MA 113 **1/23/17**

1. Log in to your REEF account using your phone, laptop, or tablet. (See my Canvas announcement about this.)
2. Today we will be testing REEF system.
3. After today, I will take attendance using REEF on random days; not every day.
4. See Canvas announcement for reminders about homework / quiz info for this week.

ANSWER: 1

Average + Inst. Velocity §2.1

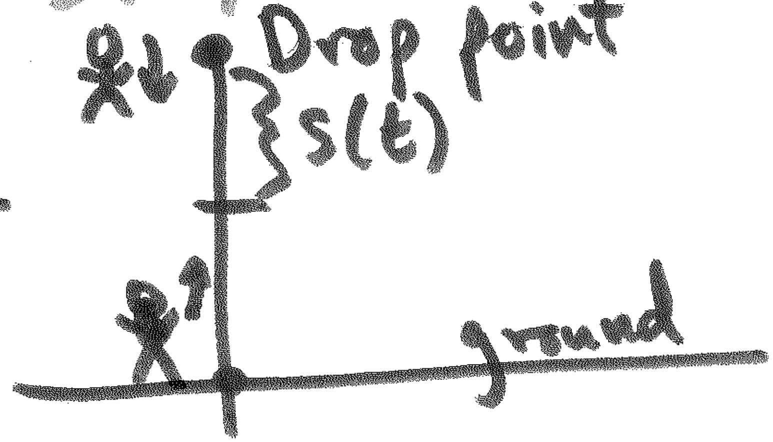
<u>Ex</u>: Suppose we drop an object from high up. Ignoring air resistance, Galileo's law applies:

If $s(t)$ = distance in meters fallen after t seconds, then

$$s(t) = 4.9 \cdot t^2 \ \underline{m/s}.$$

← units!

<u>Picture:</u>



<u>Q</u>: Do you want to consider $s(t)$ as a positive or as a negative quantity?

<u>A</u>: There is no clean answer. We must decide.

This leads to <u>distance</u> i.e. total amount traveled, no sign.

+ <u>displacement</u>, i.e. distanced traveled w/ a sign indicating direction.

So, distance $= |s(t)|$, displacement, i.e. sign, depends on observer.

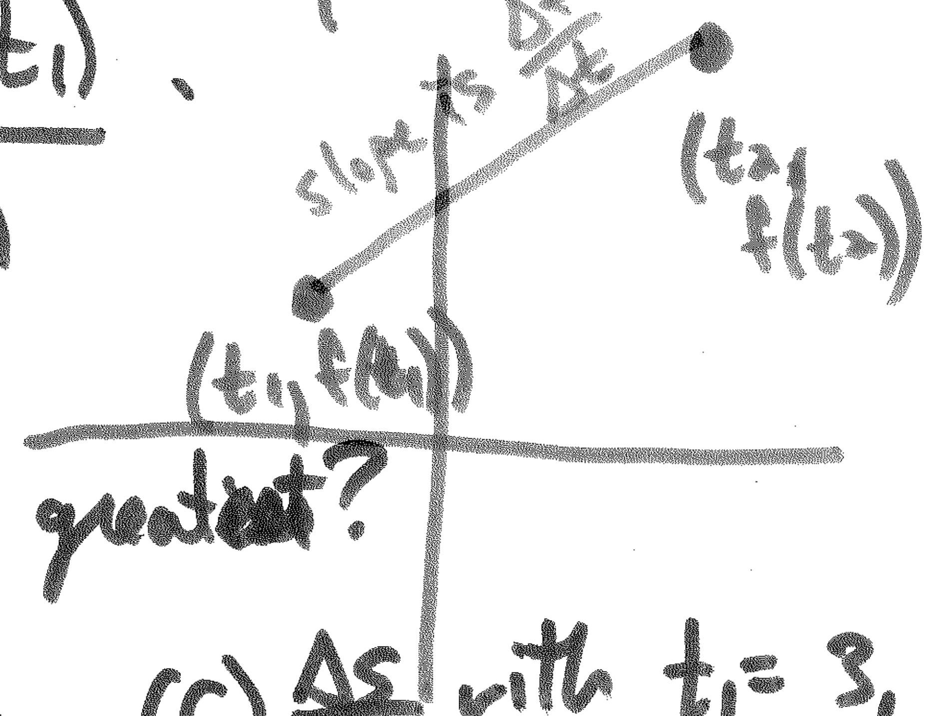
Average Velocity: Given an object with position $f(t)$ meters at t seconds, the **average velocity** of the object between time t_1 and time t_2 is

