MA 614 – Enumerative Combinatorics[1]
Spring 2015

1. General Information

Dr. Benjamin Braun
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Office Phone: 257-6810
Class Time/Location: 12:00-12:50PM, MWF, CB 347
Office Location/Hours: 831 POT, by appointment and via piazza.com.

2. Texts

Primary Text:

- *Enumerative Combinatorics, Volume 1, 2nd edition*, Richard Stanley.

Additional Resources:

- *Enumerative Combinatorics, Volume 2*, Richard Stanley
- *The Two Cultures of Mathematics*, Timothy Gowers. Available electronically at
  www.dpmms.cam.ac.uk/∼wtg10/2cultures.pdf
- *The Many Faces of Modern Combinatorics*, by Christian Lenart. Available electronically at
  http://www.albany.edu/∼lenart/articles/combin1.pdf
- *A Course in Combinatorics*, Van Lint and Wilson, 2001.
- *A Course in Enumeration*, Martin Aigner, 2007.
- *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory, 2nd edition*, by Miklos Bona, 2006.

3. Course Description

> Having vegetated on the fringes of mathematical science for centuries, combinatorics has now burgeoned into one of the fastest growing branches of mathematics . . . The mathematical world had been attracted by the success of algebra and analysis and only in recent years has it become clear . . . that combinatorics, the study of finite sets and finite structures, has its own problems and principles. These are independent of those in algebra and analysis but match them in difficulty, practical and theoretical interest, and beauty.
> László Lovász, *Combinatorial Problems and Exercises*

The basic problem of enumerative combinatorics is that of determining the number of elements of a finite set. This enterprise is interesting, subtle, surprising, and very challenging. Enumeration is part of the larger discipline of combinatorics and has connections to algebra, analysis, topology, statistics, geometry, and a host of other areas in mathematics and science. Enumeration, and combinatorics in general, has a different feel than these other, more "traditional," areas of mathematics, a difference articulated by Timothy Gowers in the following passage.

> If the processes of abstraction and generalization, which are so important in mathematics, are of only limited use in combinatorics, then how can the subject be transmitted to future generations? One way of thinking about this question is to ask what the requirements of tomorrow's combinatorialists are likely to be . . . their priority is likely to be solving problems, so their interest in one of today's results

[1] I reserve the right to change or amend this syllabus at any time for any reason.

> will be closely related to whether, by understanding it, they will improve their own problem-solving ability. And this brings us straight to the heart of the matter. The important ideas of combinatorics do not usually appear in the form of precisely stated theorems, but more often as general principles of wide applicability.
> Timothy Gowers, *The Two Cultures of Mathematics*

Given this quality of combinatorial theory, the student learning goals in this course are the following:

- To understand general principles of wide applicability for solving problems involving enumeration of finite sets, and
- To develop the ability to apply these principles to solve such problems.

We will develop a theory of enumeration based on common enumerative structures. Enumerative theory consists of understanding each of these common structures along with techniques for studying them. We will focus on the following ideas and principles.

(1) Basic structures of enumeration
  - Common enumerative families: binomial coefficients, Catalan numbers, Fibonacci numbers, Eulerian numbers, Bell numbers, Stirling numbers.
  - Lattice paths.
  - Compositions and partitions.
  - Recurrences.
  - Permutations and permutation statistics.
  - q-analogues.
  - The twelvefold way.

(2) Generating functions
  - Ordinary and exponential generating functions.
  - Tree enumeration and Cayley's theorem.
  - The exponential formula.
  - Lagrange inversion formula.

(3) Sieve Methods
  - The principle of inclusion-exclusion.
  - Unimodality and log-concavity of sequences.
  - The involution principle and sign-reversing involutions.
  - The Gessel-Viennot theorem.

(4) Partially Ordered Sets
  - Partially ordered sets (posets).
  - Lattices and their refinements.
  - The incidence algebra of a poset.
  - The zeta and Möbius functions for posets.
  - Computational methods for Möbius functions and the Möbius inversion formula.
  - Möbius algebras of lattices, Weisner's theorem, and the crosscut theorem.
  - Rank-selection, flag f- and h-vectors, and R-labelings.
  - Eulerian posets and duality.

