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Ehrhart Theory for Lattice Polytopes

by

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Poetry is no less mysterious than the other elements of the orb. A lucky line here and there should not make us think any higher of ourselves, for such lines are the gift of Chance or the Spirit; only the errors are our own. I hope the reader may find in my pages something that merits being remembered; in this world, beauty is so common.

In Praise Of Darkness

Jorge Luis Borges

Chapter 1

Introduction

> The best way to teach real mathematics, I believe, is to start deeper down, with the elementary ideas of number and space. ...in fact, arithmetic, algebra, and geometry can never be outgrown...by maintaining ties between these disciplines, it is possible to present a more unified view of mathematics, yet at the same time to include more spice and variety.
>
> *Numbers and Geometry*
>
> John Stillwell

1.1 Overture

This dissertation is a study of the roots and coefficients of Ehrhart polynomials, a class of polynomials that count lattice points in integral dilates of lattice polytopes. In Chapter 2, we discuss lattice polytopes and the theory of Ehrhart polynomials, including connections with commutative algebra. Chapter 3 presents results regarding the roots of Ehrhart polynomials, including some joint work with Mike Develin of the American Institute of Mathematics. We end with Chapter 4, where we focus on reflexive lattice polytopes and show that the coefficients of their Ehrhart polynomials behave nicely with respect to the free sum operation. Along the way, we will encounter many connections between discrete geometry, combinatorics, and algebra. In Chapters 2-4, we assume our reader is at the mathematical level of a beginning graduate student.

The remainder of this chapter is a gentle introduction to counting lattice points in lattice polygons, i.e. Ehrhart theory in dimension two. We write this section with the mathematically untrained reader in mind, the reader for whom later chapters might be intimidating or unfamiliar. Yet fear not the higher dimensions, for they contain endless beauty!

1.2 Ehrhart Theory in Dimension Two

We begin with an example. Let $\mathbb{R}^2$ denote a plane, where points in the plane are given by pairs (a, b) where a and b are both real numbers.

Example 1.1. Say that P_5 is the pentagon in the plane shown in Figure 1.1.

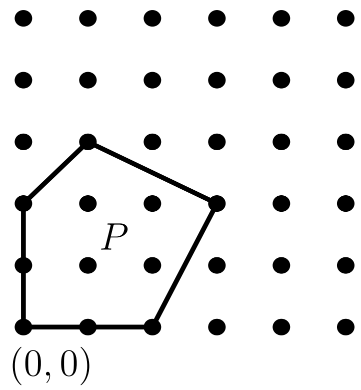


Figure 1.1: A pentagon P_5.

The corners of P_5 all have integer coordinates. The lower left corner is $(0, 0)$, the next corner clockwise is $(0, 2)$, the next clockwise is $(1, 3)$, then $(3, 2)$, and finally

$(2, 0)$. Coordinates with integer entries are very important, so they earn the following designation.

Definition 1.2. *A* lattice point *is a point* (a, b) *in the plane with* a *and* b *integers.*

The following properties of P_5 are easy to verify:

1. P_5 has area $\frac{13}{2}$.

2. P_5 has 7 lattice points on its boundary.

3. P_5 contains 11 lattice points (including those on its boundary).

These numbers are related in a curious way. In particular,

$$11 = \frac{13}{2} + \frac{7}{2} + 1,$$

that is, the number of lattice points in P_5 is equal to the area of P_5 plus one-half the number of boundary lattice points plus one. In the late 1800's, G. Pick discovered that this is true for any convex lattice polygon. To make his statement precise, we need to introduce some terms and notation.

Definition 1.3. *A* convex lattice polygon *is a convex polygon* P *in* $\mathbb{R}^2$ *whose corners are lattice points.*

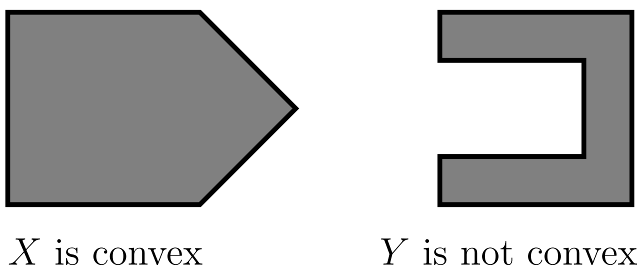


Figure 1.2: Convex and non-convex sets.

Being convex means that any two points in P are connected by a straight line segment contained in P. For example, Figure 1.2 provides examples of a convex region X and a non-convex region Y in $\mathbb{R}^2$. No matter which two points in X are chosen, the straight line segment between them stays in X. However, if two points in separate "arms" of Y are chosen, the straight line segment between them leaves Y before returning to it.

Definition 1.4. *The polygon tP is the polygon resulting from scaling P by a factor of t.*

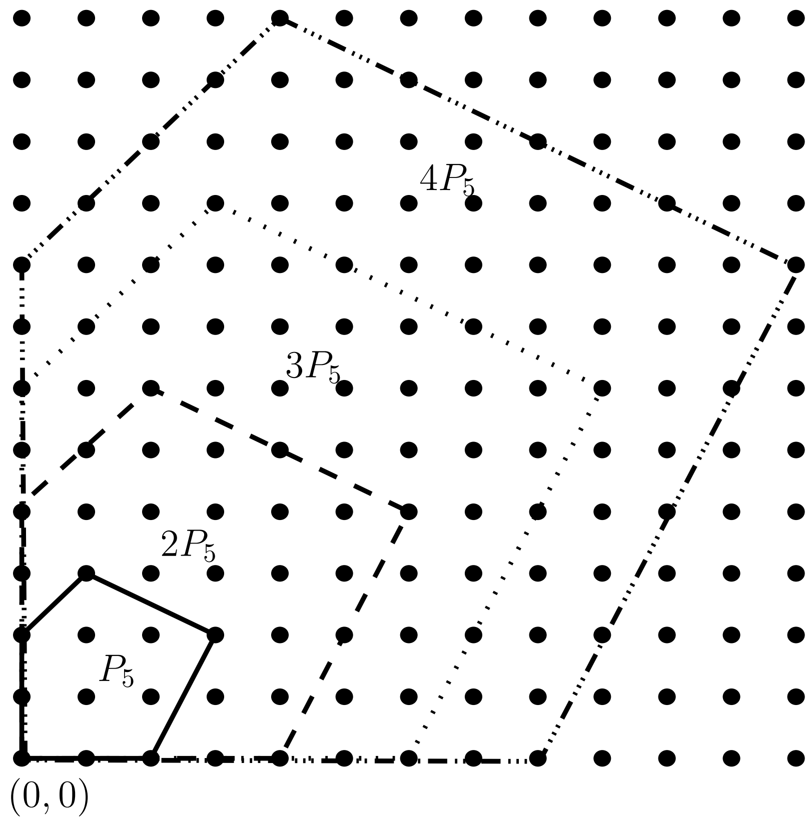


Figure 1.3: P_5 scaled by $2, 3$ and 4.

Definition 1.5. *For a convex lattice polygon, let $A(P)$ denote the area of P, $L_P(t)$ denote the number of lattice points in tP (including boundary points), and $B(P)$ denote the number of lattice points on the boundary of P.*

Theorem 1.6. (Pick's Theorem) *For any convex lattice polygon,*

$$L_P(1) = A(P) + \frac{B(P)}{2} + 1 \tag{1.1}$$

.

Pick's theorem is a gem, and it has a beautiful extension that was discovered by E. Ehrhart in the 1960's.

Theorem 1.7. (Ehrhart's Theorem in dimension 2, see [4]) *Let P be a convex lattice polygon and let t be a positive integer. The following equality always holds:*

$$L_P(t) = A(P)t^2 + \frac{B(P)}{2}t + 1. \tag{1.2}$$

We illustrate this theorem with several examples, for which the interested reader is encouraged to draw pictures of the lattice polygons and verify our equations for small values of t.

Example 1.8. For our earlier example P_5, we have

$$L_{P_5}(t) = \frac{13}{2}t^2 + \frac{7}{2}t + 1.$$

Example 1.9. Let Δ_2 be the polygon with corners at $(0,0)$, $(1,0)$ and $(0,1)$. Then $A(\Delta_2) = \frac{1}{2}$ and $B(\Delta_2) = 3$, thus

$$L_{\Delta_2}(t) = \frac{1}{2}t^2 + \frac{3}{2}t + 1.$$

Example 1.10. Let $\Diamond_2$ be the polygon with corners at $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$. Then $A(\Diamond_2) = 2$ and $B(\Diamond_2) = 4$, thus

$$L_{\Diamond_2}(t) = 2t^2 + 2t + 1.$$

Example 1.11. Let $\square_2$ be the polygon with corners at $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$. Then $A(\square_2) = 4$ and $B(\square_2) = 8$, thus

$$L_{\square_2}(t) = 4t^2 + 4t + 1.$$

We will see higher-dimensional versions of Examples 1.9-1.11 in the next chapter.