change in f →

$$\frac{\Delta f}{\Delta t} = \frac{f(t_2)-f(t_1)}{t_2-t_1}.$$

change in time →

You've seen this!



Q: For $s(t) = 4.9t^2$, which is greatest?

(a) $\frac{\Delta s}{\Delta t}$ with $t_1=0, t_2=1$

(b) $\frac{\Delta s}{\Delta t}$ with $t_1=\frac{1}{2}, t_2=\frac{3}{4}$

(c) $\frac{\Delta s}{\Delta t}$ with $t_1=3, t_2=3.1$

(a) is 4.9 m/s

(b) 6.125 m/s

(c) 29.89 m/s greatest! (even though smallest time change...)

What do we really want?: How fast is the object falling at a specific instant in time?

To answer this, we consider average velocities for very small time intervals.

Ex: $s(t) = 4.9t^2$ m/s, estimate the "instantaneous" velocity at 3.2 seconds.

- use time interval $[3.2, 3.21]$ + get

$$\frac{s(3.21) - s(3.2)}{3.21 - 3.2} = \boxed{31.409 \text{ m/s.}}$$

- use time interval $[3.199, 3.2]$ + get

$$\frac{s(3.2) - s(3.199)}{3.2 - 3.199} = \boxed{31.3551 \text{ m/s.}}$$

- Use time interval $[3.15, 3.25]$ + get

$$\frac{s(3.25) - s(3.15)}{3.25 - 3.15} = \boxed{31.36 \text{ m/s}}$$

Def$^{\underline{n}}$: The <u>instantaneous velocity</u> <u>at a seconds</u> of an object with position $s(t)$ meters at t seconds is the limiting value of

"difference quotient" → $$\frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

for very small values of h (if a limiting value exists).

<u>NEXT BIG Q</u>: What does "limiting value" mean?
we will tell you a half-truth in response.

The Limit of a Function

Peter A. Perry, Posing as Ben Braun

University of Kentucky

January 25, 2017

Bill of Fare

1. Why Study Limits, Anyhow?
2. The Limit of a Function
3. One-Sided Limits
4. Infinite Limits

Deadlines, Deadlines!

- Webwork A3 due Wednesday
- Webwork A4 due Friday
- Quiz #2 in recitation Thursday
- Written Assignments due Friday

Why Study Limits, Anyhow?

The *concept* of limit in Calculus seeks to make sense out of what we did in the Falling Body problem, where we wanted to compute instantaneous velocity at $t = 1$ if the signed distance from the starting point is given by

$$g(t) = -16t^2 + 64t$$

measuring t in seconds and distance in feet.

We found that the average velocity between $t = 1$ and $t = 1 + h$ seconds is

$$\frac{g(1+h) - g(1)}{h} = \frac{32h + 16h^2}{h} = 32 + 16h$$

Why Study Limits, Anyhow?

We found that the average velocity between $t = 1$ and $t = 1 + h$ seconds is

$$\frac{g(1+h) - g(1)}{h} = \frac{32h + 16h^2}{h} = 32 + 16h$$

Thus, although the "average velocity between $t = 1$ and $t = 1$ doesn't make any sense, the *instantaneous velocity* is certainly what you get as h "tends to zero." What's going on here?

