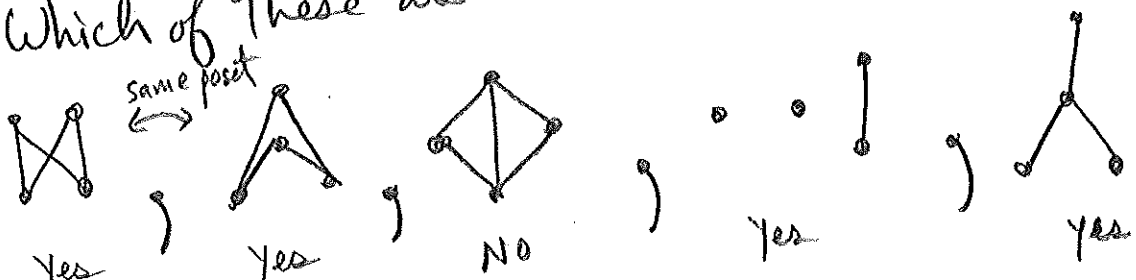
Ex: Chains are the finite total orders.

Defⁿ: For $s, t \in P$, we say t covers s if $s < t$ and there is no element $z \in P$ with $s < z < t$. If t covers s , write $s \lessdot t$ or $s \dot{\leq} t$.

Ex: In B_4 above, I have drawn edges for every cover relation.

Remark: The Hasse diagram of a finite poset P is the graph whose vertices are elements of P and edges are cover relations, drawn so that if $s \lessdot t$, then s is "below" t .

Q: Which of these are valid Hasse diagrams?



Defⁿ: For $x, y \in P$, the closed interval from x to y is

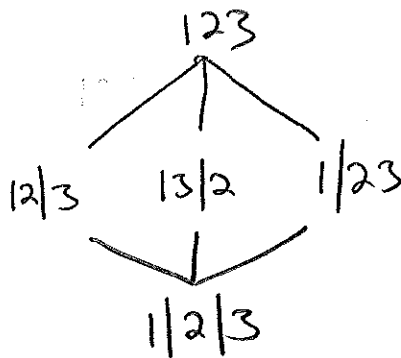
$[x, y] := \{z \in P : x \leq z \leq y\}$, and the open interval is $(x, y) := \{z \in P : x < z < y\}$.

Remark: Intervals are subsets of P , see pg 243 in EC1 for definition of subposet & induced subposet.

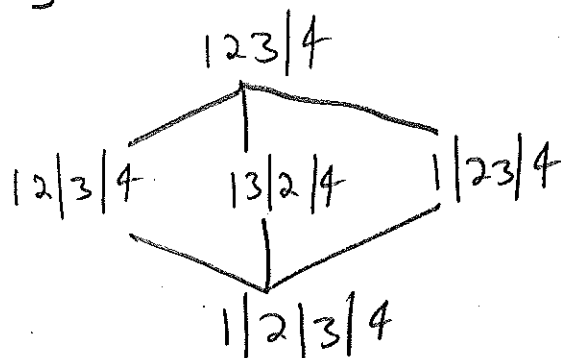
Examples (cont.): (2) The partition lattice Π_n . The elements of Π_n are all set partitions $\Pi \vdash [n]$, where $\leq$ is given by refinement of partitions, i.e.

$B_1 | B_2 | \dots | B_k \leq C_1 | C_2 | \dots | C_e$ if each B_i is contained in some C_j .

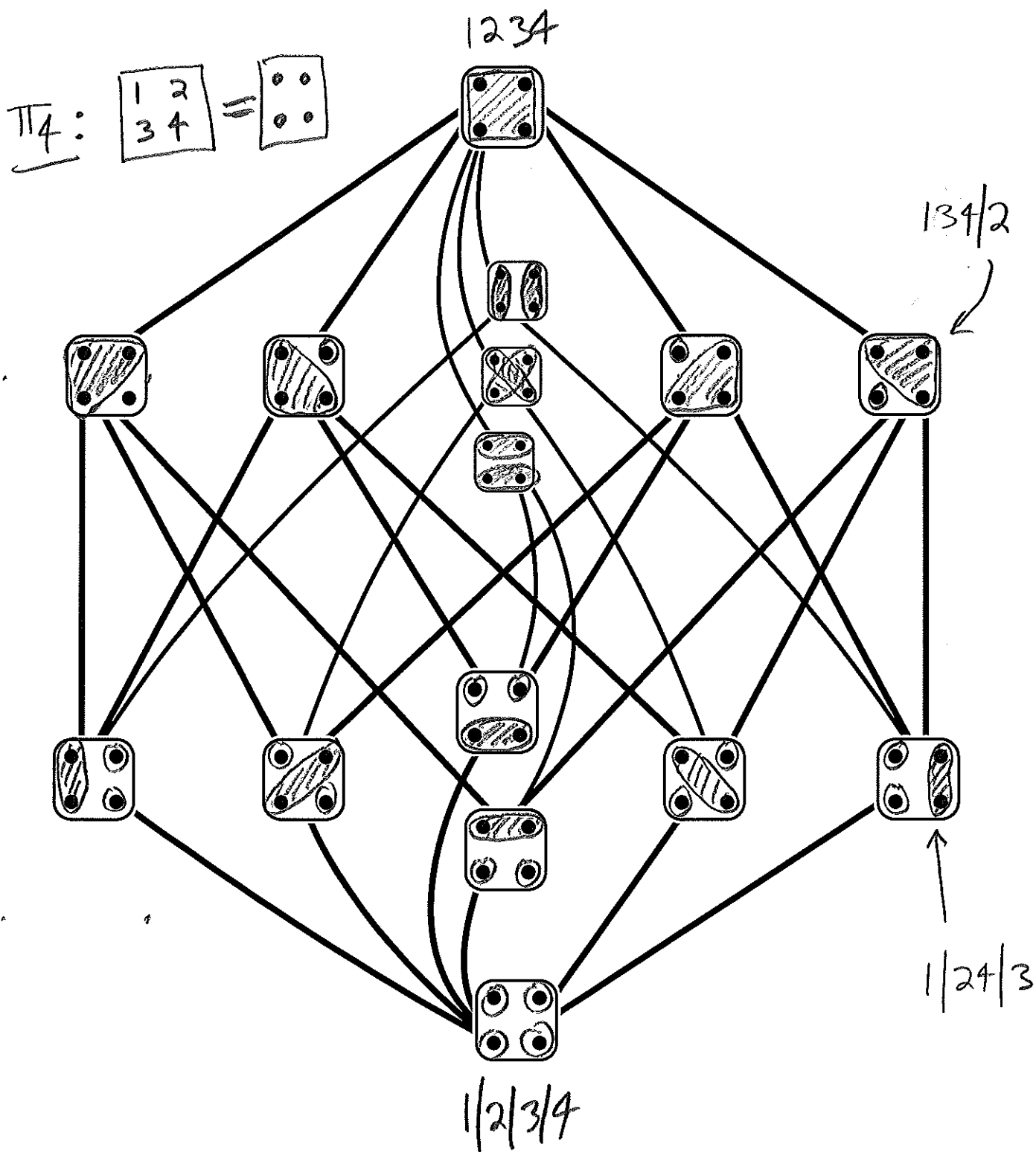
Ex: $\Pi_3 =$



observe that the interval $[1|2|3|4, 123|4]$ in Π_4 "looks like" Π_3 :



The full Π_4 poset gets complicated, see next page.

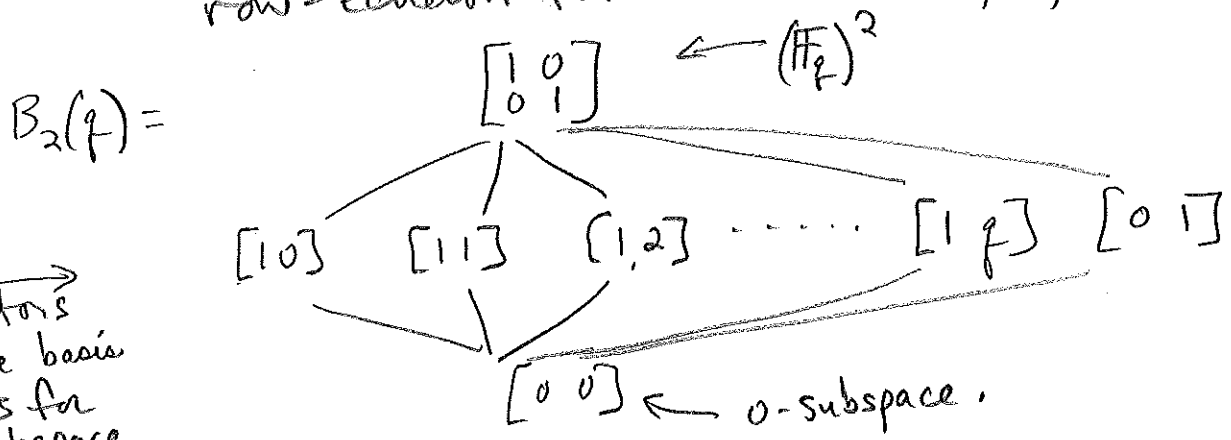


③ The lattice of subspaces $B_n(q)$ has elements all subspaces of $(\mathbb{F}_q)^n$, where $W \leq V$ if W is a subspace of V .

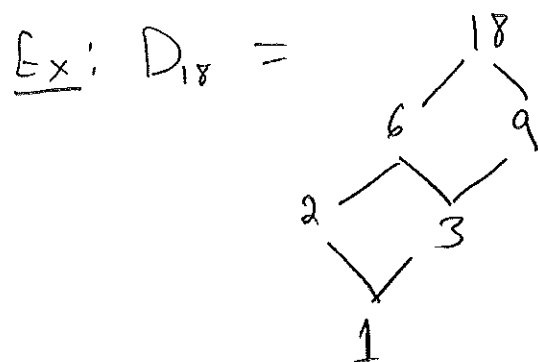
Ex: The poset $B_3(2)$ of all subspaces of $(\mathbb{F}_2)^3$

is shown on pg 244 in EC1.

Given a fixed basis for $(\mathbb{F}_q)^n$, each subspace $W \in B_n(q)$ can be uniquely represented in row-echelon form. For example,



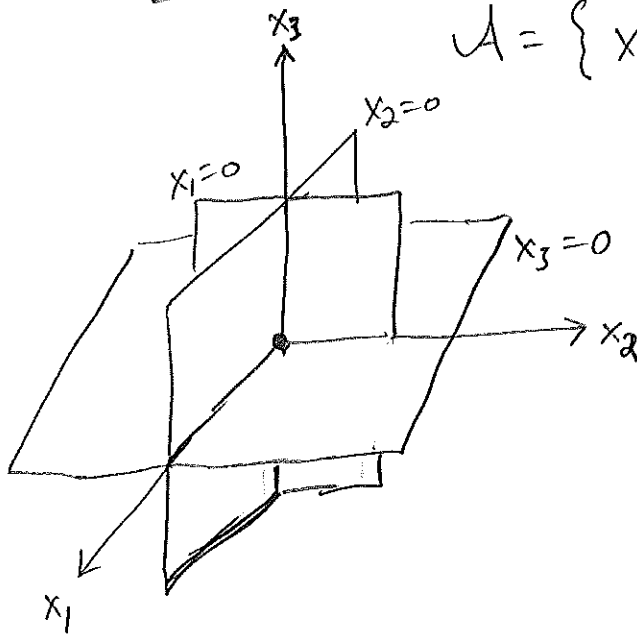
④ The divisor lattice, D_n . For $n \in \mathbb{Z}_{>0}$, the elements of D_n are all divisors of n , ordered by $x \leq y$ if $x|y$, i.e. x divides y .



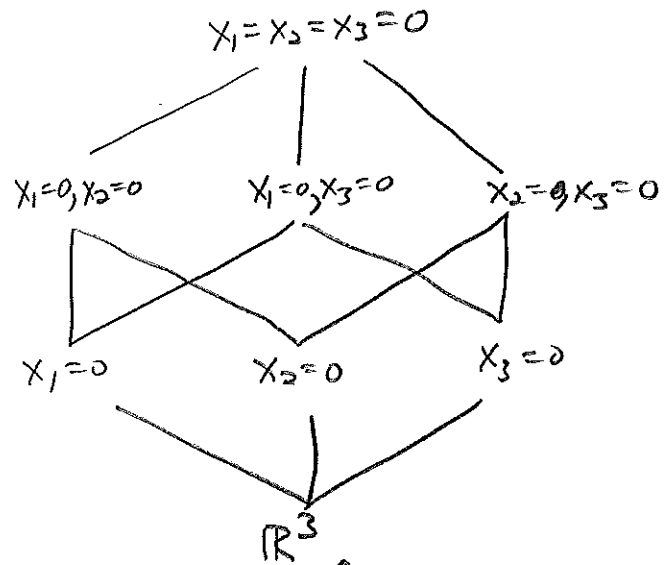
⑤ Hyperplane arrangements. A hyperplane arrangement in $\mathbb{R}^n$ is a collection $\mathcal{A}$ of hyperplanes, i.e.
 $H = \{x \in \mathbb{R}^n : x \cdot a = b\}$ for fixed $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Ex: The coordinate arrangement in $\mathbb{R}^3$:

$$\mathcal{A} = \{x_1=0, x_2=0, x_3=0\} \leftarrow \text{Abusing notation}$$



↑
Arrangement



$L(\mathcal{A})$ ↑ lattice

Defⁿ: $L(\mathcal{A})$ is the lattice whose elements are finite intersections of elements of $\mathcal{A}$, ordered by reverse containment.