One amazing aspect of Ehrhart's theorem is that if we know the area of P and the number of lattice points on the boundary of P, we have enough information to count the number of lattice points in tP for any integer t. We get so much from so little! In honor of Ehrhart, the polynomial in (1.2) is called the *Ehrhart polynomial* of P. Pick's theorem is obtained from Ehrhart's theorem by setting t equal to one.

There are a number of interesting observations to make about $L_P(t)$. The coefficients of $L_P(t)$ come from the geometry of P. The leading term is the area of P, which is simple enough. The second coefficient is half of the "length" of the boundary of P, if we measure the length of an edge as the number of intervals between lattice points and observe that there is one such interval for each lattice point on the boundary. The constant term is always one, which is forced on us by Pick's theorem. Also, the degree of $L_P(t)$ is always two, since the area of a convex polygon is never zero; we will see later that this corresponds to P being a two dimensional object (as opposed to a line segment or a cube which are one and three dimensional, respectively).

These observations are even more interesting when one begins with the sequence $\{L_P(1), L_P(2), L_P(3), \ldots\}$ instead of the polynomial. This sequence records the number of lattice points for every positive integral dilate of P. Ehrhart's theorem says that this sequence is expressible as a polynomial evaluated at positive integers. We then discover that this "counting" polynomial has coefficients and degree reflecting geometric properties of P, properties which at first glance do not seem related to counting lattice points. Ehrhart's general theorem is remarkable in that it extends this point of view to higher dimensions when we study lattice polytopes, the higher-dimensional analogues of lattice polygons. In the next chapter, we will see how one can start with the sequence recording the number of lattice points in positive integral dilates of lattice polytopes and end up with a higher degree Ehrhart polynomial sharing many of the properties of Ehrhart polynomials of degree two.

Chapter 2

Lattice Polytopes and Ehrhart Theory

In this chapter we discuss the basics of convex lattice polytopes and Ehrhart theory as well as connections with commutative algebra. There are a variety of excellent sources from which we draw this material, in particular [4], [13], [25], and others noted in the text.

2.1 Convex Polytopes

We begin with the definition of convexity.

Definition 2.1. *A subset X of $\mathbb{R}^n$ is* convex *if $(1-t)x + ty \in X$ for every $x, y \in X$ and $t \in [0,1]$. Given a finite set C in $\mathbb{R}^n$, the* convex hull *of C, denoted* $\operatorname{conv}\{C\}$ *is the intersection of all convex sets containing C. Equivalently,*

$$\operatorname{conv}\{C\} = \left\{ \sum_{c \in C} \alpha_c c : \alpha_c \in \mathbb{R}_{\geq 0}, \sum_{c \in C} \alpha_c = 1 \right\}.$$

The mathematical objects we are interested in are convex polytopes. There are two ways to define a general convex polytope in $\mathbb{R}^n$, the first being the "halfspace" description.

Definition 2.2. *A* halfspace *in $\mathbb{R}^n$ is the set of solutions to a linear inequality of the form $a \cdot x \leq b$ for some $a \in \mathbb{R}^n, b \in \mathbb{R}$.*

Definition 2.3. *An* affine subspace $A \subseteq \mathbb{R}^n$ *of dimension d is a translate by some fixed $y \in \mathbb{R}^n$ of a d-dimensional linear subspace of $\mathbb{R}^n$.*

Definition 2.4. *Given a subset $V \subseteq \mathbb{R}^n$, the* affine span *of V,* $\operatorname{aff}(V)$*, is the intersection of all affine subspaces of $\mathbb{R}^n$ containing V.*

It is an elementary exercise to see that the affine span of a subset of $\mathbb{R}^n$ is an affine subspace. The boundary of any halfspace is clearly an affine subspace. Figure 2.1 shows the halfspace $H = \{x + y \leq 1 : (x, y) \in \mathbb{R}^2\}$ whose boundary is the 1-dimensional affine subspace $A = \{(x, -x) + (0, 1) : x \in \mathbb{R}\} = \operatorname{aff}(\{(0, 1), (1, 0)\})$.

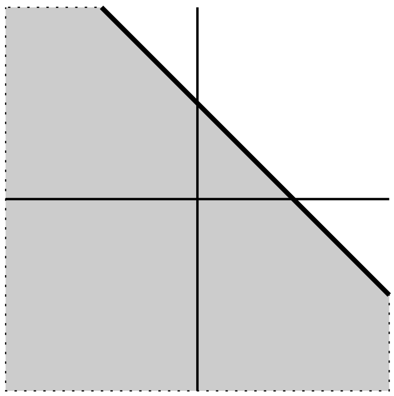


Figure 2.1: $H = \{x + y \leq 1 : (x, y) \in \mathbb{R}^2\}$.

Definition 2.5. *An $\mathcal{H}$*-polytope *of dimension d in $\mathbb{R}^n$ is a bounded intersection P of a finite number of halfspaces in $\mathbb{R}^n$ such that P has affine span of dimension d.*

The second definition of a convex polytope is the "vertex" description; the meaning of the word "vertex" will be explained in Lemma 2.9.

Definition 2.6. *A* $\mathcal{V}$-polytope *of dimension d in $\mathbb{R}^n$ is the convex hull P of a finite set of points $V \subset \mathbb{R}^n$ such that P has affine span of dimension d.*

The theory of $\mathcal{H}$- and $\mathcal{V}$-polytopes are linked by the following classical theorem.

Theorem 2.7. *Any $\mathcal{V}$-polytope is also an $\mathcal{H}$-polytope. Similarly, any $\mathcal{H}$-polytope is also a $\mathcal{V}$-polytope.*

Proofs of this theorem may be found many places in the literature, see Chapter 5 of [17] and Chapter 1 of [25] for two different approaches. It is extremely valuable theoretically to be able to pass from one description of a convex polytope to the other, though in practice computing an $\mathcal{H}$-description of a polytope from a $\mathcal{V}$-description and vice versa is a challenging problem. For our purposes, however, we only need to know that these descriptions are equivalent. For the rest of this dissertation, the word *polytope* will be used without the $\mathcal{H}$- and $\mathcal{V}$- modifiers.

A nice property of polytopes is that their topological boundaries are made of up polytopes of smaller dimension called the *faces* of P.

Definition 2.8. *A linear inequality $a \cdot x \leq b$, where $a \in \mathbb{R}^n, b \in \mathbb{R}$, is* valid *for P if it is satisfied for all points $x \in P$. A* face *of P is any set of the form*

$$F = P \cap \{x \in \mathbb{R}^n : a \cdot x = b\}$$

where $a \cdot x \leq b$ is a valid inequality for P. The dimension *of a face is the dimension of its affine span.*

Given a d-dimensional polytope P, the faces of dimensions $0, 1$ and $d-1$ are called the *vertices, edges,* and *facets* of P, respectively. The vertices of P are particularly special, due to the following lemma.

Lemma 2.9. *Any polytope P is the convex hull of its vertices. Further, if $P =$ $\operatorname{conv}\{V\}$, then V contains the vertices of P.*

Thus, in Definition 2.6, the finite set V may be taken to be the vertices of P.

Example 2.10. Let e_i denote the i^{th} standard basis vector in $\mathbb{R}^d$. The *standard d-simplex* is

$$\Delta_d := \operatorname{conv}\{0, e_1, \dots, e_d\} = \left\{ x \in \mathbb{R}^d : x \cdot e_i \geq 0, \sum_{i=1}^{d} x \cdot e_i \leq 1 \right\}.$$

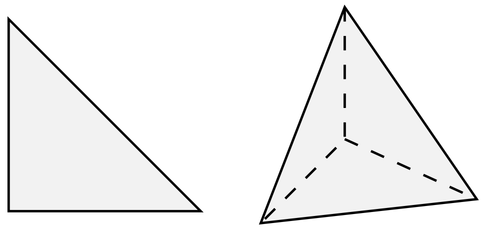


Figure 2.2: The 2- and 3-dimensional standard simplices.

By examining Figure 2.2, one can see that Δ_2 has three vertices and three edges while Δ_3 has four vertices, six edges, and four facets.