The answer is that there is a difference between the *value of a function at a point* and the *limit of a function as its argument approaches a point*

The Limit of a Function

Consider the rational function

$$f(x) = \frac{x^2 + x - 2}{x - 1}$$

whose domain is the set of all real numbers $x \neq 1$. So, there is *no such thing* as $f(1)$.

However...

x	$f(x)$
1.01	3.01
1.001	3.001
1.0001	3.0001

x	$f(x)$
0.99	2.99
0.999	2.999
0.9999	2.9999

so that, the closer x gets to 1, the closer $f(x)$ gets to 3, i.e.,

$$\lim_{x \to 1} f(x) = 3$$

An Important Note

$$f(x) = \frac{x^2 + x - 2}{x - 1}$$

x	$f(x)$
1.01	3.01
1.001	3.001
1.0001	3.0001

x	$f(x)$
0.99	2.99
0.999	2.999
0.9999	2.9999

In finding $\lim_{x\to 1} f(x)$, it is important to examine the behavior of $f(x)$ *both* as x approaches 1 from the right *and* as x approaches 1 from the left!

What's the Trick?

Of course,

$$f(x) = \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x+2)}{x-1} = x + 2$$

on its domain, so without looking at a table of data, we could use this formula to predict

$$\lim_{x \to 1} f(x) = 3$$

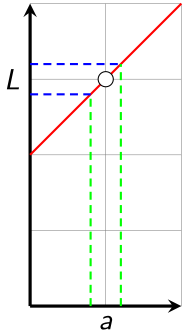
The crucial point is, *even though* $x = 1$ *does not belong to the domain of* f, it still makes sense to talk about the *limit* of $f(x)$ as x tends to 1.

The Intuitive Definition of Limit

We say that a number L is the **limit** of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L$$

if the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a.



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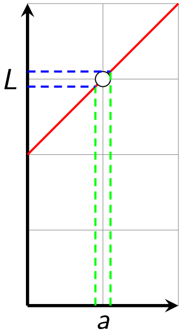

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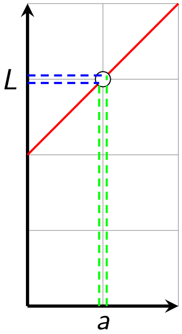
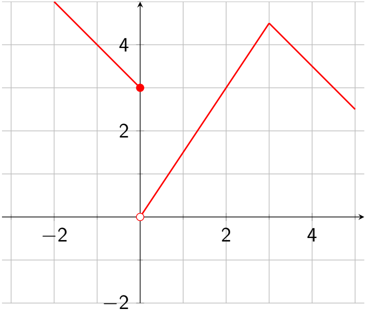


Find the Limits!

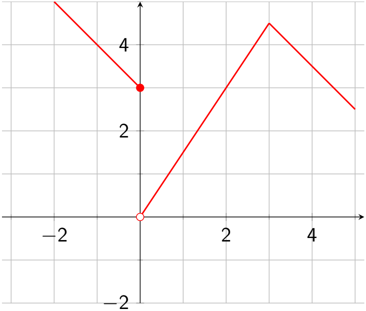


For the graph shown:

Find $\lim_{x\to 2} f(x)$

A. 2
B. 3
C. 4
D. The limit does not exist

Find the Limits!

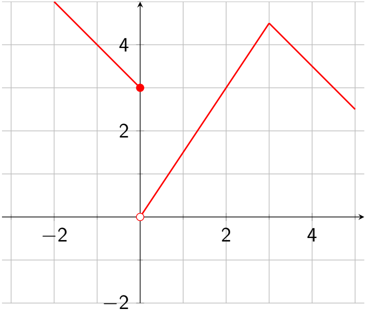


For the graph shown:

Find $\lim_{x \to 3} f(x)$

A. 3.5
B. 5.5
C. 3
D. The limit does not exist

Find the Limits!