More Poset Basics

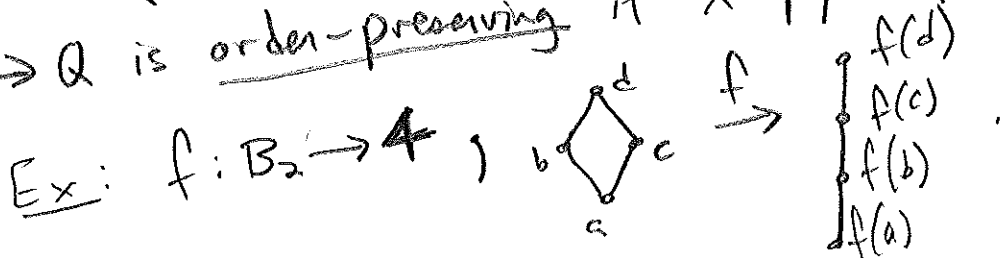
Defⁿ: We say a poset P has a $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq t$ for all $t \in P$. P has a $\hat{1}$ if there is $\hat{1} \in P$ such that $t \leq \hat{1}$ for all $t \in P$. We write $\hat{P}$ for the poset obtained by adding a $\hat{0}$ and a $\hat{1}$ to P .

Ex:

	$\mathbb{N}$	B_n	Π_n	$B_n(f)$	D_n	$\mathcal{L}(\mathcal{A})$
$\hat{0}$	1	$\emptyset$	$1 2 \dots n$	$\vec{0}$	1	$\mathbb{R}^n$
$\hat{1}$	n	$[n]$	$12\dots n$	$(\mathbb{F}_f)^n$	n	$\bigcap_{H \in \mathcal{A}} H$

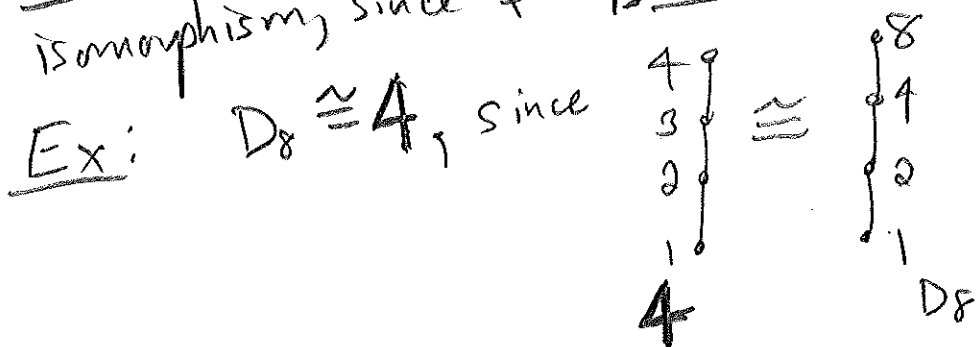
Recall: $[1|2|3|4, 123|4] \subseteq \Pi_4$ "looked like" Π_3 . Let's make this formal.

Defⁿ: If $(P, \leq_P)$ and $(Q, \leq_Q)$ are posets, then $f: P \rightarrow Q$ is order-preserving if $x \leq_P y$ implies $f(x) \leq_Q f(y)$.



Defⁿ: f is an isomorphism if f is order-preserving and bijective, with f^{-1} also order-preserving; write $P \cong Q$.

Ex: Our example $f: B_2 \rightarrow \mathbf{4}$ is not an isomorphism, since f^{-1} is not order-preserving.



Here are more examples (exercise to prove):

- For p prime, $D_{p^n} \cong n+1$.
- For $S \subseteq T \in \mathcal{B}_n$, $[S, T] \cong \mathcal{B}_{|T \setminus S|}$.
- For $c, d \in D_n$, $[c, d] \cong D_{d/c}$.
- For $W \leq V \in \mathcal{B}_n(q)$, $[W, V] \cong \mathcal{B}_{\dim V - \dim W}(q)$.

Remark: As this shows, many meaningful posets have strong self-similarity properties.

Next: Let's look at how to study "internal" properties of P , by studying subposets of P isomorphic to chains of length n .

Defⁿ: If $x, y \in P$, a chain from x to y of length l is $C := x = x_0 < x_1 < x_2 < \dots < x_l = y$.

$l+1$ elts for a chain of length l .

C is saturated if $x_i \leq x_{i+1}$ for all i . C is maximal if it is saturated and runs from a minimal element of P to a maximal one.

Ex: $\emptyset < 1 < 13 < 134 < 1234$ is a maximal chain in B_4 .

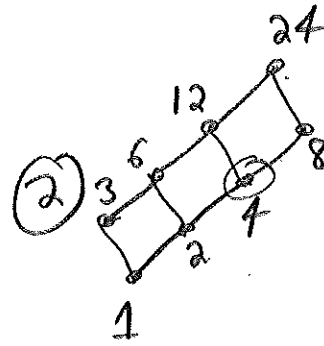
Defⁿ: A multichain is $C := x = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_l = y$,
i.e. a chain w/ repetition allowed.

Ex: $\emptyset \leq \emptyset \leq 12 \leq 124 \leq 124 \leq 124$ is a length 5 multichain in B_4 .

Defⁿ: P is graded if every maximal chain has the same length. For graded P , the rank of $x \in P$ is the length of any saturated chain from a minimal element $m \in P$ to x . We set $\text{rank}(P) := \text{rank}(m)$ for any maximal element of P .

Ex:

(1)  is not graded.



The rank of 4 in D_{24} is 2. The rank of D_{24} is 4.

Defⁿ: For a graded poset P of rank n , the rank-generating function is

$$F(P, x) := \sum_{t \in P} x^{\text{rank}(t)}.$$

Note: We denote the coefficient $[x^i] F(P, x)$ by $w_i(P)$, and call it the i^{th} Whitney number of the 2nd kind for P .

<u>Examples:</u>	n	B_n	Π_n	$B_n(q)$	$D_n, n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$
$\text{rank}(t), t \in P$	$t-1$	$ t $	$n-k$, if $t = B_1 \dots B_k$	$\dim t$	$\text{rank}(p_1^{l_1} \dots p_k^{l_k}) = l_1 + l_2 + \dots + l_k.$
$\text{rank}(P)$	$n-1$	n	$n-1$	n	$\sum_{i=1}^k a_i$
$w_k(P)$	1	$\binom{n}{k}$	$S(n, n-k)$	$\begin{bmatrix} n \\ k \end{bmatrix}_q$	Various
$F(P, q)$	$\sum_{i=0}^{n-1} q^i = \begin{bmatrix} n \end{bmatrix}_q$	$\sum_{k=0}^n \binom{n}{k} q^k = (1+q)^n = [2]_q^n$	$\sum_{k=0}^{n-1} S(n, n-k) q^k$	$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^k$ field size variable	$\prod_{i=1}^k [a_i + 1]_q$

New Posets from Old

In this section, let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. Here are a variety of ways to produce new posets from old ones.

Duals: The dual of P , denoted P^* , is the elements of P with order reversed.

Ex: $\left(\begin{array}{c} \bullet \\ | \\ \bullet \\ / \backslash \\ \bullet \end{array} \right)^* = \begin{array}{c} \bullet \\ / \backslash \\ \bullet \end{array}$

A poset is self-dual if $P \cong P^*$.

Ex:

Poset	n	B_n	Π_n	$B_n(p)$	D_n
Self-dual?	Yes	Yes	no	Yes	Yes
isomorphism	$i \mapsto n-i+1$	$S \mapsto [n] \setminus S$		$W \mapsto W^\perp$ for some bilinear form on $(\mathbb{F}_2)^n$	$d \mapsto \frac{n}{d}$

Disjoint Union: The direct sum of P and Q is $(P+Q, \leq)$ with

i) Elements $P \uplus Q$

$\uparrow$ disjoint union

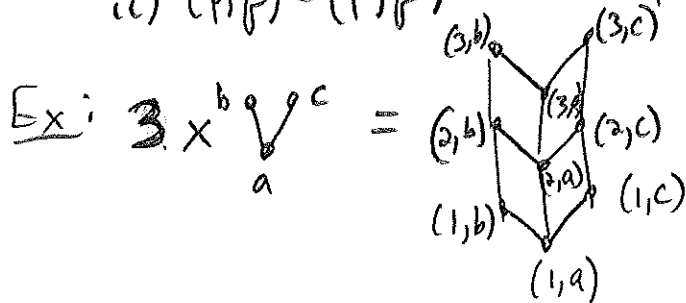
ii) Order $x \leq y$ iff $\begin{cases} x \leq_P y & \text{if } x, y \in P \\ \text{or} \\ x \leq_Q y & \text{if } x, y \in Q. \end{cases}$

Ex: $3 + B_2 = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ / \backslash \\ \bullet \end{array}$

Notation: $nP = \underbrace{P + P + \dots + P}_{n \text{ times}}$

Cartesian Product: The Cartesian product of P and Q is $(P \times Q, \leq)$ with

- i) elements $P \times Q$, as sets
- ii) $(p, f) \leq (p', f')$ iff $p \leq_P p'$ and $f \leq_Q f'$.



Notation: $P^n = \underbrace{P \times P \times \dots \times P}_{n \text{ times}}$

Remark: There are also ordinal sums, $\oplus$, and ordinal products, $\otimes$, defined on pgs 246-247 of EC1. You should review these on your own.

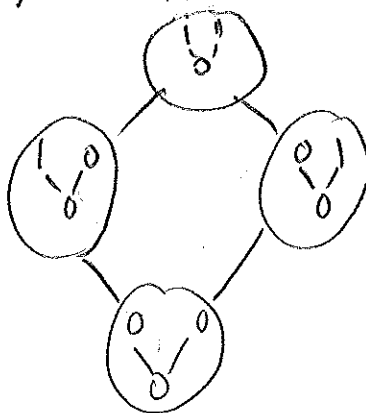
Power Posets: The power poset P^Q is $(P^Q, \leq)$ with

- i) elements all order-preserving maps $f: Q \rightarrow P$
- ii) $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in Q$.

Note: If P, Q are anti-chains, i.e. sets w/ no order, i.e. $\circ \circ \circ \dots \circ$, then $|P^Q| = |P|^{|Q|}$.

Ex: $P = \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}$, $Q = \begin{smallmatrix} y \\ x \\ z \end{smallmatrix}$.

$P^Q =$



Question: How do $+$, $\times$, and Powers interact?

Proposition: ① $+$ and $\times$ are associative and commutative.

② $P \times (Q + R) \cong P \times Q + P \times R$ (distributive).

③ $P^{Q+R} \cong P^Q \times P^R$

④ $(P^Q)^R \cong P^{Q \times R}$

Pf: Exercise. 

Note: This proposition shows that these operations behave "as expected," from an arithmetic perspective.

Proposition: Let $F(P, x)$ be the rank-generating function, and let P and Q be graded posets.

① $F(P + Q, x) = F(P, x) + F(Q, x)$

② $F(P \times Q, x) = F(P, x) \times F(Q, x)$.

Pf: Exercise. 

Ex: • $F\left(\begin{smallmatrix} \bullet & & \bullet \\ | & & | \\ \bullet & & \bullet \end{smallmatrix}, x\right) = F\left(\begin{smallmatrix} \bullet \\ | \\ \bullet \end{smallmatrix}, x\right) + F\left(\begin{smallmatrix} & \bullet \\ | & \\ & \bullet \end{smallmatrix}, x\right).$

$$2 + 2x + x^2 = (1 + x + x^2) + (1 + x).$$

• $F\left(\begin{smallmatrix} & \bullet & & \bullet \\ & | & & | \\ \bullet & & \bullet & \\ & | & & | \\ & \bullet & & \bullet \end{smallmatrix}, x\right) = F\left(\begin{smallmatrix} \bullet \\ | \\ \bullet \end{smallmatrix}, x\right) \cdot F\left(\begin{smallmatrix} & \bullet \\ | & \\ & \bullet \end{smallmatrix}, x\right)$

$$1 + 2x + 2x^2 + x^3 = (1 + x + x^2) \cdot (1 + x).$$

Remark: $F(P^Q, x)$ is hard to compute in general.