Example 2.11. The d-dimensional *crosspolytope* is

$$\Diamond_d := \operatorname{conv}\{\pm e_1, \dots, \pm e_d\} = \left\{ x \in \mathbb{R}^d : x \cdot v \leq 1, \text{ for all } v \in \{\pm 1\}^d \right\}.$$

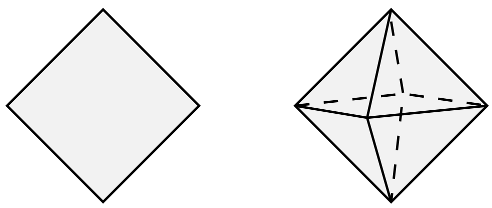


Figure 2.3: The 2- and 3-dimensional crosspolytopes.

By examining Figure 2.3, one can see that $\Diamond_2$ has four vertices and four edges while $\Diamond_3$ has six vertices, twelve edges, and eight facets.

Example 2.12. The d-dimensional *cube* is

$$\square_d := \operatorname{conv}\left\{x \in \mathbb{R}^d : x \in \{\pm 1\}^d\right\} = \left\{x \in \mathbb{R}^d : -1 \leq x \cdot e_i \leq 1, 1 \leq i \leq d\right\}.$$

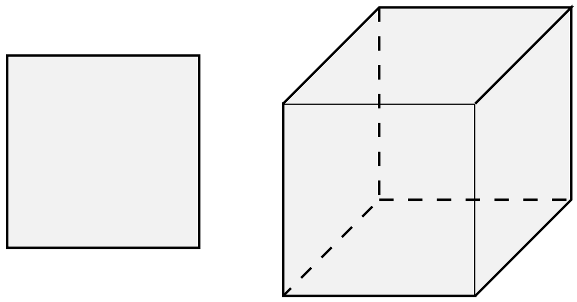


Figure 2.4: The 2- and 3-dimensional cubes.

By examining Figure 2.4, one can see that $\square_2$ has four vertices and four edges while $\square_3$ has eight vertices, twelve edges, and six facets.

A fundamental construction in the theory of convex polytopes is that of the cone over a polytope.

Definition 2.13. *Given a polytope P in $\mathbb{R}^n$, the* cone over P*, denoted $\mathcal{C}(P)$, is constructed as follows. Set $(P,1) := \{(p,1) : p \in P\} \subset \mathbb{R}^{n+1}$ and define $\mathcal{C}(P) := \{\sum_{i=0}^m \alpha_i p_i : m \in \mathbb{N}, \alpha_i \in \mathbb{R}_{\geq 0}, p_i \in (P,1)\}$. Thus, $\mathcal{C}(P)$ is the set of all non-negative real linear combinations of elements of $(P,1)$.*

Figure 2.5 shows the cone in $\mathbb{R}^3$ over the simplex Δ_2.

While convex polytopes have been studied since antiquity, many mathematicians have recently focused on the following class of convex polytopes.

Definition 2.14. *If $P = \operatorname{conv}\{V\}$, where $V \subset \mathbb{Z}^n$, then P is called a* lattice polytope.

Note that all of the polytopes in Examples 2.10-2.12 are lattice polytopes while, for example, $\operatorname{conv}\left\{0, \frac{1}{2}\right\} \subset \mathbb{R}$ is not. Lattice polytopes will be the focus of this

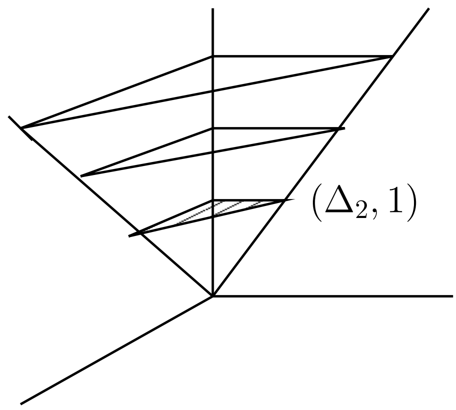


Figure 2.5: The cone over Δ_2.

dissertation, our approach being from the combinatorial and geometric viewpoint of Ehrhart theory. Our discussions will also highlight some connections between lattice polytopes and commutative algebra.

2.2 Rational Generating Functions

For our foray into Ehrhart theory we will need a basic fact about rational generating functions of a single variable. While we will only need one crucial lemma, the subject of generating functions forms an extensive part of combinatorics and the interested reader should see Chapter 4 of [23] for more details as well as the proof of Lemma 2.16.

Definition 2.15. *The generating function for a sequence $\{a(t) : t \in \mathbb{Z}_{\geq 0}\}$ of complex values is the power series*

$$\sum_{t \in \mathbb{Z}_{\geq 0}} a(t)x^t \in \mathbb{C}[[x]].$$

Lemma 2.16. *A function $f(t) : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree d if and only if* *there exist complex values h_j^* so that*

$$\frac{\sum_{j=0}^{d} h_j^* x^j}{(1-x)^{d+1}} = \sum_{t\in\mathbb{Z}_{\geq 0}} f(t)x^t, \quad \text{and} \quad \sum_j h_j^* \neq 0.$$

Example 2.17. Let $f(t) = 4t^2 + 4t + 1$. Then

$$\begin{aligned}
\sum_{t\in\mathbb{Z}_{\geq 0}} f(t)x^t =& 4\sum_{t\in\mathbb{Z}_{\geq 0}} t^2x^t + 4\sum_{t\in\mathbb{Z}_{\geq 0}} tx^t + \sum_{t\in\mathbb{Z}_{\geq 0}} x^t \\
=& 4(x\frac{d}{dx})^2 \sum_{t\in\mathbb{Z}_{\geq 0}} x^t + 4(x\frac{d}{dx}) \sum_{t\in\mathbb{Z}_{\geq 0}} x^t + \sum_{t\in\mathbb{Z}_{\geq 0}} x^t \\
=& 4(x\frac{d}{dx})^2 \frac{1}{1-x} + 4(x\frac{d}{dx}) \frac{1}{1-x} + \frac{1}{1-x} \\
=& \frac{4(x^2+x)}{(1-x)^3} + \frac{4(x-x^2)}{(1-x)^3} + \frac{(1-x)^2}{(1-x)^3} \\
=& \frac{1+6x+x^2}{(1-x)^3}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\frac{1+6x+x^2}{(1-x)^3} =& (1+6x+x^2)\sum_{t\in\mathbb{Z}_{\geq 0}} \binom{t+2}{2} x^t \\
=& \sum_{t\in\mathbb{Z}_{\geq 0}} \left(\binom{t+2}{2} + 6\binom{t+1}{2} + \binom{t}{2} \right) x^t \\
=& \sum_{t\in\mathbb{Z}_{\geq 0}} \left(\frac{(t+2)(t+1)}{2} + \frac{6(t+1)t}{2} + \frac{t(t-1)}{2} \right) x^t \\
=& \sum_{t\in\mathbb{Z}_{\geq 0}} (4t^2+4t+1)x^t.
\end{aligned}$$

As suggested by our computations in Example 2.17, one consequence of Lemma 2.16 is that any degree d polynomial $f(t)$ can be expressed as

$$f(t) = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d},$$

where

$$\binom{t+d-j}{d} := \frac{\prod_{i=0}^{d-1}(t+d-j-i)}{d!}$$

and the h_j^*-coefficients are those in the numerator of the rational generating function for $f(t)$. This is easily seen by using the geometric series identity

$$\frac{1}{(1-x)^{d+1}} = \sum_{t\in\mathbb{Z}_{\geq 0}} \binom{t+d}{d} x^t$$

to expand the rational function for $f(t)$ as a formal power series as in Example 2.17. Thus, the rational generating function for $f(t)$ encodes the change of coefficients for $f(t)$ from the standard monomial basis

$$\left\{1, t, t^2, \ldots, t^d\right\}$$

to the basis

$$B_d := \left\{ \binom{t+d-j}{d} : 0 \leq j \leq d \right\}.$$

2.3 Ehrhart Theory

The idea of Ehrhart theory is to study convex lattice polytopes by studying the number of lattice points in integral dilates of P, hence the following definition.