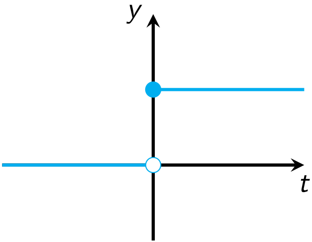
For the graph shown:

Find $\lim_{x \to 0} f(x)$

A. 3
B. 0
C. 1.5
D. The limit does not exist

One-Sided Limits

The *Heaviside Function* (known informally as the "off-on function") is defined as:



$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

The value of $H(t)$ approaches 0 at t tends to zero from the left

$$\lim_{t \to 0^-} H(t) = 0$$

The value of $H(t)$ approaches 1 as t tends to zero from the right

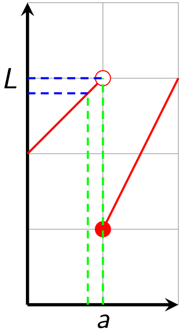
$$\lim_{t \to 0^+} H(t) = 1$$

Left-Hand Limits

We say that

$$\lim_{x \to a^-} f(x) = L$$

and say that **the left-hand limit of** $f(x)$ **as** x **approaches** a **is** L if the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a, but *less than* a



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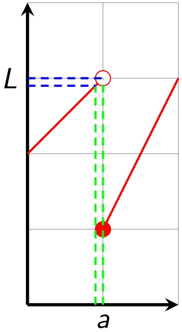

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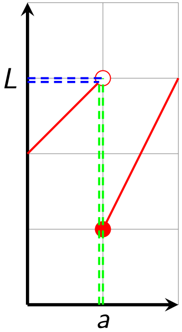
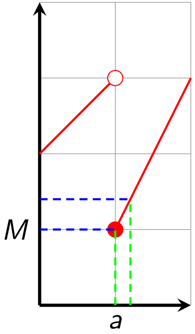


Right-Hand Limits

We say that

$$\lim_{x \to a^+} f(x) = M$$

and say that **the right-hand limit of** $f(x)$ **as** x **approaches** a **is** M if the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a, but *greater than* a

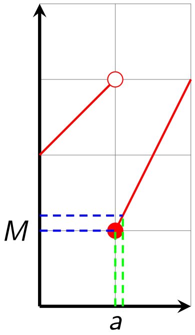


Right-Hand Limits

We say that

$$\lim_{x \to a^+} f(x) = M$$

and say that **the right-hand limit of** $f(x)$ **as** x **approaches** a **is** M if the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a, but *greater than* a

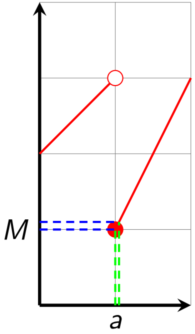


Right-Hand Limits

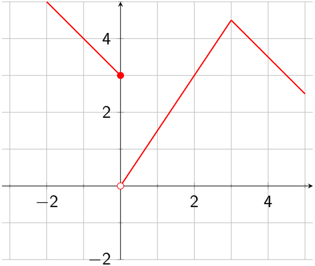
We say that

$$\lim_{x \to a^+} f(x) = M$$

and say that **the right-hand limit of** $f(x)$ **as** x **approaches** a **is** M if the value of $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to a, but *greater than* a



Find the Left- and Right-Hand Limits!

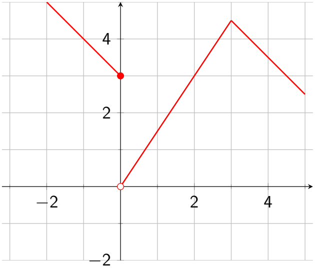


For the graph shown:

Find $\lim_{x \to 0^-} f(x)$

A. 0
B. 2
C. 1.5
D. 3
E. Does not exist

Find the Left- and Right-Hand Limits!