Lattices

Motivation: In many posets, given any two elements we can take their "intersection" or "sum/union".

Ex: • In B_n , $T \wedge S$ and $T \vee S$.

• In $B_n(f)$, $W \wedge V$ and $W \vee V$.

• In D_n , $\gcd(x, y)$ and $\text{lcm}(x, y)$.

In general, these can be viewed in posets as least upper bounds and greatest lower bounds, hence the following.

Defⁿ: A poset L is a lattice if the following two conditions hold:

i) For any $x, y \in L$, x and y have a greatest lower bound, or meet, denoted $x \wedge y$.

ii) For any $x, y \in L$, x and y have a least upper bound, or join, denoted $x \vee y$.

Remarks: A meet of x and y must satisfy

$$\textcircled{1} \quad x \wedge y \leq x \quad \text{and} \quad x \wedge y \leq y$$

$$\textcircled{2} \quad z \leq x \text{ and } z \leq y \text{ imply } z \leq x \wedge y.$$

Similarly for joins,

$$\textcircled{1} \quad x \vee y \geq x \text{ and } x \vee y \geq y,$$

$$\textcircled{2} \quad z \geq x \text{ and } z \geq y \text{ imply } z \geq x \vee y.$$

Examples: • $B_n, B_n(f)$, and D_n given above.

• In T_n , meets are "common refinement", joins are set partitions with "least required common merging."

$$\pi = 125|38|4|67|9, \quad \mu = 124|3|679|5|8 \Rightarrow$$

$$\pi \wedge \mu = 12|3|4|5|67|8|9, \quad \pi \vee \mu = 1245|38|679.$$

Remarks: • All lattices have a $\hat{0}$ and a $\hat{1}$; For example, if x and y were both maximal in L , we would require $x \vee y$ to exist, obtaining a contradiction.

- For lattices L and M , the following are lattices:
 L^* , $L \times M$, $\widehat{L+M}$.
- See EC1 §3.3 for defⁿ of meet- and join-semilattices.

We will often be interested in graded lattices, due to connections with enumeration. So, we are often interested in understanding the interaction of $\wedge$, $\vee$, and ρ for a graded lattice with rank function $\rho: L \rightarrow \mathbb{Z}_{\geq 0}$.

Ex: • B_n : $\rho(T) = |T|$, and
 $\rho(T \cap S) + \rho(T \cup S) \leq \rho(T) + \rho(S)$

• $B_n(q)$: $\rho(V) = \dim(V)$, and

$\rho(V \cap W) + \rho(V + W) \leq \rho(V) + \rho(W)$

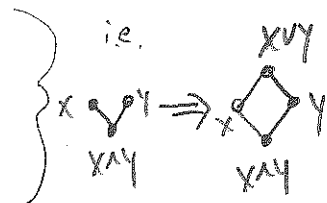
Actually equality.

Defⁿ/Propⁿ: If L is a finite lattice, we say L is a finite upper-semimodular lattice if one of the following equivalent conditions hold:

(i) L is graded and ρ satisfies

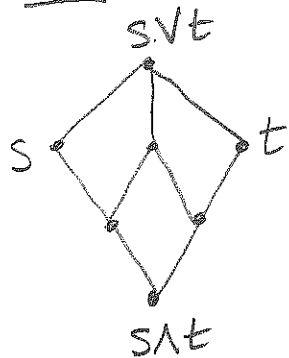
$\rho(x \wedge y) + \rho(x \vee y) \leq \rho(x) + \rho(y)$ for all $x, y \in L$.

(ii) If $x \geq x \wedge y$ and $y \geq x \wedge y$, then
 $x \vee y \geq x$ and $x \vee y \geq y$.



Note: Lower semimodularity is defined similarly, w/ inequalities reversed.

Ex: The following lattice is semimodular only in the upper sense:



We see that $svt \geq s$ and $svt \geq t$, but $s \not\geq sat$ so this is not lower-semimodular. However, it is quick to check upper holds.

Ex: Both B_n and $B_n(p)$ are upper and lower-semimodular, leading to the following.

Defⁿ: A lattice L is modular if it is both upper and lower semimodular, i.e. if for all $x, y \in L$ we have

$$\rho(x \wedge y) + \rho(x \vee y) = \rho(x) + \rho(y).$$

Propⁿ: (Exercise 3.30b in EC1): A lattice is modular if and only if it does not contain a sublattice isomorphic to .

Proof of Defⁿ/Propⁿ equivalence

We need to show (i) iff (ii).

(i) $\Rightarrow$ (ii): Say s and t cover sat . Then $\rho(s) = \rho(t) = \rho(sat) + 1$. Further, $\rho(svt) > \rho(s) = \rho(t)$, since $s \neq t$ in this case (otherwise sat is s or t). Thus, (i) implies $\rho(svt) = \rho(s) + 1 = \rho(t) + 1$, and hence svt covers both s and t .

(ii) $\Rightarrow$ (i): We need to show that (A) L is graded and (B) ρ satisfies our inequality.

PF of (A): Go by contradiction. If L is not graded, assume $[u, v]$ is a minimal length ungraded interval in L . Thus, there are $s_1, s_2 \in [u, v]$ covering u such that all maximal chains in $[s_i, v]$ have length l_i for $i = 1, 2$ and $l_1 \neq l_2$. (over)

By (ii), there are saturated chains in $[s_i, v]$ of the form $s_i < s_1 < s_2 < \dots < t_k = v$, contradicting $l_1 \neq l_2$, since $[s_i, v]$ must be graded by the minimality assumption on $[u, v]$.

Pf of (B): Again, we go by contradiction. Suppose $s, t \in L$ satisfy $\rho(s) + \rho(t) < \rho(s \wedge t) + \rho(s \vee t)$. Consider all pairs of s, t satisfying this where $l(s \wedge t, s \vee t)$ is minimal, then select an s and t with $\rho(s) + \rho(t)$ minimal. (Recall $l(u, v)$ is the length of the interval $[u, v]$, being the length of the longest chain in the interval.) By (ii), not both s and t can cover $s \wedge t$, so assume without loss of generality there is some $s' \in L$ with $s \wedge t < s' < s$. By the minimality of our choices,

$$\rho(s') + \rho(t) \geq \rho(s' \wedge t) + \rho(s' \vee t).$$

But, $s' \wedge t = s \wedge t$, thus by combining equations we get

$$\rho(s) + \rho(s' \vee t) < \rho(s') + \rho(s \vee t).$$

Now, $s \wedge (s' \vee t) \geq s'$ and $s \vee (s' \vee t) = s \vee t$. Setting $u = s$, $T = s' \vee t$, we have constructed $u, T \in L$ such that

$$\rho(u) + \rho(T) < \rho(u \wedge T) + \rho(u \vee T), \text{ but}$$

$l(u \wedge T, u \vee T) < l(s \wedge t, s \vee t)$, contradicting minimality of s, t in our choice. ▢

Finite Distributive Lattices

Remark: Read about Geometric Lattices in EC1, which are closely related to matroid theory.

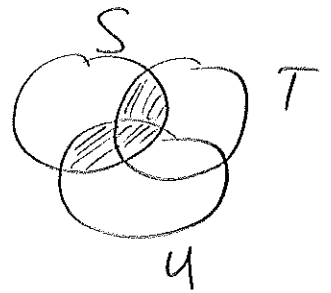
Defⁿ: For a lattice L , the distributive laws are

$$(1) \quad s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u) \quad \text{and}$$

$$(2) \quad s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$$

Remark: (1) holds $\Leftrightarrow$ (2) holds.

Example: For $U, T, S \subseteq [n]$, in B_n we have
 $S \wedge (T \vee U) = (S \wedge T) \vee (S \wedge U)$.



Non-Example: For $W, V, Y \in B_n(q)$, this is not always true. For example, $\mathbb{F}_2^2$ has four elements: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Let $W = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$, $V = \langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle$, and $Y = \langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle$. Then

$$W \wedge (V \vee Y) = W \wedge (\mathbb{F}_2^2) = W, \text{ but}$$

$$(W \wedge V) + (W \wedge Y) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

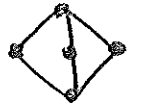
Thus, distributivity distinguishes B_n from $B_n(q)$.

Defⁿ: A lattice L is distributive if it satisfies the distributive laws.

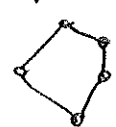
Propⁿ: L distributive $\Rightarrow L$ modular.

Pf: In the first law, assume $s \leq u$. Then $s \vee u = u$, so we get
 $s \vee (t \wedge u) = (s \vee t) \wedge u$ for all $s, t, u \in L$ with $s \leq u$.
The result follows from the following claim. (over) (1)

Claim: (see EC1 pg. 250) L is modular if and only if for all $s, t, u \in L$ with $s \leq u$, $sv(t \wedge u) = (s \vee t) \wedge u$.
Pf: omitted.

Propⁿ: L is distributive if and only if L is modular and has no sublattice isomorphic to .

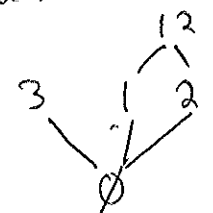
Pf: Omit. This is part of Exercise 3.31 in EC1. 

Remark: modular avoids , distributive avoids this one and .

Observation: The distributive condition seems very strict! However, there are good sources of finite distributive lattices.

Defⁿ: An order ideal of a poset P is a subset $I \subseteq P$ satisfying if $t \in I$ and $s \leq t$, then $s \in I$, aka lower order ideal.

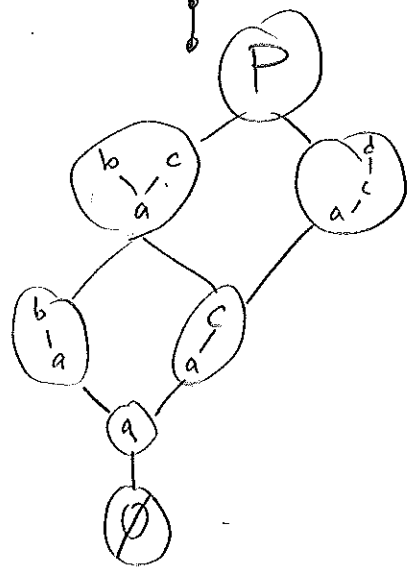
A dual order ideal of a poset P is a subset $I \subseteq P$ satisfying if $t \in I$ and $s \geq t$, then $s \in I$, aka upper order ideal.

Ex: In B_3 , consider the order ideal generated by 3 and 12:
 $\langle 3, 12 \rangle =$ , where $\langle a_1, \dots, a_n \rangle := \{z \in P; z \leq a_i \text{ for some } i\}$.

Remark: If $I = \langle a \rangle$, then I is called a principal order ideal.

Note: See page 246 in EC1 for more formal definitions.

Defⁿ: For any poset P , let $\mathcal{J}(P)$ be the poset of order ideals in P , ordered by inclusion.

Ex: $P =$  $J(P) \cong$  where

Remark: $J(P)$ is graded,
with $\rho(I) = |I|$ for
the rank function.

Propⁿ: $J(P)$ is a lattice with union and intersection
as the operations. Thus, $J(P)$ is a sublattice of
 2^P , and it is distributive.

Pf: It is straightforward to verify that if I, J are ideals,
so is $I \cap J$ and $I \cup J$. From this and the
fact that 2^P is a dist. lattice with these operations, we are
done. $\square$

Fundamental Theorem of Finite Distributive Lattices (FTFDL):

For every finite distributive lattice L , there is a unique
poset P such that $L \cong J(P)$.

Idea of Proof: Given such an L , we can construct $J(P)$
by letting P be the set of "join-irreducible" elements of L .

Defⁿ: Given a distributive lattice L , a join-irreducible element $s \in L$ is an
element satisfying:

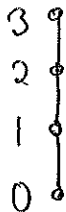
(i) $s \neq \hat{0}$.

(ii) $s \neq t \vee u$ for any $t, u < s$ and $t \neq u$.

Let $cl(L)$ denote the set of join-irreducible elements of L .

(3)

Ex: C_3

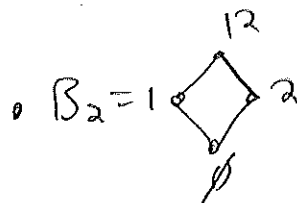


$$\text{cl}(C_3) = \{1, 2, 3\}.$$

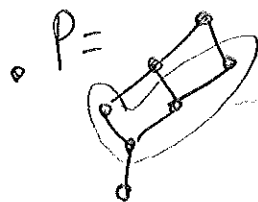
Recall: $s \neq \hat{0}$.

Note: An elt in L is join-irred. iff it covers exactly one element in L , for L finite.

Pf: Exercise. $\square$



$$\text{cl}(B_2) = \{1, 2\}.$$



$$\text{cl}(P) =$$

Note: previously, we saw $J(V) =$

Remark: In all three cases above, $\text{cl}(J(P)) = P$. This is true in general!