Definition 2.18. *For P, a dimension d lattice polytope in $\mathbb{R}^n$, set $L_P(t) := |tP \cap \mathbb{Z}^n|$ for $t \in \mathbb{Z}_{\geq 1}$. The* Ehrhart series of P *is*

$$\mathrm{Ehr}_P(x) := 1 + \sum_{t\in\mathbb{Z}_{\geq 1}} L_P(t) x^t.$$

Example 2.19. Recall that $\Delta_d := \left\{ x \in \mathbb{R}^d : x \cdot e_i \geq 0, \sum_{i=1}^d x \cdot e_i \leq 1 \right\}$, and thus $t\Delta_d = \left\{ x \in \mathbb{R}^d : x \cdot e_i \geq 0, \sum_{i=1}^d x \cdot e_i \leq t \right\}$. It is a well known exercise in combinatorics that the number of non-negative integral solutions to $\sum_{i=1}^{d} x \cdot e_i \leq t$ is $\binom{t+d}{d}$ for positive integers t. Thus,

$$L_{\Delta_d}(t) = \binom{t+d}{d}$$

and hence

$$\mathrm{Ehr}_{\Delta_d}(x) = \frac{1}{(1-x)^{d+1}}.$$

Example 2.20. Recall that

$$\Diamond_d := \left\{x \in \mathbb{R}^d : x \cdot v \leq 1, \text{ for all } v \in \{\pm 1\}^d\right\} = \left\{x \in \mathbb{R}^d : \sum_i |x \cdot e_i| \leq 1\right\}.$$

Thus,

$$t\Diamond_d = \left\{x \in \mathbb{R}^d : \sum_i |x \cdot e_i| \leq t\right\}.$$

Counting the number of lattice points in $t\Diamond_d$ is another combinatorial exercise, which we can interpret geometrically: Consider the case $d = 2$. We group the integral solutions to $|x| + |y| \leq t$ by the number of coordinates that are positive. See Figure 2.6 for the grouping with $t = 4$.

The number of integral solutions $(x, y) \in \mathbb{Z}^2_{\geq 1}$ to $|x|+|y| \leq t$ is $\binom{t}{2}$. The number of integral solutions $(x, y) \in (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\leq 0}) \bigcup (\mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 1})$ to $|x|+|y| \leq t$ is $2\binom{t+1}{2}$. Finally, the number of integral solutions $(x, y) \in \mathbb{Z}^2_{\leq 0}$ to $|x| + |y| \leq t$ is $\binom{t+2}{2}$. As these sets partition the set of integral solutions, we have shown that $L_{\Diamond_2}(t) = \sum_{j=0}^{2} \binom{2}{j}\binom{t+2-j}{2}$. This line of argument generalizes to show that

$$L_{\Diamond_d}(t) = \sum_{j=0}^{d} \binom{d}{j}\binom{t+d-j}{d},$$

hence

$$\mathrm{Ehr}_{\Diamond_d}(x) = \frac{(1+x)^d}{(1-x)^{d+1}}.$$

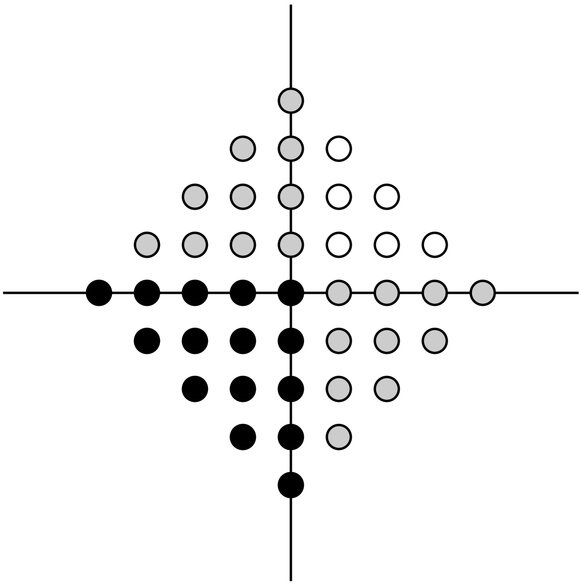


Figure 2.6: Our grouping of the solutions to $|x| + |y| \leq 4$.

Example 2.21. Recall that $\square_d = \left\{x \in \mathbb{R}^d : -1 \leq x \cdot e_i \leq 1, 1 \leq i \leq d\right\}$, hence $t\square_d = \left\{x \in \mathbb{R}^d : -t \leq x \cdot e_i \leq t, 1 \leq i \leq d\right\}$. It is immediate that

$$L_{\square_d}(t) = (2t+1)^d.$$

Computing the Ehrhart series for $\square_d$ for every d is somewhat complicated, so we omit the computation. Example 2.17 illustrated the computation of $\mathrm{Ehr}_{\square_2}(x)$, as $L_{\square_2}(t) = 4t^2 + 4t + 1$.

The examples above are specific instances of the following theorem and its corollary.

Theorem 2.22. (Ehrhart, [10]) *Let P be a lattice polytope of dimension d in $\mathbb{R}^n$. There exist complex values h_j^*, $0 \leq j \leq d$, such that the Ehrhart series for P is a rational generating function of the following form:*

$$\mathrm{Ehr}_P(x) = \frac{\sum_{j=0}^{d} h_j^* x^j}{(1-x)^{d+1}}, \sum_j h_j^* \neq 0. \tag{2.1}$$

Corollary 2.23. *$L_P(t)$ is expressible as a polynomial of degree d in the variable t.*

Definition 2.24. *The polynomial in Corollary 2.23 is called the* Ehrhart polynomial of P *and also denoted by $L_P(t)$. The vector $(h_0^*, h_1^*, \dots, h_d^*)$ of coefficients of the numerator of* $\mathrm{Ehr}_P(x)$ *is called the* h-star vector *for P.*

Textbook presentations of the proof of Theorem 2.22 may be found in [4] and [13]. The standard coefficients of $L_P(t)$ and the h^*-vector for P have received a lot of attention because of their connections with the geometry of P, as demonstrated by the following theorem.

Definition 2.25. *Given a d-dimensional lattice polytope $P \subset \mathbb{R}^n$, the* relative volume *of a facet F of P is*

$$relvol(F) := \lim_{t\to\infty} \frac{1}{t^{d-1}} |tF \cap \mathbb{Z}^n|.$$

Theorem 2.26. (see [4]) *If $P \subset \mathbb{R}^d$ is a d-dimensional lattice polytope and $L_P(t) = \sum_{k=0}^{d} c_k t^k = \sum_{j=0}^{d} h_j^* \binom{t+d-j}{d}$, then:*

(i) $c_d = vol(P)$.

(ii) $c_{d-1} = \frac{1}{2}\sum_F relvol(F)$, *where the sum is over facets F of P.*

(iii) $c_0 = 1$.

(iv) $h_0^* = 1$.

(v) $h_1^* = L_P(1) - d - 1$.

(vi) $h_d^* = L_P(1) - |\mathbb{Z}^d \cap \partial P|$.

Parts *i-iii* of Theorem 2.26 generalize the coefficient behavior seen in (1.1) to higher dimensions. A major goal of current research, and one motivating question for this dissertation, is to understand what geometric and combinatorial data the other Ehrhart polynomial coefficients and h^*-coefficients of P might encode.

Arguably the most important theorem in Ehrhart theory, after Ehrhart's original theorem, is the non-negativity theorem of R. Stanley.

Theorem 2.27. (Stanley's Non-negativity Theorem, [21]) *For any d-dimensional lattice polytope P, the h^*-vector of P satisfies $(h_0^*, \ldots, h_d^*) \in (\mathbb{Z}_{\geq 0})^{d+1}$.*

The non-negativity of the h^*-vector for P is a driving force behind many of the current results regarding Ehrhart polynomials, including the content of Chapter 3.

The final major theorem about Ehrhart series and polynomials that we will mention is a reciprocity theorem.

Definition 2.28. *For a d-dimensional polytope P in $\mathbb{R}^n$, the* relative interior *of P, denoted P°, is the interior of P with respect to the embedding of P into its affine span, in which P is full dimensional. Set $L_{P^\circ}(t) := |tP^\circ \cap \mathbb{Z}^n|$ and $\mathrm{Ehr}_{P^\circ}(x) := \sum_{t\in\mathbb{Z}_{\geq 1}} L_{P^\circ}(t)x^t$.*

Theorem 2.29. (Ehrhart-Macdonald Reciprocity, [16]) *For any dimension d lattice polytope P, the following (equivalent) equalities hold:*

$$L_{P^\circ}(t) = (-1)^d L_P(-t)$$

and

$$\mathrm{Ehr}_P(\frac{1}{x}) = (-1)^{d+1}\mathrm{Ehr}_{P^\circ}(x).$$

Ehrhart-Macdonald reciprocity is interesting because it demonstrates a connection between the topology of P and $L_P(t)$. The passage from a polytope to its interior is algebraically captured by changing "$\leq$"-signs to "$<$"-signs in the defining inequalities for the polytope. Ehrhart-Macdonald Reciprocity is crucial for our understanding of the reflexive polytopes discussed in Chapter 4.