For the graph shown:

Find $\lim_{x\to 0^+} f(x)$

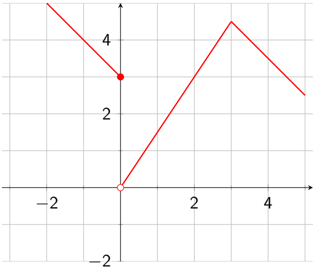
A. 0

B. 2

C. 1.5

D. It cannot be determined from the information given

Find the Left- and Right-Hand Limits!



For the graph shown:

Find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$

A. $\lim_{x\to 3^-} f(x) = +2.5$ and $\lim_{x\to 3^+} f(x) = -2.5$

B. $\lim_{x\to 3^-} f(x) = 4.5$ and $\lim_{x\to 3^+} f(x) = -4.5$

C. $\lim_{x\to 3^-} f(x) = 4.5$ and $\lim_{x\to 3^+} f(x) = 4.5$

D. $\lim_{x\to 3^+} f(x) = 3$ and $\lim_{x\to 3^+} f(x) = -3$

Left-Hand Limits, Right-Hand Limits, and Limits

$$\lim_{x \to a} f(x) = L$$

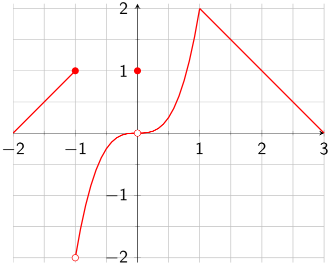
if and only if

$$\lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

In words: The limit of $f(x)$ as x tends to a is a number L *if and only if*

- the left- and right-hand limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist
- Both the left- and the right-hand limits are equal to the number L

Have We Reached the Limit Yet?



To the left is the graph of a function $f(x)$.

For which points a are the left- and right-hand limits of $f(x)$ as $x \to a$ equal?

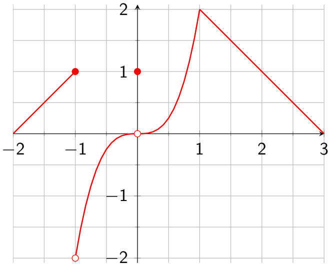
A. $a = -1$ and $a = 1$

B. $a = 0$ and $a = 1$

C. $a = -1$ only

D $a = -1$, $a = 0$, and $a = 1$

Have We Reached the Limit Yet?



To the left is the graph of a function $f(x)$.

For which points a does the limit as $x \to a$ of $f(x)$ exist?

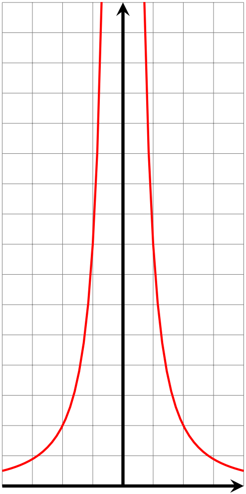
A. $a = -1$ and $a = 1$

B. $a = 0$ and $a = 1$

C. $a = -1$ only

D $a = -1$, $a = 0$, and $a = 1$

Infinite Limits



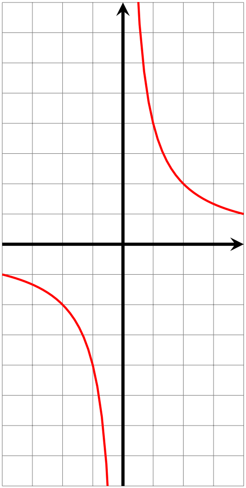
Let's look at the graph of the function

$$f(x) = \frac{1}{x^2}$$

To describe this function's behavior at zero, we can say that $f(x) \to +\infty$ as $x \to 0$ or write