One major challenge for students as they proceed to advanced study in mathematics is reading dense texts independently. Stanley's two-volume set on enumerative combinatorics, paired with his book *Combinatorics and Commutative Algebra*, aka "the green book," have the interesting reputation as being both A) the definitive works on this subject and B) very difficult for the beginning combinatorialist to read. To help you gain experience with reading mathematics at an advanced level, and to help you gain familiarity with the first of Stanley's books, daily reading assignments are required for this course. Here are some suggestions for how to approach the reading.

First: understand the story. Even if you don't understand all the words, you can understand a lot by skimming the expository paragraphs. Is this portion of the text about a specific example? a general phenomenon? Does the author say it is related to something you know about? Does the section contain a lot of detailed proofs, or mainly a discussion? What words are defined in the section?

Second: understand the broad ideas. Read the definitions. Create small examples and non-examples. Read the theorems. Create small examples and non-examples to illustrate the theorem. Skip all the proofs. Summarize the text in your own words.

Third: understand the details. Read the proofs. Create larger examples and non-examples. Create generalizations of the definitions and theorems. Try to prove your generalizations.

Continually repeat this cycle. Read the section again, skipping all proofs. Create a short summary of the text in your own words. Create a short outline of the text.

4. **Course Expectations**

4.1. **Attendance.** You must be present at, prepared for, and engaged in class each day. If you need to miss class for some reason, please notify me ahead of time.

4.2. **Homework.**

- No late work will be accepted.
- Homework will be due on a regular basis. A partial selection of problems from each problem set will be graded.
- Your homework must be typed in Latex. Latex is a fantastic system for typesetting mathematics and is the standard typesetting method for professional mathematicians. For more information, see: http://www.latex-project.org
- You may collaborate with your classmates in *developing ideas* regarding homework problems; however, do not let cooperation degenerate into one person solving the problem and other people copying their answers. While it is important to celebrate mathematics as a social and cultural endeavor, it is also important that *you work out the details* for solutions on your own. You must write up your own answers to all the questions. *For each homework problem, indicate in your solution the people you shared ideas with.*
- Searching the library or internet for solutions to problems is not allowed. The act of copying a written answer from another student and submitting it as your own will be considered cheating and will be dealt with according to the procedures referenced in Section 7.

4.3. **Exams.** There will be one take-home midterm exam and a take-home cumulative final. You are expected to work on these on your own and to follow all instructions regarding them. *You are not allowed to collaborate with other students on the exams.*

5. **Course Grades**

Your total grade will be determined by your homework and exams. The grading scale will be no stricter than the usual A>89.9, B>79.9, C>69.9, D>59.9, E otherwise, weighted as follows:

- Problem Sets: 40%
- Midterm Exam: 30%
- Final Exam: 30%

6. **Tentative Schedule**

For each day there is an assigned reading; you are expected to complete the reading for each day *before* class. In this schedule, "EC 2–5" refers to pages 2–5 of Stanley's Enumerative Combinatorics Vol 1, 2nd edition.

Sets, subsets, and integer compositions

Jan 14: Reading: Syllabus, EC 1–3
Discussion: Introduction to counting
Jan 16: Reading: EC 13–17 (start at bottom paragraph on page 13, OMIT examples 1.1.16 and 1.1.17)
Discussion: Pascal's triangle, the binomial recurrence, and the binomial theorem
HW # 1 due

Jan 19: MLK Holiday – no class
Jan 21: Reading: EC 17–19 (only first half of 19, stop after the second direct proof).
Discussion: Binomial coefficients and integer compositions; multisets.
Jan 23: Reading: EC 3–8
Discussion: Generating functions, the binomial series, applications to binomial coefficients and compositions
HW # 2 due