2.4 Affine Semigroup Algebras

In this section we highlight some connections between Ehrhart theory and commutative algebra. The primary algebraic objects of interest for our purposes are affine semigroup algebras. We begin with definitions of semigroups and semigroup algebras.

Definition 2.30. *A* semigroup *is a set with an associative binary operation. An* affine semigroup *is a semigroup containing an identity element which is finitely generated and can be embedded in* $\mathbb{Z}^n$ *for some* $n \in \mathbb{N}$.

Example 2.31. The set $\{3, 4, 5, \ldots\}$ is a semigroup under addition. However, it is not an affine semigroup as it lacks an identity element.

The most important affine semigroups for our purposes are those arising as a result of the coning operation.

Lemma 2.32. *Given a lattice polytope* $P \subset \mathbb{R}^n$*, the set* $\mathbb{Z}^{n+1} \cap \mathcal{C}(P)$ *is an affine semigroup.*

The proof of Lemma 2.32 relies on the fact that $\mathcal{C}(P)$ is closed under addition and on Gordon's Lemma, which implies finite generation and can be found in Chapter X of [13].

Definition 2.33. *A vector space* A *over* $\mathbb{C}$ *is called a* (commutative) $\mathbb{C}$-algebra *if* A *has a multiplicative structure such that for every* $x, y, z \in A$ *and* $\alpha, \beta \in \mathbb{C}$*, we have*

(i) $xy = yx$;

(ii) $x(yz) = (xy)z$;

(iii) $x(y + z) = xy + xz$;

(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

(v) $\alpha(\beta x) = (\alpha\beta)x$.

A is called an $\mathbb{N}$-graded $\mathbb{C}$-algebra *if A has a decomposition $A = \oplus_{t\in\mathbb{Z}_{\geq 0}} A_t$ as a vector space over $\mathbb{C}$ such that $A_0 = \mathbb{C}$ and $A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 0$.*

Example 2.34. The polynomial ring $\mathbb{C}[x_1,\ldots,x_n]$ is an $\mathbb{N}$-graded $\mathbb{C}$-algebra, where A_i consists of the linear span of all monomials with exponent entries summing to i. This is called the *standard* grading on $\mathbb{C}[x_1,\ldots,x_n]$.

Definition 2.35. *Given an affine semigroup Γ, the* affine semigroup algebra associated with Γ *is the algebra $\mathbb{C}[\Gamma] = \left\{\sum_{\gamma\in G} c_\gamma\gamma : G \subset \Gamma, |G| < \infty, c_\gamma \in \mathbb{C}\right\}$ with multiplication defined by*

$$\left(\sum_{\gamma\in G} c_\gamma\gamma\right)\left(\sum_{\xi\in H} c_\xi\xi\right) = \sum_{\substack{\gamma\in G\\ \xi\in H}} c_\gamma c_\xi \gamma\xi.$$

A convenient way to think of $\mathbb{C}[\Gamma]$ is the following. If Γ is presented as a sub-semigroup of $\mathbb{Z}^n$, then $\mathbb{C}[\Gamma]$ is isomorphic to a sub-algebra of the Laurent polynomial algebra $\mathbb{C}[x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}]$, where the lattice points of the semigroup and the monomials in the algebra are in correspondence via

$$(a_1,\ldots,a_n) \leftrightsquigarrow x_1^{a_1}\cdots x_n^{a_n}.$$

Definition 2.36. *Given a lattice polytope $P \subset \mathbb{R}^n$, the* Ehrhart algebra of P *is the affine semigroup algebra $\mathbb{C}[P] := \mathbb{C}[\mathbb{Z}^{n+1} \cap \mathcal{C}(P)]$.*

The Ehrhart algebra of P is an $\mathbb{N}$-graded $\mathbb{C}$-algebra where $\mathbb{C}[P]_i$ is defined to be the $\mathbb{C}$-linear span of

$$\left\{(a_1,\ldots,a_{n+1}) \in \mathbb{Z}^{n+1} \cap \mathcal{C}(P) : a_{n+1} = i\right\}.$$

One key observation of this section is a connection between the Ehrhart series for P and the Hilbert series of the Ehrhart algebra of P.

Definition 2.37. *The* Hilbert series *of a finitely generated* $\mathbb{N}$*-graded* $\mathbb{C}$*-algebra* $A = \oplus_{t \in \mathbb{Z}_{\geq 0}} A_t$ *is* $H(A; x) := \sum_{t=0}^{\infty} dim_{\mathbb{C}}(A_t)x^t$.

Lemma 2.38. *For any convex lattice polytope* P, $H(\mathbb{C}[P]; x) = \mathrm{Ehr}_P(x)$.

The proof of this lemma is straightforward, as the lattice points at height t in the cone $\mathcal{C}(P)$ are in one-to-one correspondence with the monomials in the t^{th} graded component of $\mathbb{C}[P]$ and the polytope $\mathcal{C}(P) \cap (\mathbb{R}^n, t) \subset \mathbb{R}^{n+1}$ is a copy of tP. Thus, the study of Ehrhart polynomials and series can be viewed as a special case of the study of Hilbert polynomials and series of graded $\mathbb{C}$-algebras.

Example 2.39. Consider the standard two-dimensional simplex, i.e. the polytope $\Delta_2 = \mathrm{conv}\{(0,0), (1,0), (0,1)\}$. The cone over Δ_2 was depicted in Figure 2.5. It is not hard to see that

$$\mathbb{C}[\Delta_2] = \left\{ \sum \alpha_{a_1,a_2,t} x_1^{a_1} x_2^{a_2} x_3^t : a_1 + a_2 \leq t, \alpha_{a_1,a_2,t} \in \mathbb{C}, |\{\alpha_{a_1,a_2,t} \neq 0\}| < \infty \right\}. \quad (2.2)$$

The number of lattice points in $t\Delta_2$ is equal to the number of non-negative integer solutions to the equation $a_1 + a_2 \leq t$. This is also the number of monomials in the t^{th} graded component of $\mathbb{C}[\Delta_2]$. Finally, if we change variables in (2.2) and write $\theta_1 = x_1x_3, \theta_2 = x_2x_3$, and $\theta_3 = x_3$, then $\mathbb{C}[\Delta_2] \cong \mathbb{C}[\theta_1, \theta_2, \theta_3]$, where $\mathbb{C}[\theta_1, \theta_2, \theta_3]$ is endowed with the standard grading.

The realization of $\mathbb{C}[\Delta_2]$ as a polynomial ring in three variables is an instance of the most important algebraic fact regarding Ehrhart algebras, namely that they are always Cohen-Macaulay. We will now introduce the basic definitions and theorems regarding Cohen-Macaulay algebras, primarily following Chapters IV and X of [13].

Definition 2.40. *Given an* $\mathbb{N}$*-graded* $\mathbb{C}$*-algebra* A*, an element* $a \in A$ *is called* homogeneous of degree i *if* $a \in A_i$ *for some* i.

Definition 2.41. *We say that homogeneous elements* $a_1, \ldots, a_k \in A$ *are* algebraically independent *over* $\mathbb{C}$ *if for all* $f \in \mathbb{C}[x_1, \ldots, x_k]$ *such that* $f(a_1, \ldots, a_k) = 0$*, we have* $f(x_1, \ldots, x_k) = 0$ *as a polynomial.*

The first tool we need to introduce is the Noether Normalization Lemma, which states that any finitely generated $\mathbb{N}$-graded $\mathbb{C}$-algebra A contains a polynomial ring $\mathbb{C}[\theta_1, \ldots, \theta_d]$ such that A is a finitely generated $\mathbb{C}[\theta_1, \ldots, \theta_d]$-module.

Theorem 2.42. (Noether Normalization Lemma) *Let* $A = \oplus_{t \in \mathbb{Z}_{\geq 0}} A_t$ *be a finitely generated* $\mathbb{N}$*-graded* $\mathbb{C}$*-algebra. Then there exist a finite number of homogeneous elements* $\theta_1, \ldots, \theta_d$ *such that*

i) *the elements* $\theta_1, \ldots, \theta_d$ *are algebraically independent over* $\mathbb{C}$*;*

ii) *there exist a finite number of homogeneous elements* $\eta_1, \ldots, \eta_s$ *such that each element* $a \in A$ *can be expressed in the form*

$$a = \sum_{i=1}^{s} \eta_i p_i(\theta_1, \ldots, \theta_d), \tag{2.3}$$

where each $p_i(\theta_1, \ldots, \theta_d)$ *is a polynomial in* $\theta_1, \ldots, \theta_d$ *which depends on* a*.*

The sequence $\theta_1, \ldots, \theta_d$ is called a *system of parameters* for A. It can be shown that the number d, called the *Krull dimension of* A, is uniquely determined by A. It is also not hard to see that the Krull dimension of A is 0 if and only if $A_t = 0$ for all sufficiently large t. A system of parameters is called *regular* if, for some choice of $\eta_1, \ldots, \eta_s$ in 2.3, the coefficients $p_i(\theta_1, \ldots, \theta_d)$ in the expression (2.3) are unique for every a in A. The existence of regular systems of parameters is an important condition, earning the following special designation.