Propⁿ: $\text{cl}(J(P)) \cong P$ for all finite posets P .

Proof: An order ideal $I \subseteq P$ is join-irreducible in $J(P)$ iff I is a principal ideal. Two principal ideals $\langle s \rangle$ and $\langle t \rangle$ satisfy $\langle s \rangle \subseteq \langle t \rangle$ iff $s \leq t$. This gives our desired isomorphism. $\square$

Cor: $J(P) \cong J(Q)$ iff $P \cong Q$.

Note: A complete proof of the FTFDL is on pg. 253 in EC1. With the background given here, the proof is tedious but straightforward to show that $L \cong J(\text{cl}(L))$.

Incidence Algebras

Let's begin by discussing algebras. Given an abelian group G , we can do two familiar things to it:

- make it a vector space by incorporating scalar multiplication over a field K ; or
- make it a ring by incorporating a multiplication operation.

In other words, the axioms defining both vector spaces and rings include an abelian group structure. When both of these extensions occur, we get an algebra; informally, a ring that is a vector space (or vice versa, as you like).

Ex: • $\mathbb{C}[x]$ is an algebra over $\mathbb{C}$.

• $U(n; \mathbb{C}) :=$ all upper-triangular matrices over $\mathbb{C}$.

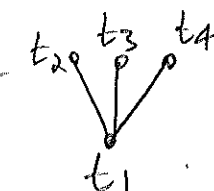
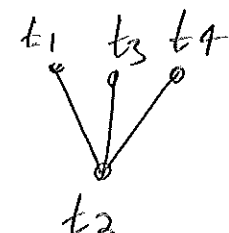
add: $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix}$

mult: $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 2 \end{bmatrix}$

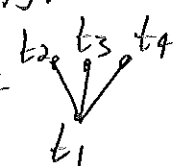
scale: $3 \cdot \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix}$

} etc.

Assume that we have a finite poset P , where $p = |P|$, and label the elements $t_1, t_2, \dots, t_p$ such that $t_i < t_j$ in $P \Rightarrow i < j$. This is called a linear extension of P .

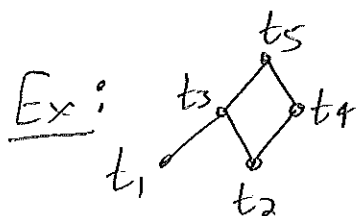
Ex: $p =$  works, but  does not.

Consider the set of all upper-triangular $p \times p$ matrices over $\mathbb{F}$ where we label the rows and columns by $t_1, t_2, \dots, t_p$ and require the $(i, j)^{\text{th}}$ entry to be zero if $t_i \not\leq t_j$.

Ex: $P =$ 

$$P = \begin{matrix} & t_1 & t_2 & t_3 & t_4 \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{matrix} & \begin{bmatrix} * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \end{matrix}$$

where $*$ denotes the entry is allowed to be non-zero.



$$P = \begin{matrix} & t_1 & t_2 & t_3 & t_4 & t_5 \\ \begin{matrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{matrix} & \begin{bmatrix} * & 0 & * & 0 & * \\ 0 & * & * & * & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \end{matrix}$$

Propⁿ/Defⁿ: For a fixed finite poset P , these matrices form a sub-algebra of $U(|P|; \mathbb{F})$ called the Incidence Algebra of P and denoted $\mathcal{I}(P; \mathbb{F})$.

Pf: That sums and scalings of these matrices are elements of $\mathcal{I}(P; \mathbb{F})$ is straight forward. The "hard" part is multiplication. Let $A, B \in \mathcal{I}(P; \mathbb{F})$, and consider $A \cdot B$. We need to show that if $t_i \not\leq t_j$, the $(i, j)^{\text{th}}$ entry of $A \cdot B$ is zero. Note that

$$(A \cdot B)_{ij} = \sum_{k=1}^p A_{ik} B_{kj}.$$

Since $t_i \not\leq t_j$, there is not any k where $t_i \leq t_k \leq t_j$. Thus, for each k , either A_{ik} or B_{kj} is zero, hence $(A \cdot B)_{ij} = 0$ for $t_i \not\leq t_j$. (2)

Remark: For a matrix $M \in I(P; \Phi)$, M_{ij} can be non-zero iff $t_i \leq t_j$. Thus, we may consider M as a function from the set of intervals $[t_i, t_j] \subseteq P$ to Φ , leading a more abstract characterization of $I(P; \Phi)$ (that extends to locally finite infinite posets).

Propⁿ: Let $I(P)$ denote the set of closed intervals in P . Then

$I(P; \Phi) = \{f: I(P) \rightarrow \Phi\}$ with the following operations:

$$(i) (f+g)([x, y]) = f([x, y]) + g([x, y]),$$

$$(ii) (cf)([x, y]) = c \cdot f([x, y]) \text{ for } c \in \Phi,$$

$$\text{and (iii) } (f * g)([x, y]) = \sum_{x \leq z \leq y} f([x, z]) \cdot g([z, y]).$$

Note: (iii) is a convolution product, and after changing notation it is exactly the same as obtaining the $(ij)^{\text{th}}$ entry of AB by the formula $\sum_{k=1}^P A_{ik} B_{kj}$.

Note: The multiplicative identity in $I(P; \Phi)$ is the identity matrix, i.e. $f([x, y]) = \delta_{xy}$, which we denote by δ .

Note: Write $f(x, y)$ for $f([x, y])$.

Propⁿ: $f \in I(P; \Phi)$ has a multiplicative inverse $\iff f(x, x) \neq 0$ for all $x \in P$.
Sketch of Pf: f corresponds to an upper-triangular matrix, and the $f(x, x)$'s are diagonal elements. These are all non-zero iff the matrix is invertible. One must also check that f^{-1} is actually in $I(P; \Phi)$, i.e. has the correct non-zero positions. $\blacksquare$

One of the most important elements in $I(P; \Phi)$ is the following.

Defⁿ: Given $I(P; \Phi)$, the zeta function is $S(x, y) := 1$ for $x \leq y$.

Remark: S is interesting because it counts things!

Propⁿ: $S^k(x, y) = \#$ multichains of length k from x to y .

Note: S^k means the k^{th} power of the matrix S , so $S^k(x, y)$ is the $(x, y)^{\text{th}}$ entry of the matrix.

Pf: $S^k(x, y) = \sum_{x=x_0 \leq x_1 \leq \dots \leq x_k=y} 1$

summed over all admissible x_i 's,
i.e. the multichains. ▀

Propⁿ: $(S - \delta)^k(x, y) = \#$ chains of length k from x to y .

Pf: $(S - \delta)(x, y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{else} \end{cases}$, So,

$(S - \delta)^k(x, y) = \sum_{x=x_0 < x_1 < \dots < x_k=y} 1$

over all admissible x_i 's,
i.e. the chains. ▀

Möbius functions

We've seen that ζ and $\zeta - \delta$ can be used to count multichains and chains in a poset P . Let's look at two more examples that illustrate this connection, but involve the inverse matrix.

$$\text{Consider } (2\delta - \zeta)(s, t) = \begin{cases} 1 & \text{if } s = t \\ -1 & \text{if } s < t \end{cases}$$

Since diagonal entries of $2\delta - \zeta$ are nonzero, $(2\delta - \zeta)^{-1}$ exists.

Defⁿ: For a finite poset P with $x, y \in P$, let $C_k(x, y)$ denote the number of chains in P from x to y of length k .

Propⁿ: $(2\delta - \zeta)^{-1}(s, t) = C_0(s, t) + C_1(s, t) + C_2(s, t) + \dots$
= total # of chains from s to t in P .

finite sum since P is finite

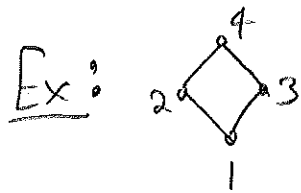
Pf: Observe that if ℓ is the length of the longest chain in $[s, t]$, then $(\zeta - \delta)^{\ell+1}(s, t) = 0$. Thus,

$$\zeta(s, t) = [\delta - (\zeta - \delta)^{\ell+1}](s, t) = [\delta - (\zeta - \delta)] [\delta + (\zeta - \delta) + (\zeta - \delta)^2 + \dots + (\zeta - \delta)^\ell](s, t) = (2\delta - \zeta) [\delta + (\zeta - \delta) + (\zeta - \delta)^2 + \dots + (\zeta - \delta)^\ell](s, t).$$

Thus, $(2\delta - \zeta)^{-1} = \delta + (\zeta - \delta) + (\zeta - \delta)^2 + \dots + (\zeta - \delta)^\ell$, and we've seen already that the summands of the right-hand side count chains.

Q: What, if anything, does ζ^{-1} count? Since $\zeta(x, x) = 1$ for all $x \in P$, ζ^{-1} exists.

Defⁿ: The Möbius function of P is $\mu := \mathcal{S}^{-1}$.



$$\mathcal{S} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$\mu = \mathcal{S}^{-1} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

So, $\mu(1,4) = 1$, $\mu(1,3) = -1$.

Claim: $\mu \cdot \mathcal{S} = \mathcal{I}$ is equivalent to recursively defining

• $\mu(x,x) := 1$ for all $x \in P$

and • $\sum_{x \leq z \leq y} \mu(x,z) = 0$.

Some prefer to write:

$$\mu(x,y) = - \sum_{x \leq z < y} \mu(x,z)$$

Pf: These two lines are precisely the row-column form of the matrix multiplication $\mu \cdot \mathcal{S} = \mathcal{I}$.

Thm (Hall's Theorem): For P a finite poset, we have that

$$\mu(x,y) = C_0(x,y) - C_1(x,y) + C_2(x,y) - C_3(x,y) + \dots$$

$$= \sum_{j \geq 0} (-1)^j C_j(x,y).$$

$\leftarrow P \text{ finite} \Rightarrow \text{finite sum}$

Pf: We use a similar technique as before.

$$\mu = \mathcal{S}^{-1} = (\mathcal{S} + (\mathcal{S} - \mathcal{S}))^{-1} \underset{\substack{\uparrow \\ \text{geometric series}}}{=} \mathcal{S} - (\mathcal{S} - \mathcal{S})^{-1} + (\mathcal{S} - \mathcal{S})^{-2} - (\mathcal{S} - \mathcal{S})^{-3} + \dots$$

We've seen that the right-hand side counts chains.

Note: If $x \neq y$, $C_0(x,y) = 0$ and $C_1(x,y) = 1$.

- Remarks:
- This shows that μ is an "Euler Characteristic" for posets.
 - For those familiar w/ topology, pages 269-274 in EC1 discuss connections with simplicial complexes, homology, and Cohen-Macaulay properties.

- Summary:
- $\mathcal{S}^k$ counts multichains of length k .
 - $(\mathcal{S}-\mathcal{S})^k$ counts chains of length k .
 - $(2\mathcal{S}-\mathcal{S})^{-1}$ counts all chains.
 - $\mu = \mathcal{S}^{-1}$ counts alternating sums of all chains.

Note: $\mu(x, y) = \sum_{\substack{C \text{ chain} \\ \text{from } x \text{ to } y}} (-1)^{\ell(C)}$. This looks like it reflects some type of sieving process, similar to PIE, and sign-reversing involutions, hence the following...

Theorem (Möbius Inversion): Let P be a finite poset, and let $f, g: P \rightarrow \mathbb{C}$. The following statements are equivalent:

$$(i) \quad g(x) = \sum_{y \leq x} f(y)$$

$$(ii) \quad f(x) = \sum_{y \leq x} g(y) \mu(y, x)$$

Remark: This is similar to PIE!

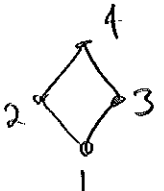
Remark: We can give a second proof of PIE, using a dual form of Möbius inversion, which we will see later in the course.

Proof : (EC1 has two proofs of this, both very short).

(i) tells us that $g(x) = \sum_{y \leq x} f(y) \cdot \mathcal{S}(y, x)$, since $\mathcal{S}(y, x) = 1$. Thus, record

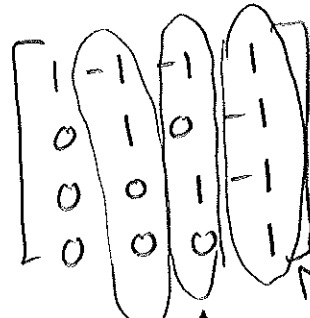
g as a (row) vector and f as a (row) vector, and we get $g = f \cdot \mathcal{S}$. Thus, $g \cdot M = f \cdot \mathcal{S} \cdot M = f$. But, $g \cdot M = f$, as a matrix acting on a row vector, is precisely what

$f(x) = \sum_{y \leq x} g(y) M(y, x)$ records.