Variations on integer compositions

Jan 26: Reading: Wikipedia pages on Fibonacci Numbers and Fibonacci polynomials:
http://en.wikipedia.org/wiki/Fibonacci_number (sections 1–3)
http://en.wikipedia.org/wiki/Fibonacci_polynomials
Discussion: Compositions using 1's and 2's, Fibonacci numbers, Fibonacci polynomials.
Jan 28: Reading: EC 10 (only Example 1.1.12), EC 464–466 (stop after Example 4.1.2)
Discussion: Recurrence relations and ordinary generating functions, Binet's formula for Fibonacci numbers
Jan 30: Reading: EC 466–467 (only Example 4.1.3)
Discussion: Combinatorial interpretations for linear recurrence relations
HW # 3 due

Feb 2: Reading: EC 58, 61–62 (stop at Prop 1.8.1), 63–64 (start with Prop 1.8.4, stop after the first proof of Prop 1.8.5)
Discussion: Partition Identities

The twelvefold way

Feb 4: Reading: EC 71–75, on page 75 stop after the proof of Entry 3.
Discussion: The twelvefold way, Bell numbers, and Stirling numbers of the second kind.
Feb 6: Reading: EC 75–76 and 79–80 (on page 79, start with the proof of Entry 4)
Discussion: The twelvefold way, continued.
HW # 4 due

Permutations

Feb 9: Reading: EC 20 (only definition of $\mathfrak{S}_S$ at bottom), EC 22–24 (stop after proof of Theorem 1.3.2)
Discussion: The Symmetric Group $\mathfrak{S}_n$, word/two-line notation and cycle notation, and cycle structure
Feb 11: Reading: EC 26–29 (for Prop 1.3.7, read *only* the first proof and fourth proof)
Discussion: (signless) Stirling numbers of the first kind.
Feb 13: Reading: EC 29–31 (read the entire subsection regarding inversions), EC 42–43 (last paragraph of 42, first paragraph of 43).
Discussion: Inversions and diagrams of permutations.
HW # 5 due

Feb 16: Reading: EC 31 (read only the definition of $D(w)$), EC 32–34 (start at last paragraph on 32, read entire proof of Prop 1.4.5)
Discussion: Descent sets and Eulerian numbers

Multinomial coefficients, lattice paths, and Catalan numbers

Feb 18: Reading: EC 20–22
Discussion: Multinomial coefficients and lattice paths.

Feb 20: Reading: http://en.wikipedia.org/wiki/Catalan_number, read the sections titled "Properties," "Applications in Combinatorics," and the *second proof* in the section "Proof of the formula."
Discussion: Catalan numbers, Dyck paths, the Catalan recurrence.
HW # 6 due

Feb 23: Reading: http://en.wikipedia.org/wiki/Catalan_number, first proof in the section "Proof of the formula", and all of http://en.wikipedia.org/wiki/Narayana_number
Discussion: Catalan numbers, generating functions, and Narayana numbers

q-analogues

Feb 25: Reading: EC 54–58 (start at section 1.7, *skip both proofs* of Prop 1.7.1 as we will prove a special case in class, read Prop 1.7.2 and its proof).
Discussion: q-multinomial polynomials, Gaussian polynomials, and multiset permutation inversions.

Feb 27: Reading: EC 59–60, omit proof of Prop 1.7.3 (we will give a straightforward proof in class).
Discussion: Gaussian polynomials, partitions, and lattice paths.
HW # 7 due

Exponential generating functions

Mar 2: Reading: None.
Discussion: Multiplication of exponential generating functions, derangements, and involutions.
EXAM: Distribute take-home midterm exam. The midterm exam covers material up to and including Feb 23.

Mar 4: Reading: None.
Discussion: The Block-Partitioned Structure Principle and the exponential formula.

Mar 6: Reading: None.
Discussion: Applications of the exponential formula
HW# 8 due

Mar 9: Reading: None.
Discussion: Recurrences for EGFs, permutations with restricted cycle types.
EXAM: Midterm exam due.

Mar 11: Reading: None.
Discussion: The Point-Removal Structure Principle and the Lagrange Inversion Formula.