Definition 2.43. *A finitely generated* $\mathbb{N}$*-graded* $\mathbb{C}$*-algebra* A *is called* Cohen-Macaulay *if some system of parameters for* A *is regular.*

The Cohen-Macaulay condition implies that A is a free module over the polynomial ring, i.e. a direct sum of polynomial rings (not all necessarily generated by degree 1 elements).

Example 2.44. Consider the following sub-algebra of $\mathbb{C}[x]$:

$$\mathcal{P}_3 := \left\langle 1, x^3, x^4, x^5 \right\rangle,$$

where $\langle a, b, c, \ldots \rangle$ indicates the algebra generated by $a, b, c, \ldots$. Following the notation of the Noether Normalization Lemma, if we let $\theta_1 = x^3$, then $\mathcal{P}_3$ is a module over $\mathbb{C}[x^3]$ generated by $\{1, x^4, x^5\}$. If we set $\eta_1 = x^4, \eta_2 = x^5, \eta_3 = 1$, then each element of $\mathcal{P}_3$ has a unique representation in the form of (2.3). This can be seen by first noting that the elements of $\mathcal{P}_3$ are linear combinations of a constant term and monomials with exponent greater than or equal to 3. By looking at the residue classes of the exponents of such monomials modulo 3, we can group terms by residue class and hence get our unique expression. Thus, $\mathcal{P}_3$ is Cohen-Macaulay of Krull dimension one. Specifically,

$$\mathcal{P}_3 \cong_{\mathbb{C}[x^3]} \mathbb{C}[x^3] \oplus \mathbb{C}[x^3](-4) \oplus \mathbb{C}[x^3](-5),$$

where the notation indicates that the grading on the second and third summands is *increased* by 4 and 5, respectively. Note that this decomposition implies that the Hilbert series of $\mathcal{P}_3$ is

$$H(\mathcal{P}_3; z) = \frac{1 + z^4 + z^5}{(1 - z^3)}.$$

Examples 2.39 and 2.44 illustrate how a finitely generated Cohen-Macaulay algebra A can be represented as a direct sum of isomorphic polynomial rings, with the number of variables in the polynomial rings determined by the Krull dimension of A. As demonstrated in Example 2.44, the grading of each summand might be shifted. One may also consider polynomial rings where the variables are not all of degree 1.

Example 2.45. Consider the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$, where the degree of x_i is denoted by e_i. It is easy to see that the Hilbert series for $\mathbb{C}[x_1, \ldots, x_d]$ collapses as the following rational generating function:

$$H(\mathbb{C}[x_1, \ldots, x_d]; x) = \frac{1}{\prod_{i=1}^{d}(1 - x^{e_i})}. \tag{2.4}$$

Given that a Cohen-Macaulay algebra A is a direct sum of polynomial rings in d variables, one would expect that the Hilbert series for A would be a (possibly shifted) sum of Hilbert series of the form in (2.4). This indeed turns out to be the case.

Theorem 2.46. *Suppose that $A = \oplus_{t \in \mathbb{Z}_{\geq 0}} A_t$ is Cohen-Macaulay of Krull dimension d. Let $\theta_1, \ldots, \theta_d$ be a system of parameters with θ_i of degree $e_i > 0$. Then for some $s \geq 0$, the Hilbert series $H(A; x)$ is of the form*

$$H(A; x) = \frac{h_0 + h_1 x + h_2 x^2 + \cdots + h_s x^s}{\prod_{i=1}^{d}(1 - x^{e_i})},$$

with each $0 \leq h_j \in \mathbb{Z}$.

Note that if $e_i = 1$ for every i, and $s \leq d - 1$, then Lemma 2.16 implies that the sequence $\dim_{\mathbb{C}}(A_t)$ is given by a polynomial of degree d, called the Hilbert polynomial of A. The theorem linking the theory of Cohen-Macaulay algebras to Ehrhart theory is the following, essentially due to Hochster in [15].

Theorem 2.47. *For any lattice polytope P of dimension d, $\mathbb{C}[P]$ is Cohen-Macaulay of Krull dimension $d + 1$.*

Of the many consequences of the Cohen-Macaulay-ness of $\mathbb{C}[P]$, Stanley's non-negativity theorem is perhaps the most striking. A natural question to ask is how the behavior of the roots and coefficients for Ehrhart polynomials and the behavior of the roots and coefficients for Hilbert polynomials of Cohen-Macaulay algebras differ. We will address this question in the next two chapters.

Chapter 3

Roots of Ehrhart Polynomials

In this chapter, we will investigate the behavior of roots of Ehrhart polynomials of fixed degree. We will address two conjectures due to M. Beck, J. De Loera, M. Develin, J. Pfeifle, and R. Stanley, and answer one of their questions regarding the difference between the root behavior of Ehrhart polynomials and the root behavior of certain Hilbert polynomials. Some of the results in this chaper are joint with Mike Develin, as noted in the text.

3.1 Previous Results

The coefficients of Ehrhart polynomials have been of interest since Ehrhart first began his study of $L_P(t)$. Theorem 2.26 indicated some of the geometric and combinatorial properties of P encoded in the coefficients of $L_P(t)$. Recent research has focused on the roots of Ehrhart polynomials as well, partially in an attempt to better understand the coefficients of $L_P(t)$. There have been a variety of results suggesting that roots of Ehrhart polynomials have interesting and unique behavior, such as the following.

Theorem 3.1. (Bump, et. al., [9]) *All the roots of $L_{\Diamond_d}(t)$ have real part $-\frac{1}{2}$.*

The proof of Theorem 3.1 given in [9] was motivated by questions arising from investigations in analysis. Theorem 3.1 is also a corollary of the following result due to F. Rodriguez-Villegas, a general statement about roots of polynomials whose rational generating functions have numerators with unimodular roots.

Theorem 3.2. (Rodriguez-Villegas, [20]) *Set*

$$h_a(x) = \begin{cases} (x+1)(x+2)\cdots(x+a), & a \geq 1 \\ 1, & a < 1 \end{cases}.$$

If $f(t)$ is a degree d polynomial such that

$$\sum_{t \in \mathbb{Z}_{\geq 0}} f(t)x^t = \frac{U(x)}{(1-x)^{d+1}},$$

where $U(x)$ is a polynomial of degree $e \leq d$ and all the roots of $U(x)$ are on the unit circle, then $f(t)$ is of the form $f(t) = h_{d-e}(t)v(t)$ where $v(\alpha) = 0$ implies $\mathrm{Re}(\alpha) = -(d-e+1)/2$.

Theorem 3.1 is obtained as a corollary to Theorem 3.2 via

$$\mathrm{Ehr}_{\diamond_d}(x) = \frac{(1+x)^d}{(1-x)^{d+1}}, \tag{3.1}$$

since $\mathrm{Ehr}_{\diamond_d}(x)$ has $e = d$ and all the roots of $(1+x)^d$ have norm 1.

In [2], M. Beck, J. De Loera, M. Develin, J. Pfeifle, and R. Stanley produced the following theorem and conjecture.

Theorem 3.3. (Beck, et. al., [2]) *If P is a dimension d lattice polytope and $L_P(\alpha) = 0$, then $|\alpha| \leq 1 + (d+1)!$.*

Conjecture 3.4. (Beck, et. al., [2]) *If $L_P(t)$ is an Ehrhart polynomial of degree d, then the roots α_i of $L_P(t)$ satisfy $-d \leq \mathrm{Re}(\alpha_i) \leq d-1$ for all i.*

We will refer to Conjecture 3.4 as the *Vertical Strip Conjecture*, for obvious reasons. The motivation for this claim was experimental data produced from polytopes of relatively low dimensions, along with the fact that all real roots of a degree d Ehrhart polynomial lie in the interval $[-d, d-1]$ (as shown in [2]). Regarding Theorem 3.3, Beck, et. al. suggested that the bound $1+(d+1)!$ might be made polynomial in d, possibly quadratic, and asked whether their bound applied only to Ehrhart polynomials or to a wider class of Hilbert polynomials of Cohen-Macaulay algebras.