Ex: $P =$  as before. $f = [f_1, f_2, f_3, f_4]$

$g = [f_1, f_1 + f_2, f_1 + f_3, f_1 + f_2 + f_3 + f_4]$

Then $g = f \cdot \mathcal{S} = [f_1, f_2, f_3, f_4] \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ↖ This is the same after multiplying.

So, $f = g \cdot M = g \cdot$ 

In g:

2nd entry subtracts f_1

3rd entry subtracts f_1

4th entry subtracts $f_1 + f_2, f_1 + f_3$, then adds f_1 back to correct double-counting.

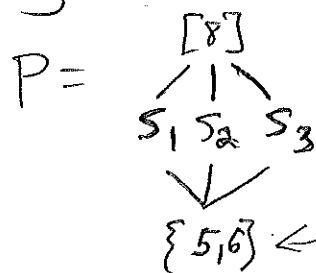
Application: Suppose we have n finite sets $S_1, \dots, S_n$.

Let P be the poset of intersections of $S_1, S_2, S_3, \dots, S_n$, ordered by inclusion, along with a maximal element $\hat{1} := S_1 \cup S_2 \cup \dots \cup S_n$.

Ex: $S_1 = \{1, 2, 5, 6\}$

$S_2 = \{3, 5, 6, 7\}$

$S_3 = \{4, 5, 6, 8\}$



$S_i \cap S_j = S_1 \cap S_2 \cap S_3 = \{5, 6\}$.

For $T \in P$, let $f(T) := \#$ elements of T that are not in any $T' \triangleleft T$.

Note that $f(S_1 \cup \dots \cup S_n) = 0$.

Similarly, let $g(T) := |T|$.

GOAL: Compute $|S_1 \cup \dots \cup S_n| = g(\hat{1}) = \sum_{T \leq \hat{1}} f(T)$, without knowing $f(T)$.

Ex: $|S_1 \cup S_2 \cup S_3| = 8 = f(S_1) + f(S_2) + f(S_3) + f(\{5, 6\}) = 2 + 2 + 2 + 2$.

Observe: $g(T) = \sum_{T' \leq T} f(T')$. Thus, Möbius inversion says

$$0 = f(\hat{1}) = \sum_{T \in P} g(T) \mu(T, \hat{1}) = g(\hat{1}) + \sum_{T < \hat{1}} |T| \mu(T, \hat{1}).$$

$$\Rightarrow |S_1 \cup S_2 \cup \dots \cup S_n| = g(\hat{1}) = - \sum_{T < \hat{1}} |T| \mu(T, \hat{1}).$$

Ex: For our example above,

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - 2|\{5, 6\}|$$

$$= 4 + 4 + 4 - 2 \cdot 2 = \boxed{8}.$$

This is what we wanted!

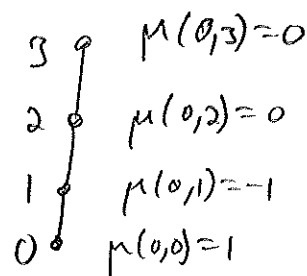
Techniques of Computation

Let C_n denote the chain of length n , $0 < 1 < 2 < \dots < n$.
Recall the recursive defⁿ of the Möbius function:

$$\mu(x, x) := 1.$$

$$\mu(x, y) := - \sum_{x \leq z < y} \mu(x, z).$$

Ex: For $i, j \in C_n$, $\mu(i, j) = \begin{cases} 1 & \text{if } i=j \\ -1 & \text{if } i+1=j \\ 0 & \text{else} \end{cases}$



Propⁿ: On C_n , Möbius inversion states

$$g(j) = \sum_{i=0}^j f(i) \text{ for all } j \geq 0 \iff \begin{aligned} f(0) &= g(0) \text{ and} \\ f(j) &= g(j) - g(j-1) \\ &= \sum_{i \leq j} g(i) \mu(i, j). \end{aligned}$$

Defⁿ: The finite difference operator for a sequence $g(0), g(1), g(2), \dots$ is $\Delta(g(j)) := g(j+1) - g(j)$.

Remarks: • This yields a discrete version of FT of Calc:

$$\int = \sum \quad \text{and} \quad \frac{d}{dx} = \Delta \quad \text{are inverses. In the finite case, this is obvious.}$$

- As with $\frac{d}{dx}$, a sequence g obtained from polynomials of degree k satisfy $\Delta^{k+1}(g) \equiv 0$.
- Chains are fundamental in many ways that we have seen. As you showed on the HW, $B_n \cong (C_1)^n$ and the divisor lattice is a product of chains of different lengths.

Thm: (Product Theorem) If P and Q are finite and $(x,y) \leq (x',y') \in P \times Q$, then

$$\mu_{P \times Q}((x,y), (x',y')) = \mu_P(x,x') \cdot \mu_Q(y,y').$$

Pf: We will show that these two functions satisfy the same recurrence and initial conditions.

Initial Conditions: $\mu_{P \times Q}((x,y), (x,y)) = 1 = \mu_P(x,x) \cdot \mu_Q(y,y).$

Recurrence: We can rewrite the recurrence for $\mu_{P \times Q}$ as

$$\sum_{(x,x') \leq (z,z') \leq (y,y')} \mu_{P \times Q}((x,x'), (z,z')) = 0 \text{ for all } (x,x') \neq (y,y'). \text{ Similarly,}$$

$$\sum_{(x,x') \leq (z,z') \leq (y,y')} \mu_P(x,x') \cdot \mu_Q(y,y') = \left(\sum_{x \leq z \leq y} \mu_P(x,z) \right) \left(\sum_{x' \leq z' \leq y'} \mu_Q(x',z') \right) = \delta_{xy} \cdot \delta_{x'y'}$$

where $\delta_{xy} = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{else} \end{cases}$. Since $(x,y) \neq (x',y')$, at least one of these is zero. $\blacksquare$

Remark: This also follows from the theory of tensor products for vector spaces; namely,

$$I(P \times Q; \mathbb{F}) \cong I(P; \mathbb{F}) \otimes I(Q; \mathbb{F}), \text{ and}$$

$$\mathcal{S}_{P \times Q} = \mathcal{S}_P \otimes \mathcal{S}_Q.$$

Corollary: For $T \leq S \in B_n$, $\mu(T,S) = (-1)^{|S| - |T|}$.

Pf: $B_n \cong (C_1)^n$, from which it follows. $\blacksquare$

Ex: $\mu(12, 1235)$ in B_5 is the same as $\mu(\underbrace{(1,1,0,0,0)}_{\{1,2,3\}}, \underbrace{(1,1,1,0,1)}_{\{1,2,3,5\}})$ in $(C_1)^5$,

which equals $\mu(1,1) \cdot \mu(1,1) \cdot \mu(0,1) \cdot \mu(0,0) \cdot \mu(0,1) =$
 $1 \cdot 1 \cdot -1 \cdot 1 \cdot -1 = (-1)^2.$

Now that we know μ_{B_n} , let's revisit P.I.E.! First:

Möbius Inversion (Dual Form): Let P be finite, $f, g: P \rightarrow \mathbb{C}$. Then

$$g(x) = \sum_{y \geq x} f(y) \quad \text{if and only if} \quad f(x) = \sum_{y \geq x} g(y) \mu(x, y).$$

Pf: our previous result was a restatement of $g = f \cdot \zeta$ iff $g\mu = f$.

This version is a restatement of $g = \zeta f$ iff $\mu g = f$. ■

P.I.E. Let $P = B_n$, and let g be $f_ =$ and f be $f_ \geq$. Then P.I.E. states

$$f_ \geq (T) = \sum_{Y \geq T} f_ = (Y)$$

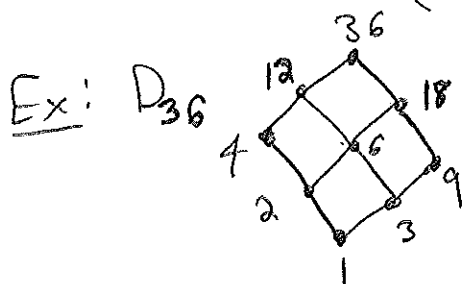
$$\text{and} \quad \underbrace{f_ = (T)}_{g(T)} = \sum_{Y \geq T} \underbrace{(-1)^{|Y| - |T|}}_{\mu_{B_n}(T, Y)} \underbrace{f_ \geq (Y)}_{f(Y)}.$$

This is simply
Möbius inversion.

Another application of the product theorem:

Let $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ where $p_1, \dots, p_k$ are primes. We saw that $D_n \cong C_{n_1} \times C_{n_2} \times \dots \times C_{n_k}$. By a similar argument to the B_n case, we can see that

$$\mu_{D_n}(r, s) = \begin{cases} (-1)^t & \text{if } s/r \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{else.} \end{cases}$$



$$\mu_{D_{36}}(2, 36) = 0, \text{ and } 36/2 = \underbrace{3 \cdot 3 \cdot 2}_{\text{not distinct}}$$

$$\mu_{D_{36}}(3, 18) = 1, \text{ and } 18/3 = \underbrace{2 \cdot 3}_{\text{two distinct primes.}}$$

Remark: With this definition of μ , which holds independent of D_n , we recover the classical number-theoretic Möbius formula: Fix an integer $N \in \mathbb{Z}_{\geq 1}$.

$$g(n) = \sum_{d|n} f(d) \quad \text{for all } n|N$$

$$\text{iff } f(n) = \sum_{d|n} g(d) \mu(d, n) \quad \text{for all } n|N.$$

This function was originally studied in the context of the multiplicative inverse of the Riemann Zeta function.

We next turn our attention to special computational techniques for lattices, namely the following two theorems.

Thm (Weisner's Thm): For L a finite lattice and $a \in L$ with $a \neq \hat{1}$,

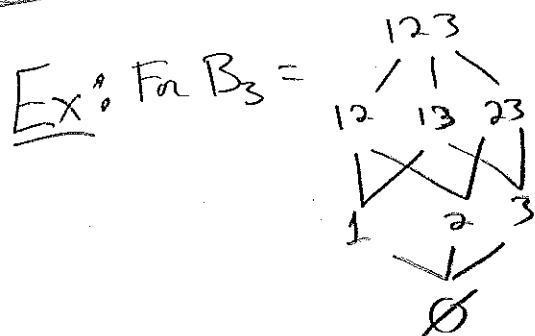
$$\sum_{t: t \wedge a = \hat{0}} \mu(t, \hat{1}) = 0$$

and $\sum_{t: t \vee a = \hat{1}} \mu(\hat{0}, t) = 0.$

Thm (Special Case of Crosscut Theorem, Cor. 3.9.5 in EC1):

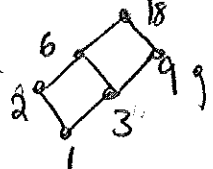
If L is a finite lattice for which $\hat{0}$ is not a meet of coatoms, then $\mu(\hat{0}, \hat{1}) = 0$. Similarly, if $\hat{1}$ is not a join of atoms, then $\mu(\hat{0}, \hat{1}) = 0$. (Here we mean $\bigvee_{a \text{ atom}} a$ when we say join of atoms.)

Recall from reading: $t \in L$ is an atom if $\hat{0} < t$, a coatom if $t < \hat{1}$.



with $a = 1$, Weisner implies

$$\begin{aligned} 0 &= \mu(\emptyset, \hat{1}) + \mu(2, \hat{1}) + \mu(3, \hat{1}) + \mu(23, \hat{1}) \\ &= (-1)^3 + (-1)^2 + (-1)^2 + (-1)^1. \end{aligned}$$

Ex: If $D_{18} = C_1 \times C_2$  we have that 2 and 3 are atoms, but $2 \vee 3 = 6 \neq 18 = \hat{1}$.

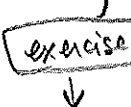
Thus, by special case of crosscut, we have $\mu(\hat{0}, \hat{1}) = 0$.

Remark:  When computing with Weisner's theorem, we want the sum to be over a small number of elements; usually we choose "a" to be an atom or coatom.

Ex: We show that for B_n , $\mu(\hat{0}, \hat{1}) = (-1)^n$. Go by induction on n . For $n=0$, $B_0 \cong \emptyset$, so $\mu(\hat{0}, \hat{1}) = (-1)^0$. For $n > 0$, Let $a = \{1, 2, \dots, n-1\}$. Suppose that $a \wedge t = \emptyset = \hat{0}$; then $t = \emptyset$ or $t = n$. Weisner then implies that $\mu(\hat{0}, \hat{1}) + \mu(\{n\}, \hat{1}) = 0$. But, $\mu(\{n\}, \hat{1}) = \mu(\emptyset, [n-1])$, which by induction is $(-1)^{n-1}$, since $[n, [n]] \cong \hat{1}_{B_{n-1}}$. Thus, $\mu(\hat{0}, \hat{1}) = (-1) \cdot (-1)^{n-1} = (-1)^n$ for B_n . 

Remark: Proving Weisner's theorem requires some work. We will first show how Crosscut follows from Weisner.