Mar 13: Reading: None.
Discussion: Tree enumeration, applications of point-removal and Lagrange inversion.
HW # 9 due

Spring Break: March 16–20 – no class

Inclusion-exclusion and sieve methods

Mar 23: Reading: http://en.wikipedia.org/wiki/Inclusion-exclusion_principle, read the "statement" section and the first two examples in the "Examples" section. NOTE: This is equivalent to

section 2.1 in EC.
Discussion: The principle of inclusion-exclusion.

Mar 25: Reading: EC 212-213, and Exercise 1.50 (on page 213 stop at the section that is "a more complicated example" and read the Exercise as if the problems were theorem statements).
Discussion: Sign-reversing involutions, unimodal sequences, and log-concave sequences.

Mar 27: Reading: EC 215–218 (start at section 2.7, skip the proof of Theorem 2.7.1 as we will prove a special case in class).
Discussion: The Gessel-Viennot theorem.
HW # 10 due

Posets and lattices

Mar 30: Reading: EC 241–243 (stop at definition of isomorphic).
Discussion: Posets.

Apr 1: Reading: EC 243–246 (stop at section 3.2).
Discussion: Posets.

Apr 3: Reading: EC 246–247.
Discussion: New posets from old.
HW # 11 due

Apr 6: Reading: EC 248–252 (read all of section 3.3).
Discussion: Lattices.

Apr 8: Reading: EC 252–253 (stop after proof of Theorem 3.4.1).
Discussion: Finite distributive lattices.

Incidence algebras, zeta functions, and Möbius functions

Apr 10: Reading: EC 261–263 (stop after discussion of $(\zeta - 1)$).
Discussion: The incidence algebra of a poset and the zeta function.
HW # 12 due

Apr 13: Reading: EC 264–265 and 263 and 268-269 (on 263 only read the first justification regarding $2 - \zeta$ and on 268-269 only read Prop 3.8.5 and its proof).
Discussion: Möbius functions and the Möbius inversion formula.

Apr 15: Reading: EC 266–267 (stop after Example 3.8.3), EC 274-275 (start at section 3.9, stop just before Theorem 3.9.2).
Discussion: Techniques of Computation: the product theorem, Weisner's theorem and the crosscut theorem.

Apr 17: Reading: EC 275–276 (start with Cor 3.9.3).
Discussion: Applications of Weisner's theorem.
HW #13 due

Apr 20: Reading: None.
Discussion: Möbius algebras for lattices

Apr 22: Reading: EC 277–280.
The Möbius function of a semi-modular lattice.

Properties inspired by polytopes and topology: rank selection, flag f and h vectors, R-labeling, and Eulerian posets

Apr 24: Reading: EC 257–258 (read only the definition of *linear extension*), 293–295 (skip the NOTE regarding simplicial complexes, and on page 334 skip Theorem 3.13.3).
Discussion: Rank selection and the flag f- and h-vector of a graded poset.
HW # 14 due

Apr 27: Reading: None.
More on Rank selection.
Apr 29: Reading: EC 295–298.
Discussion: R-labelings.
May 1: Reading: EC 310 and 311–312 (on 310 only read the definition of Eulerian at the top and on 311–312 only read from Lemma 3.16.3 through Cor 3.16.6).
Discussion: Eulerian posets and duality
HW # 15 due
EXAM: Final exam distributed – due Tuesday, May 5, at 5PM to my office.

7. **Academic Integrity and Classroom Demeanor**

All students are expected to follow the academic integrity standards as explained in the University Senate Rules, particularly Chapter 6, found at:

http://www.uky.edu/USC/New/SenateRulesMain.htm

Turn off all cell phones, pagers, etc. prior to entering the classroom. ***You are not to use your cell phones, pagers, or other electronic devices during class.*** An attitude of respect for and civility towards other students in the class and the instructor is expected at all times.

8. **Classroom and Learning Accommodations**

Any student with a disability who is taking this course and needs classroom or exam accommodations should contact the Disability Resource Center, 257-2754, room 2 Alumni Gym, jkarnes@uky.edu.