In this chapter we will provide an improved bound on the norm of roots of Ehrhart polynomials that is quadratic in d, indicate that it is essentially optimal via a result of C. Bey, M. Henk, and J. Wills, and analyze the growth rates for the roots of certain families of polynomials. We will also prove Conjecture 3.4 for small d and provide a potential counterexample for $d = 26$.

3.2 Norm Bounds and Root Growth

Recall that Stanley's non-negativity theorem states that the h^*-vectors for lattice polytopes consist of non-negative integers, being a special instance of Theorem 2.46. Polynomials with this property are special with regards to their root behavior, earning them the following designation. Recall from Lemma 2.16 that any degree d polynomial $f(t)$ over $\mathbb{C}$ is defined by the complex values h_j^*, $0 \leq j \leq d$, in the numerator of the rational generating function for $f(t)$.

Definition 3.5. *A non-zero polynomial satisfying the condition that* $(h_0^*, \ldots, h_d^*) \in (\mathbb{R}_{\geq 0})^{d+1}$ *is called a* Stanley non-negative, or SNN*, polynomial.*

The non-negativity of the h_j^*'s is enough to bound the roots of any SNN polynomial.

Theorem 3.6. (Braun, [6]) *If $f(t)$ is a degree d SNN polynomial, then all the roots of $f(t)$ lie inside the closed disc with center $\frac{-1}{2}$ and radius $d(d-\frac{1}{2})$.*

A proof of this theorem appeared in [6]; we will obtain it in this dissertation as a special case of the following more general theorem.

Theorem 3.7. *Suppose that $p_0, p_1, \ldots, p_d$ is a basis for the vector space of degree d polynomials over $\mathbb{C}$, where each p_j is monic with roots $\alpha_1^j, \ldots, \alpha_d^j$. Suppose further that $\beta \in \mathbb{C}$, $\nu \in \mathbb{R}_{>0}$ satisfy $\alpha_k^j \in \{z : |z-\beta| \leq \nu\}$ for all $0 \leq j \leq d, 1 \leq k \leq d$. If*

$$0 \neq f(t) = \sum_{j=0}^{d} r_j p_j,$$

where $r_j \in \mathbb{R}_{\geq 0}$ for all j, then all the roots of $f(t)$ are contained in the disc of radius $d\nu$ with center β.

For any SNN polynomial $f(t)$, Theorem 3.6 follows from Theorem 3.7 by multiplying $f(t)$ by $d!$ (thus leaving the roots unchanged) then setting $p_j(t) = d!\binom{t+d-j}{d}$ and observing that $\beta = -\frac{1}{2}$ and $\nu = d - \frac{1}{2}$.

Proof. Let d be a positive integer, let $D_d := \{z : |z-\beta| \leq d\nu\}$, and let $f(t)$ be as given in the theorem. It is enough to show that for any complex number z not in D_d there exists an open half-plane with zero on the boundary containing $B_d(z) := \{p_j(z) : 0 \leq j \leq d\}$, since this implies that $f(z)$ is a non-trivial, non-negative linear combination of elements in a common open half-plane with zero on the boundary and is hence non-zero.

Each $p_j(z) = \prod_{i=1}^{d}(z - \alpha_i^j)$ and each $z - \alpha_i^j$ is contained in the disc $D(z)$ of diameter ν centered at $z - \beta$. Consider the two points a_+ and a_- where the tangent lines to $D(z)$ through the origin meet the boundary of $D(z)$. If we take any d points in $D(z)$ and multiply them together, they are contained in the cone C with vertex the origin and sides passing through $(a_+)^d$ and $(a_-)^d$. We will now show that if we

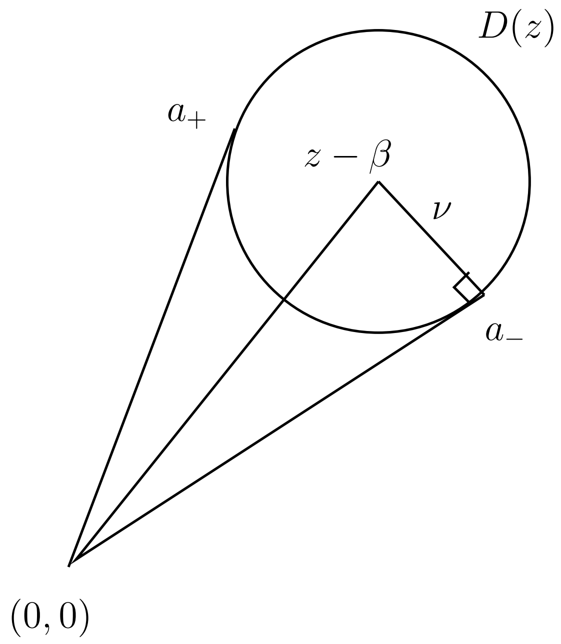


Figure 3.1: $D(z)$ and the tangent lines under consideration.

choose z outside D_d, the angular width of C is less than π. Thus, all the points $p_j(z)$ are contained in a common half-plane containing zero on the boundary.

To verify the bound on the angular width of C, consider the triangle formed by the origin, a_-, and $z-\beta$. This is a right triangle with hypotenuse of length $|z-\beta|$ and a side of length ν opposite the interior angle at the origin. Hence, the interior angle at the origin is $\sin^{-1}\left(\frac{\nu}{|z-\beta|}\right)$, and thus the total angular width of $D(z)$ is $2\sin^{-1}\left(\frac{\nu}{|z-\beta|}\right)$. Finally, we see that

$$2\sin^{-1}\left(\frac{\nu}{|z-\beta|}\right) < 2\sin^{-1}\left(\frac{\nu}{d\nu}\right) = 2\sin^{-1}\left(\frac{1}{d}\right) < \frac{\pi}{d}.$$

The total angular width of the cone C is equal to $2d\sin^{-1}\left(\frac{\nu}{|z-\beta|}\right)$, hence the angular width of C is less than π. $\square$

By invoking Stanley non-negativity, we obtain a positive answer to the question of Beck, et. al. as a corollary to Theorem 3.6.

Corollary 3.8. *If P is d-dimensional and $L_P(\alpha) = 0$, then $|\alpha + \frac{1}{2}| \le d^2 - \frac{d}{2}$.*

The fact that $-\frac{1}{2}$ is the center of the disc containing the roots of any degree d SNN polynomial has a heuristic explanation, as noted in [6]. All the polynomials in

$$B_d := \left\{ \binom{t+d-j}{d} : 0 \le j \le d \right\}$$

have roots contained in $\{-d, -d+1, \ldots, d-1\}$. For $1 \le j \le d$, the number of polynomials in B_d with $-j$ as a root is equal to the number with $-1+j$ as a root. Thus, the roots of the elements of B_d are highly symmetric with respect to the point $\frac{-1}{2}$, and the location of the center of the disc in our proposition should not come as a surprise. The critical line on which the roots of $L_{\diamond_d}(t)$ lie is also $x = \frac{-1}{2}$.

In response to Theorem 3.6, C. Bey, M. Henk and J. Wills proved the following theorem.

Theorem 3.9. (Bey, et. al., [5]) *The polynomial $S_d(t) = \sum_{j=0}^{d} \binom{t+d-j}{d}$ is an Ehrhart polynomial whose roots all have real part $\frac{-1}{2}$. Further, if α_d is the root of $S_d(t)$ of maximal norm, then*

$$\left| \alpha_d + \frac{1}{2} \right| = \frac{d(d+2)}{2\pi} + O(1)$$

as $d \to \infty$.

Thus, a norm bound for Ehrhart polynomial roots cannot be better than quadratic in d. It was suggested in [5] that $S_d(t)$ possesses the roots of maximal norm among all dimension d polytopes with interior lattice points, and this was proved for $d = 2, 3$. In response to the analogous question for SNN polynomials, Mike Develin and I produced the following theorem and conjecture.

Theorem 3.10. (Braun, Develin, [8]) *The polynomial $M_d(t) = \binom{t+d}{d} + \binom{t}{d}$ satisfies the following:*

(i) $M_d(t)$ *is not an Ehrhart polynomial;*

(ii) If β_d *is the root of* $M_d(t)$ *of maximal norm, then*

$$|\beta_d + \frac{1}{2}| = \frac{d^2}{\pi} + O(1),$$

as $d \to \infty$.

Proof. Regarding (i), for any lattice polytope P, h_d^* is equal to the number of interior lattice points in P. If this is non-zero, then h_1^*, which records $L_P(t) - d + 1$ when P is d-dimensional, must also be non-zero. As this condition is not satisfied by $M_d(t)$, $M_d(t)$ is not an Ehrhart polynomial.