Pf of Crosscut Special Case: Assume Weisner holds. Let A denote the set of atoms in L , and assume $\forall t =: a \neq \hat{1}$. We want to show that $\mu(\hat{0}, \hat{1}) = 0$. Let $\forall t = a$ be the a in Weisner's theorem. We have

$\sum_{x: x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0$ implies $\mu(\hat{0}, \hat{1}) = - \sum_{\substack{x \neq \hat{0} \\ x \wedge a = \hat{0}}} \mu(x, \hat{1})$. On the right-hand-side,  $x \neq \hat{0}$ implies $x \geq a^*$ for some atom $a^* \in t$. Hence, $x \wedge a = x \wedge \bigvee_{t \in A} t \geq \bigvee_{t \in A} (x \wedge t) \geq x \wedge a^* = a^* > \hat{0}$. Thus, there is no element meeting with a to $\hat{0}$, hence the right-hand-side is 0, as desired.  (5)

Ex: Let $L = J(P)$ be a finite distributive lattice. What is $\mu_L(I, I')$? Remember that these objects behave similarly to B_n , and are highly structured, so we hope that their Möbius functions are nice.

Consider the interval $[I, I']$, regarded as a sublattice of L . The join of all atoms in $[I, I']$ is the order ideal $I \vee M$, where M is the set of minimal elements of the poset $I' \setminus I$.



(Recall that every ideal covering I is of the form $I \vee \{t\}$ for some element t .) As the elements of M form an antichain in P , we have that the interval $[I, I \vee M]$ is a Boolean algebra. Further, if $[I, I']$ is a Boolean algebra, then $I' \setminus I$ must be an antichain, and $I' = I \vee M$. Hence:

$[I, I']$ a Boolean algebra $\iff I'$ a join of atoms in $[I, I']$.

By the special case of crosscut, it follows that

$$\mu_L(I, I') = \begin{cases} (-1)^{|I' \setminus I|} & \text{if } [I, I'] \text{ is a Boolean algebra} \\ 0 & \text{else.} \end{cases}$$

In this way, we have a further connection between B_n and general finite distributive lattices.

Möbius Algebras for Lattices

To prove Weisner's theorem, and hence crosscut, we will develop a theory of Möbius algebras.

Defⁿ: Let L be a finite lattice. The Möbius algebra, $A(L)$, is formed as follows:

① $A(L)$ is a vector space over $\mathbb{C}$ with basis given by $\{x_1, \dots, x_n\}$, the elements of L . (This is a formal basis.)

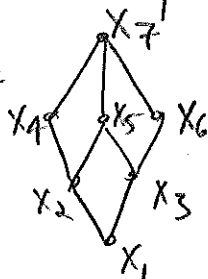
Thus, $A(L) = \left\{ \sum_{i=1}^n c_i x_i : x_i \in L, c_i \in \mathbb{C} \right\}$.

② The ring multiplication in $A(L)$ is given on basis elements by $x_i \cdot x_j = x_i \wedge x_j$, and extended by linearity and distributivity.

Since $|L| = 7$, $\dim_{\mathbb{C}}(A(L)) = 7$.

Ex:

Let $L =$



Examples of elts of $A(L)$:

- $3 \cdot x_2 + 4 \cdot x_4 - x_6$

- $x_5 - x_6$

- $2x_1 + (7+i)x_6 - \frac{\pi}{4}x_7$

To multiply:

$$(3x_2 + 4x_4 - x_6)(x_5 - x_6) = (3x_2 + 4x_4 - x_6) \cdot x_5 - (3x_2 + 4x_4 - x_6) \cdot x_6 =$$

$$3 \cdot x_2 x_5 + 4x_4 x_5 - x_6 x_5 - 3x_2 x_6 - 4x_4 x_6 + x_6 x_6 =$$

$$3x_2 \wedge x_5 + 4x_4 \wedge x_5 - x_6 \wedge x_5 - 3x_2 \wedge x_6 - 4x_4 \wedge x_6 + x_6 \wedge x_6 =$$

$$3x_2 + 4x_2 - x_3 - 3x_1 - 4x_1 + x_6 = 7x_2 - 7x_1 - x_3 + x_6$$

Remark: For all $x \in L$, $x \wedge x = x$. Thus, in $A(L)$, x is an idempotent. We want a different basis for $A(L)$ where the idempotents are orthogonal, i.e. pairwise products are zero for pairs of distinct basis elements.

Defⁿ: For $t \in L$, set $\delta_t := \sum_{s \leq t} \mu(s, t) \cdot s \in A(L)$.

Propⁿ: $D := \{\delta_t : t \in L\}$ is a vector-space basis for $A(L)$.

Pf: Since $|D| = |L| = \dim_{\mathbb{F}}(A(L))$, we only need to show that D spans $A(L)$. This follows from:

Claim: $\delta_t = \sum_{s \leq t} \mu(s, t) \cdot s$ implies $t = \sum_{s \leq t} \delta_s$.

Thus, every $t \in A(L)$ is a sum of elements in D , hence D spans $A(L)$. □

Proof of Claim: $\sum_{s \leq t} \delta_s = \sum_{s \leq t} \sum_{r \leq s} \mu(r, s) \cdot r =$

$\sum_{r \leq t} \sum_{s \leq t} \mu(r, s) \cdot r = \sum_{r \leq t} \mu(r, t) \cdot r$. Consider a fixed r :
 If $r = t$, $\mu(t, t) = 1$.
 If $r \neq t$, then $\sum_{s \leq t} \mu(r, s) = 0$.
 (Note: In the original image, there are arrows pointing to the summation indices r and s in the second sum, labeled "fixed".)

Ex: In B_2 , we get $\delta_{\emptyset} = \mu(\emptyset, \emptyset) \cdot \vec{\emptyset} = \vec{\emptyset}$, where $\vec{\cdot}$ indicates an element of $A(L)$. Similarly,

$$\delta_1 = \mu(\emptyset, 1) \vec{\emptyset} + \mu(1, 1) \vec{1} = -\vec{\emptyset} + \vec{1}.$$

$$\delta_2 = -\vec{\emptyset} + 2 \quad \text{by same reasoning, and similarly}$$

$$\delta_{12} = \vec{\emptyset} - \vec{1} - \vec{2} + \vec{12}.$$

At this point, we know:

- $A(L)$ has two bases, $\{t\}_{t \in L}$ and $\{\delta_t\}_{t \in L}$.
- $t = \sum_{s \leq t} \delta_s$ and $\delta_t = \sum_{s \leq t} \mu(s, t) \cdot s$.

- $t \cdot s = t \wedge s$.

We want to show next that $\delta_s \cdot \delta_t = \begin{cases} \delta_s & \text{if } s=t \\ 0 & \text{else} \end{cases}$, meaning these elements are orthogonal in an algebra sense.

To show this, we will arbitrarily define a new algebra $B(L)$ having a basis $\{e_t\}_{t \in L}$ with the properties we want, namely $e_t \cdot e_s := \begin{cases} e_s & \text{if } s=t \\ 0 & \text{else} \end{cases}$, and extend by linearity.

Defⁿ: Let $\Theta: A(L) \rightarrow B(L)$ be the linear transformation defined by $\Theta(\delta_t) = e_t$, extending by linearity.

Propⁿ: Θ is actually an isomorphism of algebras, i.e. $\Theta(rs) = \Theta(r) \cdot \Theta(s)$ and $B(L) \cong_{\mathbb{F}} A(L)$.

Corollary: The elements $\{\delta_t\}_{t \in L}$ form an orthogonal basis.

Pf of Propⁿ: Since $\dim_{\mathbb{F}}(B(L)) = |L| = \dim_{\mathbb{F}}(A(L))$ and Θ maps a basis of $B(L)$ to a basis of $A(L)$, we see that $B(L) \cong_{\mathbb{F}} A(L)$.

Next, we will compute $\Theta(st)$ and $\Theta(s) \cdot \Theta(t)$ and observe they are equal, which will finish the proof.

$$\Theta(st) = \Theta(s \wedge t) = \Theta\left(\sum_{r \leq s \wedge t} \delta_r\right) = \sum_{r \leq s \wedge t} e_r.$$

Next,

$$\theta(s)\theta(t) = \theta\left(\sum_{r \leq s} \delta_r\right) \theta\left(\sum_{q \leq t} \delta_q\right) = \left(\sum_{r \leq s} e_r\right) \cdot \left(\sum_{q \leq t} e_q\right) = \sum_{\substack{r \leq s \\ q \leq t}} e_r \cdot e_q = (*)$$

Since $e_r \cdot e_q = 0$ if $r \neq q$, we see that the only terms in $(*)$ that remain are e_r for $r \leq s \wedge t$, meaning

$$(*) = \sum_{r \leq s \wedge t} e_r, \text{ which is equal to our calculation above.}$$

Thus, $\theta(r \cdot s) = \theta(r)\theta(s)$, and we are done. $\square$

Remark: This is a strange result! It implies that if $|L| = |M|$ for two lattices L and M , then $A(L) \cong A(M)$ as algebras. Thus, the structure of $A(L)$ is determined by only the size of L . However, this is still a useful algebra.

Proof of Weisner's Theorem: Our goal is to prove that if $a \neq \hat{1}$ in L , we have $\sum_{t: t \wedge a = \hat{0}} \mu(t, \hat{1}) = 0$. Let $a \neq \hat{1}$, and expand $\vec{a} \cdot \delta_{\hat{1}}$ in $A(L)$ in

two ways.

First: $\vec{a} \cdot \delta_{\hat{1}} = \left(\sum_{b \leq a} \delta_b\right) \cdot \delta_{\hat{1}} = 0$, since $a < \hat{1}$.

Second: $\vec{a} \cdot \delta_{\hat{1}} = \vec{a} \cdot \left(\sum_{t \leq \hat{1}} \mu(t, \hat{1}) \vec{t}\right) = \sum_{t \leq \hat{1}} \mu(t, \hat{1}) \vec{a} \cdot \vec{t} =$

$$\sum_{t \leq \hat{1}} \mu(t, \hat{1}) \vec{a} \wedge \vec{t}.$$

Now, from both the first and second, extract the coefficients of $\vec{0}$, to get

$$0 = \sum_{\substack{t \leq \hat{1} \\ t \wedge a = \hat{0}}} \mu(t, \hat{1}) \text{ as desired.}$$

$\square$

Möbius functions for semimodular lattices

For a lattice L , recall that we have various nice properties:

distributive $\Rightarrow$ modular $\Rightarrow$ semimodular

For these highly structured lattices, the Möbius function has extra structure as well.

Propⁿ: If L is a finite semimodular lattice of length n , then $(-1)^n \mu(L) \geq 0$, where $\mu(L) := \mu(\hat{0}, \hat{1})$.

Example: $\mu(B_n) = (-1)^n$, so $(-1)^n \cdot \mu(B_n) = 1 \geq 0$.

Proof: Assume L is lower semimodular, WLOG. Now use Weisner's Theorem. Let L be rank n , with rank function ρ , and let $a \in L$ be an atom. Weisner implies

$$\textcircled{*} \quad \mu(\hat{0}, \hat{1}) = - \sum_{t: t \vee a = \hat{1}} \mu(\hat{0}, t).$$

Now, if $t \vee a = \hat{1}$ and $t \geq a$, then $t = \hat{1}$. Since a is an atom, it follows that if $t \vee a = \hat{1}$, either $t \wedge a = \hat{0}$ or $t = \hat{1}$.

Claim: If $t \wedge a = \hat{0}$, t is a coatom.

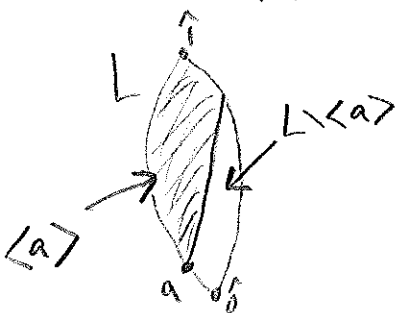
Pf: Since $\rho(t) + \rho(a) \geq \rho(t \wedge a) + \rho(t \vee a) = n$, we have $\rho(t) + \rho(a) = \rho(t) + 1 \geq n = \text{rank}(L)$. Thus, t is a coatom. $\square$

Hence, we can rewrite $\textcircled{*}$ as

$$\textcircled{*} \Rightarrow \mu(\hat{0}, \hat{1}) = - \sum_{\substack{t: t \text{ a coatom} \\ t \neq a}} \mu(\hat{0}, t).$$

(over)

(1)



Claim: Any interval $[x, y]$ in a semimodular lattice L is also semimodular.

pf: $[x, y]$ is ranked, since L is ranked, it is an induced sublattice of L , and $\forall$ in $[x, y] \Rightarrow \diamond$ in $[x, y]$ since this is true for L . $\square$

Now, induct on $n = \text{rank}(L)$.