$$\lim_{x \to 0} f(x) = +\infty$$

Infinite Limits



Let's look at the graph of the function

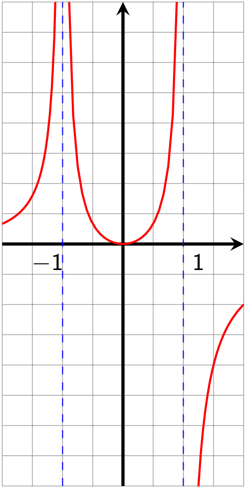
$$f(x) = \frac{1}{x}$$

To describe this function's behavior at zero, we can say that

$$\lim_{x \to 0^-} f(x) = -\infty$$

and

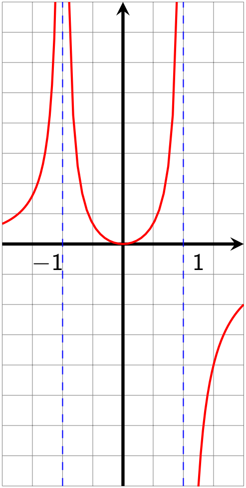
$$\lim_{x \to 0^+} f(x) = +\infty$$



At left the the graph of a function $f(x)$. Find the following infinite limits or say that they don't exist:

1. $\lim_{x\to -1} f(x)$
2. $\lim_{x\to 1^-} f(x)$
3. $\lim_{x\to 1^+} f(x)$
4. $\lim_{x\to 1} f(x)$

Notice that the lines $x = -1$ and $x = +1$ are both *vertical asymptotes* for the function f



At left the the graph of a function $f(x)$. Find the following infinite limits or say that they don't exist:

1. $\lim_{x\to -1} f(x)$ $+\infty$
2. $\lim_{x\to 1^-} f(x)$ $+\infty$
3. $\lim_{x\to 1^+} f(x)$ $-\infty$
4. $\lim_{x\to 1} f(x)$ DNE

Notice that the lines $x = -1$ and $x = +1$ are both *vertical asymptotes* for the function f

The Limit Laws

Peter A. Perry, Posing as Ben Braun

University of Kentucky

January 27, 2016

Bill of Fare

Why the Limit Laws?

Remember the limit in the Falling Body Problem?

We found that the average velocity between $t = 1$ and $t = 1 + h$ seconds is

$$\frac{g(1+h) - g(1)}{h} = \frac{32h + 16h^2}{h} = 32 + 16h$$

The instantaneous velocity is

$$\lim_{h \to 0} (32 + 16h) = 32$$

Our goal is to develop a way of calculating limits given *any* expression involving h (or x, or $\ldots$)

A Big Long List of Rules to Know

If f and g are functions and c is a constant:

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x)$$

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

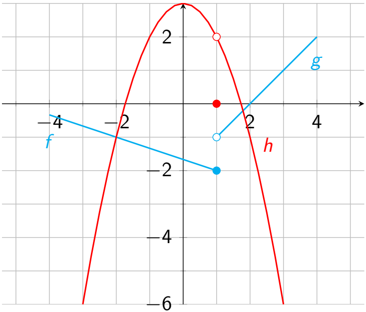
$$\text{provided } \lim_{x\to a} g(x) \neq 0$$

How on earth are you supposed to remember all these rules?

(1) By using them and

(2) by rephrasing them in English

Use The Rules!



For f, g and h as shown on the left, find:

1. $\lim_{x\to -2}\left(f(x)+3h(x)\right)$
2. $\lim_{x\to 1^-}\left(h(x)+2f(x)\right)$
3. $\lim_{x\to 0}\dfrac{f(x)}{h(x)}$
4. $\lim_{x\to 4}\left(3g(x)+2\right)$
5. $\lim_{x\to 1^+}\left(h(x)g(x)\right)$

Use The Rules!

If $\lim_{x\to 3} f(x) = -5$ and $\lim_{x\to 3} g(x) = 7$, find:

1. $\lim_{x\to 3} f(x) + 3g(x)$
2. $\lim_{x\to 3} \dfrac{f(x)}{g(x)+1}$
3. $\lim_{x\to 3} 4f(x) - 2g(x)$
4. $\lim_{x\to 3^-} f(x)g(x)$

Use The Rules!