Base Case: $n=1$: $L = \begin{smallmatrix} \hat{1} \\ \hat{0} \end{smallmatrix}$, $\mu(L) = -1$, and $(-1)^1 \cdot (-1) \geq 0$.

Inductive step: Let $n > 1$. For an atom $a \in L$, $\otimes$ says

$$\mu(\hat{0}, \hat{1}) = - \sum_{\substack{t: t \text{ atom} \\ t \neq a}} \mu(\hat{0}, t). \text{ By induction on the}$$

semimodular lattice $[\hat{0}, t]$, we have $(-1)^{n-1} \mu(\hat{0}, t) \geq 0$.

Hence,

$$(-1)^n \mu(\hat{0}, \hat{1}) = (-1)^n \cdot (-1) \cdot \sum_{\substack{t: t \text{ atom} \\ t \neq a}} \mu(\hat{0}, t) = \sum_{\substack{t \text{ atom} \\ t \neq a}} (-1)^{n-1} \mu(\hat{0}, t) \geq 0.$$



Three important classes of semimodular lattices:

(1) B_n : $\mu(B_n) = (-1)^n$

(2) $B_n(p)$: $\mu(B_n(p)) = (-1)^n q^{\binom{n}{2}}$. [Example 3.10.2 in EC1.]

(3) Π_n : $\mu(\Pi_n) = (-1)^{n-1} (n-1)!$ [Example 3.10.4 in EC1.]

Remark: We've proved (1). (2) is not bad. (3) requires careful thought about structure of intervals in Π_n .

Observations and Questions:

(A) If L is semimodular, and $x, y \in L$, then $(-1)^{l(x,y)} \mu(x,y)$ is a non-negative integer.

(B) For B_n , $B_n(q)$, and T_n , these values are combinatorial: 1 , $q^{\binom{n}{2}}$, $(n-1)!$.

Q1: Can we directly connect $|\mu|$ with counting problems in special classes of posets?

Q2: Does μ have special structure for other classes of posets?

Answers:

A1: For distributive lattices, we can connect descent sets for elements of B_n with various Möbius values.

A2: For posets with an "R-labeling", which includes distributive lattices, modular lattices, T_n , etc, we can generalize this result.

A3: μ can be used to define a special structure of a poset being Eulerian, as follows.

Defⁿ: A finite, graded poset P is Eulerian if $\mu_P(s,t) = (-1)^{l(s,t)}$ for all $s \leq t$ in P .

Ex: Face posets of polytopes are Eulerian.

Goal: Our final goals for this course are to expand on A1, A2, and A3 in a bit more detail.

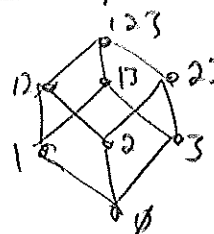
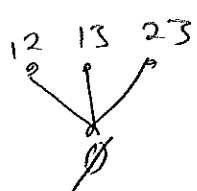
Rank Selection, flag f- and h-vectors

Our first task is to reinterpret Hall's theorem that $\mu(x,y) = C_0 - C_1 + C_2 - C_3 + \dots$ in terms of rank-selection.

Defⁿ: For P a finite, graded poset of rank n , with rank function $\rho: P \rightarrow [0,n]$, for $S \subseteq [0,n]$ we define

$$P_S := \{t \in P : \rho(t) \in S\},$$

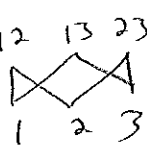
the rank-selected subposet of P , with order induced from P .

Ex: $B_3 =$  $(B_3)_{\{0,2\}} =$ 

Defⁿ: Let $\alpha_P(S)$ be the number of maximal chains in P_S , for P as above and $S \subseteq [0,n]$, where $\alpha_P(\emptyset) := 1$. The flag f-vector of P is the function $\alpha_P: 2^{[0,n]} \rightarrow \mathbb{Z}$. The flag h-vector of P is the function $\beta_P: 2^{[0,n]} \rightarrow \mathbb{Z}$ defined by

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T).$$

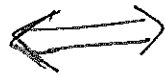
Example:
 B_3 as above.

S	P_S	$\alpha_P(S)$	$\beta_P(S)$
$\emptyset$	$\emptyset$	1	1
1	1 2 3	3	$2 = 3 - 1$
2	12 13 23	3	$2 = 3 - 1$
1,2		6	$4 = 6 - 3 - 3 + 1$

Note symmetry of the β_P values.

Remarks: ① By Möbius inversion (or P.I.E, take your pick),

$$\beta_P(s) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T)$$



$$\alpha_P(s) = \sum_{T \subseteq S} \beta_P(T).$$

② If P has a $\hat{0}$ and a $\hat{1}$, then

(i) $\alpha_P(s) = \alpha_P(S \cap [n-1]).$

pf: If 0 or $n \in S$, $\hat{0}$ or $\hat{1}$ must be in every chain, and there are no other choices at those ranks. $\square$

(ii) $\beta_P(s) = 0$ if $S \not\subseteq [n-1]$, i.e. if either $0 \in S$ or $n \in S$.

pf: Homework exercise! $\square$

③ The idea here is that for a graded poset P , computing the alternating sum of chains can be done for each choice of ranks through which a chain might pass.

④ The terms "flag f " and "flag h " come from connections to face (hence " f ") enumeration for polytopes and simplicial complexes.

Propⁿ: Suppose P has a $\hat{0}$ and a $\hat{1}$. Set $\hat{P}_S = P_S \cup \{\hat{0}, \hat{1}\}$ for all $S \subseteq [n-1]$; in this case, we have

$$\beta_P(s) = (-1)^{|s|-1} \mu_{\hat{P}_S}(\hat{0}, \hat{1}).$$

Remark: Thus, $\beta_P(s)$ gives an alternating Möbius values.

②

Proof: $\beta_P(s) = \sum_{T \leq s} (-1)^{|s|-|T|} \alpha_P(T) \stackrel{\text{use}}{=} (-1)^{|s|-1} \cdot \sum_{T \leq s} (-1)^{|T|+1} \alpha_P(T) = (*)$

$(-1)^{|T|} = (-1)^{-|T|}$

Observe that if $C_k := \#$ of length k chains from $\hat{0}$ to $\hat{1}$ in $\hat{P}_s$, then $C_0 = 0$, $C_1 = \alpha_P(\emptyset)$, $C_2 = \sum_{\substack{T \leq s \\ |T|=1}} \alpha_P(T)$, $C_3 = \sum_{\substack{T \leq s \\ |T|=2}} \alpha_P(T)$, etc.

Hence, $(*) = (-1)^{|s|-1} \cdot (C_0 - C_1 + C_2 - C_3 + \dots)$

$= (-1)^{|s|-1} \mu_{\hat{P}_s}(\hat{0}, \hat{1})$

To this point, what we have done is:

- ① defined β_P and α_P for a graded poset.
- ② Observed their relationship via Möbius inversion.
- ③ Restated Hall's Theorem using β_P 's and α_P 's in the graded case.

Our next step is to identify posets where $\beta_P(s)$ are non-negative and have combinatorial interpretations.

First, an Aside: For a finite graded poset P , the face numbers of P

are

$$f_i := \sum_{s: |s|=i+1} \alpha_P(s)$$

The h-numbers of P are

$$h_i := \sum_{s: |s|=i} \beta_P(s)$$

} These are the f - and h -numbers of the "order complex" of the poset P .

Thus, the "flag" f and h quantities refine the regular f - and h -quantities.

End Aside

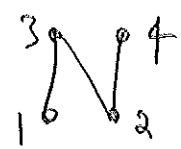
③

Rank-Selection for Finite Distributive Lattices

Recall: L is a finite distributive lattice

$L \cong J(P) \iff P$ the poset of join-irreducibles of L .

Defⁿ: A natural labeling of P is an order-preserving bijection $v: P \rightarrow \{1 < 2 < \dots < n\}$ where $|P| = n$. Given a natural labeling of P , a linear extension of P is a permutation of $[n]$, $\pi \in G_n$, such that $\pi \circ v: P \rightarrow [n]$ is also a natural labeling.

Ex:  has 5 linear extensions.

1234
2134
2143
2413
1243

Defⁿ: For a naturally labeled P , the Jordan-Hölder Set, $\mathcal{L}(P)$, of P is $\{\sigma \in G_n: \sigma \text{ is a linear extension of } P\}$.

• The main theorem connecting flag h -vectors and finite distributive lattices is the following.

Thm (Stanley's PhD Thesis): If $L = J(P)$ and $|P| = n$, with $S \subseteq [n-1]$, then

$$\beta_L(S) = \# \text{ of permutations } w \in \mathcal{L}(P) \text{ with } \text{Des}(w) = S.$$

Note: β is for L , while $\mathcal{L}$ is for P .

Ex: $B_3 = J(\begin{smallmatrix} & & \\ \circ & \circ & \circ \\ & & \end{smallmatrix})$.

S	$\emptyset$	1	2	1,2
$\beta_{B_3}(S)$	1	2	2	1
$w \in G_3 \text{ with } \text{Des}(w) = S$	123	213 312	132 231	321

Pf: Let $S = \{i_1, \dots, i_k\}$ with $i_1 < i_2 < \dots < i_k$. By defⁿ, we have $\alpha_L(S)$ equal to the number of chains $I_1 \subset I_2 \subset \dots \subset I_k$ of order ideals of P with $|I_m| = i_m$, since $L = J(P)$ and $\text{rank}_L(I) = |I|$.

Step 1: Define a permutation $\pi \in \mathcal{L}(P)$ as follows: arrange the elements of I_1 in increasing order, then arrange elements of $I_2 \setminus I_1$ in increasing order, then $I_3 \setminus I_2$, etc. Finally, we list elements of $P \setminus I_k$.

Key Observation: Going from the elements of $I_m \setminus I_{m-1}$ to $I_{m+1} \setminus I_m$ is the only possible way to obtain a descent in this list of elements.

Exercise: Check that this is actually a bijection between maximal chains of L_S and $\pi \in \mathcal{L}(P)$ with $\text{Des}(\pi) \subseteq S$.

Step 2: Observe that if $\gamma(T)$ denotes the number of $\pi \in \mathcal{L}(P)$ with $\text{Des}(\pi) = T$, then $\alpha_L(S) = \sum_{T \subseteq S} \gamma(T)$. Möbius Inversion

gives us $\gamma(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_L(S)$. Hence, it must be

that $\beta_L(S) = \gamma(S) = \# \text{elts of } \mathcal{L}(P) \text{ with descent set } S,$
as desired. 

Corollary: For $S \subseteq [n-1]$, $\beta_{B_n}(S) = \# \text{permutations in } B_n \text{ with descent set } S.$

• While $B_n(q)$ is not distributive, it is still interesting as well.

Thm: For $S \subseteq [n-1]$, $\beta_{B_n(q)}(S) = \sum_{\pi} q^{\text{inv}(\pi)}$ where π ranges

over all $\pi \in B_n$ with descent set S .

Pf: See Thm 3.13.3 in EC1. 

R-labelings

Our next goal is to generalize the previous examples to a wider class of posets. Our main theorem will say roughly:

"If P is R -labelable, then $\beta_P(S) = (-1)^{n-1} \mu_P(\hat{0}, \hat{1})$ counts the size of a certain set of permutations with fixed descents."

Idea: Label edges of a poset P with integers such that intervals contain unique "rising" chains; hence, " R " stands for "rising".

Defⁿ: Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. An R -labeling of P is a function $\lambda: H(P) \rightarrow \mathbb{Z}$, where $H(P) := \{\text{edges of Hasse diagram of } P\}$, such that for every interval $[s, t]$ of P , there is a unique saturated chain

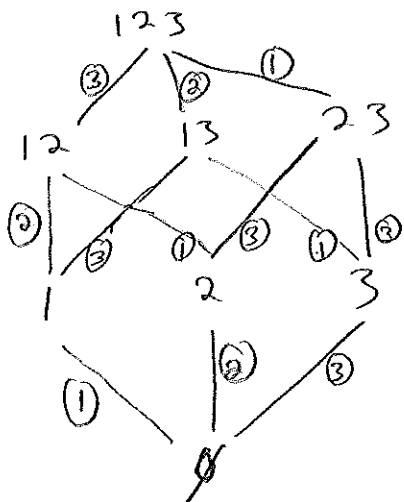
$$s = t_0 < t_1 < t_2 < \dots < t_r = t$$

such that

$$\lambda(t_0, t_1) \leq \lambda(t_1, t_2) \leq \lambda(t_2, t_3) \leq \dots \leq \lambda(t_{r-1}, t_r).$$

We call this the increasing chain for λ . If P admits an R -labeling, we say P is R -labelable.

Ex: B_n is R -labelable. For $S \subseteq [n]$, if $S \leq S'$, then $S' = S \cup \{a\}$ with $a \notin S$. Let $\lambda(S, S') = a$.



In interval $[2, 123]$, the unique rising chain is

$$\begin{array}{c} 123 \\ | \\ 132 \\ | \\ 12 \\ | \\ 1 \\ 2 \end{array}$$

(1)

Propⁿ: This is an R-labeling on B_n .

Pf: Let $[X, Y] \subseteq B_n$ be an interval. Suppose $Y - X = \{b_1, b_2, \dots, b_k\}$ with $b_1 < \dots < b_k$. For any saturated chain in $[X, Y]$, the edge labels form a permutation of $\{b_1, \dots, b_k\}$. Only one of these is strictly increasing. $\blacksquare$

More generally, we can prove a similar result for any distributive lattice.

Thm: Let P be a poset with n elements naturally labeled by $[n]$, and let $L = J(P)$. Then for $I \leq I'$ in $L = J(P)$, let $\lambda(I, I') = I' \setminus I$. This is an R-labeling on $J(P)$.

Pf: This argument is almost identical to the one just given for $B_n = J(\begin{smallmatrix} \circ & \circ & \dots & \circ \\ 1 & 2 & \dots & n \end{smallmatrix})$. $\blacksquare$

We now come to our main theorem regarding R-labelings.

Thm: Let P be an R-labelable poset of rank n . Let λ be an R-labeling of P , and let $S \subseteq [n-1]$. Then $\beta_P(S)$ is equal to the number of maximal chains m of the form

$$\hat{0} = t_0 < t_1 < t_2 < \dots < t_n = \hat{1}$$

such that

$$\lambda(m) = (\lambda(t_0, t_1), \lambda(t_1, t_2), \lambda(t_2, t_3), \dots, \lambda(t_{n-1}, t_n))$$

thought of as a word in G_n

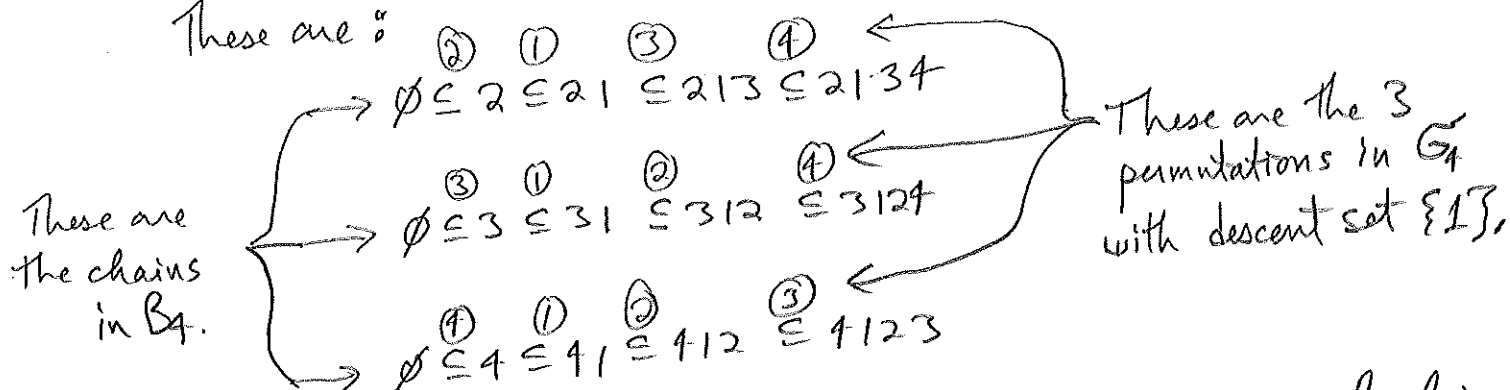
has descent set S , i.e.

$$D(\lambda(m)) = \{i : \lambda(t_{i-1}, t_i) > \lambda(t_i, t_{i+1})\} = S.$$

Example: In B_4 , what is $\beta_{B_4}(\{1\})$?

Since $\beta_{B_4}(\{1\}) = \alpha_{B_4}(\{1\}) - \alpha_{B_4}(\emptyset) = 4 - 1 = 3$, there should be three maximal chains in B_4 with descent set $\{1\}$.

These are:



Pf of Theorem: Let $c = \hat{0} < u_1 < \dots < u_s < \hat{1}$ be a maximal chain in $\hat{P}_S$.

Claim: c extends uniquely to a maximal chain m of P satisfying $\text{Des}(\lambda(m)) \subseteq S$.

Pf: We will construct $m = \hat{0} = t_0 < t_1 < t_2 < \dots < t_n = \hat{1}$. First, assume $S = \{a_1 < a_2 < \dots < a_s\}$, and set $t_{a_i} := u_i$ so that m extends c . Now, in each interval $[t_{a_i}, t_{a_{i+1}}]$, since P is R -labelable, there is a unique rising chain

$$t_{a_i} < \tau_1 < \tau_2 < \dots < t_{a_{i+1}}.$$

Concatenating these yields an unique m with $\text{Des}(\lambda(m)) \subseteq S$. $\square$

Let $\alpha'_P(s)$ be the number of maximal chains m of P such that $\text{Des}(\lambda(m)) \subseteq S$. By our claim, we must have $\alpha'_P(s) = \alpha_P(s)$.

Thus, $\alpha'_P(s) = \alpha_P(s) = \sum_{T \subseteq S} \beta_P(T)$. Also, if $\beta'_P(s)$ is the number of maximal chains in P such that $\text{Des}(\lambda(m)) = S$, clearly

$\alpha_P(s) = \alpha'_P(s) = \sum_{T \subseteq S} \beta'_P(s)$. Möbius inversion implies $\beta_P(s) = \beta'_P(s)$, as desired. $\square$ (3)

Examples of R-labelable posets

① Finite Semimodular lattices

Let L be a finite upper semimodular lattice, and let P be the subset of join-irreducibles of L . Let

$$t_1 < t_2 < \dots < t_k$$

be a linear extension of P , where $|P| = k$. For an edge $(s, t) \in H(L)$, set $\lambda(s, t) = \min \{i : s \vee t_i = t\}$.

Propⁿ: This is an R-labeling of L .

Pf: Exercise.



② Supersolvable finite lattices

A finite lattice L is supersolvable if it has a maximal chain $\underline{c}$, called an M-chain, such that the sublattice of L generated (via meets and joins) by $\underline{c}$ and any other chain in L is distributive.

Examples:

- Modular lattices

- Π_n

- lattice of subgroups of a finite supersolvable group.

See example 3.17.4 in EC1 for a canonical R-labeling in this class of posets.

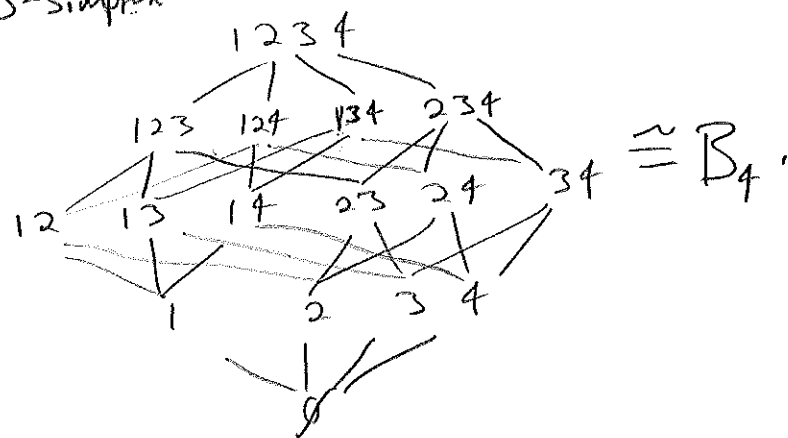
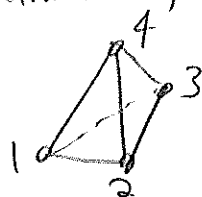
Eulerian Posets

Our final topic of the course is at hand!

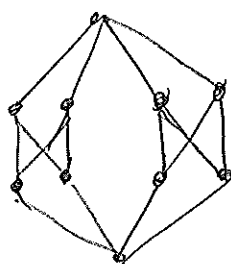
Recall: A finite, graded poset P with $\hat{0}, \hat{1}$ is Eulerian if $\mu_P(s, t) = (-1)^{l(s, t)}$ for all $s \leq t \in P$, where $l(s, t)$ is the length of $[s, t]$.

Main Example: Let P be the face lattice of a convex polytope (of any dimension). Then P is Eulerian.

E.g. Tetrahedron, aka 3-simplex



Example (not from a polytope):



Exercise: Check this is Eulerian.

Remark: P is Eulerian if and only if every interval of rank at least 1 contains an equal number of elements of even rank as of odd rank. This follows from the definition of μ_P : Let $[s, t]$ be an interval with $s < t$.

$$0 = \sum_{s \leq z \leq t} \mu(s, z) = \sum_{s \leq z \leq t} (-1)^{l(s, z)} = \sum_{z \in [s, t]} (-1)^{\text{rank}(z)} \cdot \text{Thus, all } 1\text{'s and } -1\text{'s must cancel, yielding the claim.}$$

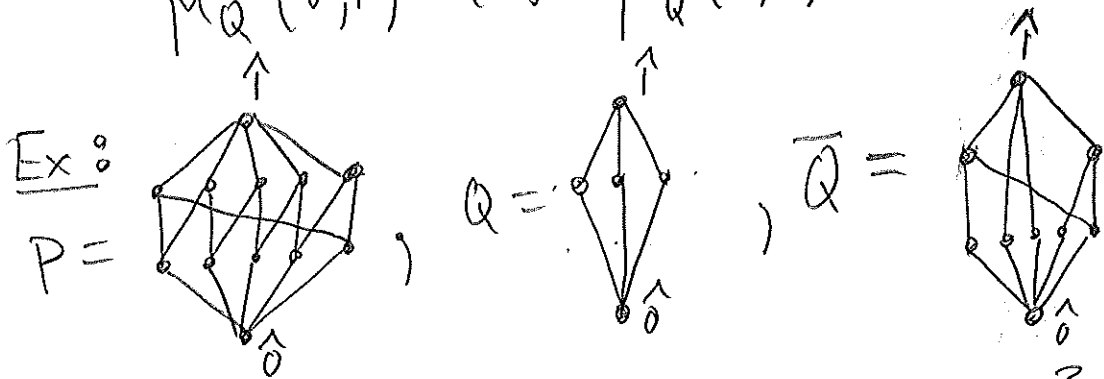
(1)

Eulerian posets satisfy strong restrictions on μ , for example:

Thm (Alexander Duality for Eulerian Posets):

Let P be a rank n Eulerian poset, and let Q be a subposet of P containing $\hat{0}$ and $\hat{1}$. Set $\bar{Q} := (P \setminus Q) \cup \{\hat{0}, \hat{1}\}$.

Then
$$\mu_Q(\hat{0}, \hat{1}) = (-1)^{n-1} \mu_{\bar{Q}}(\hat{0}, \hat{1}).$$



$$\mu_Q(\hat{0}, \hat{1}) = 2, \quad \mu_{\bar{Q}}(\hat{0}, \hat{1}) = -2 = (-1)^3 \cdot 2$$

Corollary: If P is Eulerian of rank n , and $S \subseteq [n-1]$, then set $\bar{S} = [n-1] \setminus S$. We have

$$\beta_P(S) = \beta_P(\bar{S}).$$

$$\begin{aligned} \text{Pf: } \beta_P(S) &= (-1)^{|S|-1} \mu_{\hat{P}_S}(\hat{0}, \hat{1}) = (-1)^{|S|-1} \cdot (-1)^{n-1} \mu_{\hat{P}_{\bar{S}}}(\hat{0}, \hat{1}) \\ &= (-1)^{(n-1-|S|)-1} \mu_{\hat{P}_{\bar{S}}}(\hat{0}, \hat{1}) = (-1)^{|\bar{S}|-1} \mu_{\hat{P}_{\bar{S}}}(\hat{0}, \hat{1}) \\ &= \beta_P(\bar{S}). \quad \blacksquare \end{aligned}$$

Remarks: The phrase "Alexander duality" comes from connections to Alexander duality for subspaces of spheres in algebraic topology; see Section 3.16 in EC1.

- The proof of Alexander Duality for Eulerian posets isn't hard, see EC1 §3.16 again, specifically 3.16.3-3.16.5.

- The condition that $\mu(s, t) = (-1)^{\ell(s, t)}$ can be strengthened to a homological condition on the order complex of a poset, namely that P be Gorenstein* (this is equivalent to P being simultaneously Eulerian and Cohen-Macaulay).

- One of the major open questions in algebraic/topological combinatorics is the Charney-Davis conjecture: For a Gorenstein* poset of even rank $n=2d$, is it always true that

$$(-1)^d (h_0 - h_1 + h_2 - \dots + h_n) \geq 0 ?$$

(Here, the h_i 's are the h -numbers of P , recall $h_i := \sum_{S: |S|=i} \beta_P(S)$). The conjecture is yes, open since 1995.