If $\lim_{x\to 3} f(x) = -5$ and $\lim_{x\to 3} g(x) = 7$, find:

1. $\lim_{x\to 3} f(x) + 3g(x)$ 16
2. $\lim_{x\to 3} \dfrac{f(x)}{g(x)+1}$ $-5/8$
3. $\lim_{x\to 3} 4f(x) - 2g(x)$ -34
4. $\lim_{x\to 3^-} f(x)g(x)$ -35

Rephrase The Rules in English

$$\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

means "the limit of a sum is the sum of the limits" '

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

means "the limit of a difference is the difference of the limits"

$$\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

means

Rephrase The Rules in English

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

means "the limit of a sum is the sum of the limits" '

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

means "the limit of a difference is the difference of the limits"

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

means "the limit of a product is the product of the limits"

Another Big Long List of Rules to Know

If f is a function, and c is a constant:

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

How to Do Things with Rules

Let's find $\lim_{x\to 4}\left(x^2 - 4x + 5\right)$

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Now try...

$$\lim_{x\to 1}\frac{x^2+x-1}{\sqrt{1+x^2}} = \frac{\lim_{x\to 1}(x^2+x-1)}{\lim_{x\to 1}\sqrt{1+x^2}}$$

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Simplify Your Life!

You are also allowed to use algebra to *simplify* expressions before taking the limit. For instance:

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Now use the same technique to find:

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$$

Is the answer...

A. 6

B. 4

C. 0

D. -1

E. All of the above

Simplify Your Life by Complicating It

Find $\lim_{h\to 0} \dfrac{\sqrt{9+h}-3}{h}$ using the identity

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$$

$$\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} =$$

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$$= \lim_{h\to 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)}$$

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&= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3}
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&= \lim_{h\to 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{9+h}+3} \\
&= \frac{1}{6}
\end{aligned}$$

Now You Try It!

Find $\lim_{u \to 2} \dfrac{\sqrt{4u+1} - 3}{u - 2}$ using the identity

$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = a - b$$

with $a = \sqrt{4u+1}$ and $b = 9$

Can I Use the Shortcut Now?

The effect of the rules we've discussed is to guarantee that the following "shortcut method" can be used to compute limits:

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Can I Use the Shortcut Now?

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Find $\lim_{x \to 2} \dfrac{x^3 + x}{\sqrt[3]{2x^2}}$

A. 10/3

B. 8/3

C. 5

D. π

On the One hand... On the Other Hand...

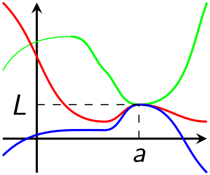
Theorem $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

Let

$$f(x) = \begin{cases} x^2 + 1 & x < 1 \\ (x-2)^2 + 1 & x \geq 1 \end{cases}$$

Does $\lim_{x\to 1} f(x)$ exist? Why or why not?

The Squeeze Theorem



The Squeeze Theorem If

$$f(x) \leq g(x) \leq h(x)$$

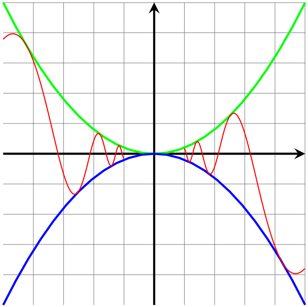
when x is near a (except possibly at $x = -a$) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

Everybody's Favorite Example of the Squeeze Theorem



y axis stretched by a factor of 4

Suppose that

$$g(x) = x^2 \sin(1/x)$$

Recall that $-1 \leq \sin\theta \leq 1$ so

$$-x^2 \leq x^2 \sin(1/x) \leq x^2$$

Find

$$\lim_{x \to 0} x^2 \sin(1/x)$$

Squeeze for Yourself

Prove that

$$\lim_{x \to 0} x^4 \cos \frac{2}{x} = 0.$$

You'll need to find functions $f(x)$ and $h(x)$ so that

$$f(x) \le x^4 \cos \frac{2}{x} \le h(x)$$

near $x = 0$, with the property that $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x)$.

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