The proof of (ii) closely follows the proof of Theorem 3.9 given in [5]. By Theorem 3.2, since the roots of the numerator of the generating function for $M_d(t)$ lie on the unit circle, all the roots of $M_d(t)$ are on the line $x = \frac{-1}{2}$. If $s = \frac{-1}{2} + bi$, $b \geq 0$, is a root of $M_d(t)$, then we have

$$(s+d)(s+d-1)\cdots(s+1) = -s(s-1)\cdots(s-d+1), \tag{3.2}$$

as any root s of $M_d(t)$ satisfies

$$-\binom{s+d}{d} = \binom{s}{d}.$$

Writing $s - j = s_j = r_j e^{i\theta_j}$ and noting that $|s+j+1| = |s-j|$ implies $s+j+1 = r_j e^{i(\pi-\theta_j)}$, we can rewrite (3.2) as

$$(-1)^{d+1} = e^{i(2\theta_0+\cdots+2\theta_{d-1})}.$$

We now substitute $\frac{\pi}{2} + \phi_j = \theta_j$, where $\phi_j \in (0, \frac{\pi}{2}]$. This gives a new equation,

$$-1 = e^{i(2\phi_0 + \cdots + 2\phi_{d-1})}.$$

Therefore, we must have, for some positive odd value of k,

$$\frac{k\pi}{2} = \sum_{0}^{d-1} \phi_j.$$

By definition, $\cot \phi_j = \frac{b}{j+\frac{1}{2}}$ for $j = 0, \ldots, d-1$. Thus, s is a root of $M_d(t)$ of maximal imaginary part if and only if

$$p_d(b) := \sum_{j=0}^{d-1} \cot^{-1}\left(\frac{b}{j+\frac{1}{2}}\right) = \frac{\pi}{2}, \tag{3.3}$$

as each $p_d(b)$ is a strictly decreasing function of b. Say that $p_d(b_d) = \frac{\pi}{2}$.

For $x > 1$, we have

$$\cot^{-1}(x) = \tan^{-1}(\frac{1}{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)(x^{2k+1})}.$$

By truncating the Taylor series after the first and second term, we have $\frac{1}{x} > \cot^{-1}(x) > \frac{1}{x} - \frac{1}{3x^3}$. Using this inequality on each summand in (3.3), substituting $x = \frac{b}{j+\frac{1}{2}}$ for each j, grouping the j^m-terms together by m, and then applying Faulhaber's formulas [24] on each of the resulting sums, we have that for $b > d + \frac{1}{2}$,

$$\frac{d^2}{2b} > p_d(b) > \frac{d^2}{2b} - \frac{d^4}{108b^3}.$$

Suppose now that $\hat{b} = \frac{d^2}{\pi} - \alpha$, where α is some large constant. In that case we have

$$p_d(\hat{b}) > \frac{d^2}{2\hat{b}} - \frac{d^4}{108\hat{b}^3} = \frac{\pi d^2}{2(d^2 - \pi\alpha)} - \frac{\pi^3 d^4}{108(d^2 - \pi\alpha)^3}. \tag{3.4}$$

The limit of the right hand side of (3.4) as d increases is $\frac{\pi}{2}$, and for all large enough d the right hand side is greater than $\frac{\pi}{2}$. As each $p_d(b)$ is decreasing in b, we see that $b_d \geq \hat{b} = \frac{d^2}{\pi} - \alpha$. As we also have $\frac{d^2}{\pi} > b_d$ for large d, we have our result. $\square$

Conjecture 3.11. (Braun, Develin, [8]) *The root of the polynomial $M_d(t)$ with largest norm, call it γ_d, has $|\mathrm{Im}(\gamma_d)|$ maximal among the imaginary parts of all roots of degree d SNN polynomials.*

Experimental data for the roots of a large number of SNN polynomials of degree less than or equal to seven form the basis for this conjecture.

We finish this section by observing that the root bounds above also apply to certain Hilbert polynomials, namely those of Cohen-Macaulay $\mathbb{N}$-graded $\mathbb{C}$-algebras whose Hilbert series have a numerator of degree less than or equal to one less than their Krull dimension. Thus, our norm bounds for Ehrhart polynomial roots can be considered a consequence of the Cohen-Macaulayness of the Ehrhart algebras.

3.3 The Vertical Strip Conjecture

In this section we consider Conjecture 3.4. All of the material in this section is joint work with Mike Develin, adapted from the preprint [8]. If Conjecture 3.4 is true, one might hope that it holds for all SNN polynomials. A step in this direction is the following.

Theorem 3.12. *For $d \geq 2$, let C_d^1 (respectively C_d^2) be the open pointed cone in $\mathbb{C}$ with vertex $d-1$ (respectively $-d$), angular width $\frac{2\pi}{d}$, and bisecting ray $(d-1, \infty)$ (respectively $(-\infty, -d)$). For any SNN polynomial f of degree d, $C_d^1 \cup C_d^2$ does not contain a root of f.*

Proof. That the real roots of such polynomials are in the interval $[-d, d-1]$ was shown in [2]. We prove that the theorem holds for complex values in C_d^1 with positive imaginary part. The proof is similar for the other cases. Let $t \in C_d^1$. Then the difference between the arguments of $\binom{t+d-j}{d}$ and $\binom{t+d-j-1}{d}$ is less than $\frac{\pi}{d}$ for all j, as this is equal to the difference between the arguments of $(t+d-j)$ and $(t-j)$, both of which have argument less than $\frac{\pi}{d}$. Therefore, the points $\binom{t+d-j}{d}$, $0 \leq j \leq d$, lie in a common half plane with zero on the boundary. A nonnegative, nonzero linear combination of such points cannot be zero. $\square$

Note that Conjecture 3.4 follows from Theorem 3.12 for $d = 2$, as in this case the angular widths of our cones are π. We can extend this as follows.

Theorem 3.13. *For any SNN polynomial of degree* 3 *or* 4*, the Vertical Strip Conjecture holds.*

Proof. If $d = 3$ or 4, we show that for every complex number z lying outside the vertical strip $\{z : -d \leq \mathrm{Re}(z) \leq d-1\}$, the numbers $\binom{z+d-j}{d}$, $0 \leq j \leq d$, all lie in a half-plane with zero on the boundary. We show this by proving that the angles $A(j)$ formed between the vectors $z-j$ and $z+d-j$, which are the angle differences between successive $\binom{z+d-j}{d}$, sum to less than π for $0 \leq j \leq d-1$. The situations for the real part of z being less than d and greater than $d-1$ are symmetric (interchanging j with $d-1-j$), so assume that the real part of z is greater than $d-1$. As these are real polynomials, complex roots will occur in conjugate pairs, hence we may assume that all relevant vectors lie in the first quadrant and therefore each $A(j)$ is increased by decreasing the real part of z. Hence, we need to show that the angle sum $\sum_j A(j)$ is at most π when the real part of z is $d-1$.

Let $z = (d-1) + ki$. An application of the Law of Cosines to the triangle with vertices 0, $z-j$, and $z+d-j$ yields the following formula, where $r = (d-1)-j$ (the x-coordinate of $z-j$) and $s = (2d-1-j)$ (the x-coordinate of $z+d-j$):

$$\cos^2 A(j) = 1 - \frac{d^2k^2}{(k^4 + (r^2+s^2)k^2 + r^2s^2)}$$

Since d is a constant, this quantity is minimized when $k^2 = rs$, and increases (thus A(j) decreases) monotonically in both directions as k gets further from this point.

Now, we move on to the specific applications for $d = 3$ and $d = 4$. For $d = 3$, simply applying the above formula yields (by numerical computation):

$$(A(0), A(1), A(2)) \leq (0.45, 0.65, 1.58);$$

the sum of these is less than π, so the relevant $\binom{z+d-j}{d}$ all lie in a half-plane.

For $d = 4$, we need to consider cases. Using the monotonicity results obtained above, we divide into the cases $k^2 \leq 2$ and $k^2 \geq 2$. For $k^2 \leq 2$ we have:

$$(A(0), A(1), A(2), A(3)) \leq (0.24, 0.38, 0.70, 1.58),$$

while for $k^2 \geq 2$ we have:

$$(A(0), A(1), A(2), A(3)) \leq (0.41, 0.53, 0.73, 1.23);$$

in both cases the sum of these angles is less than π, and so in both cases the relevant $\binom{z+d-j}{d}$ again all lie in a half-plane. This completes the proof that all zeroes lie in the vertical strip for $d = 3, 4$. $\square$

Experimental data suggests that the vertical strip conjecture might hold for SNN polynomials of higher degree, as demonstrated in Figures 3.2 and 3.3.

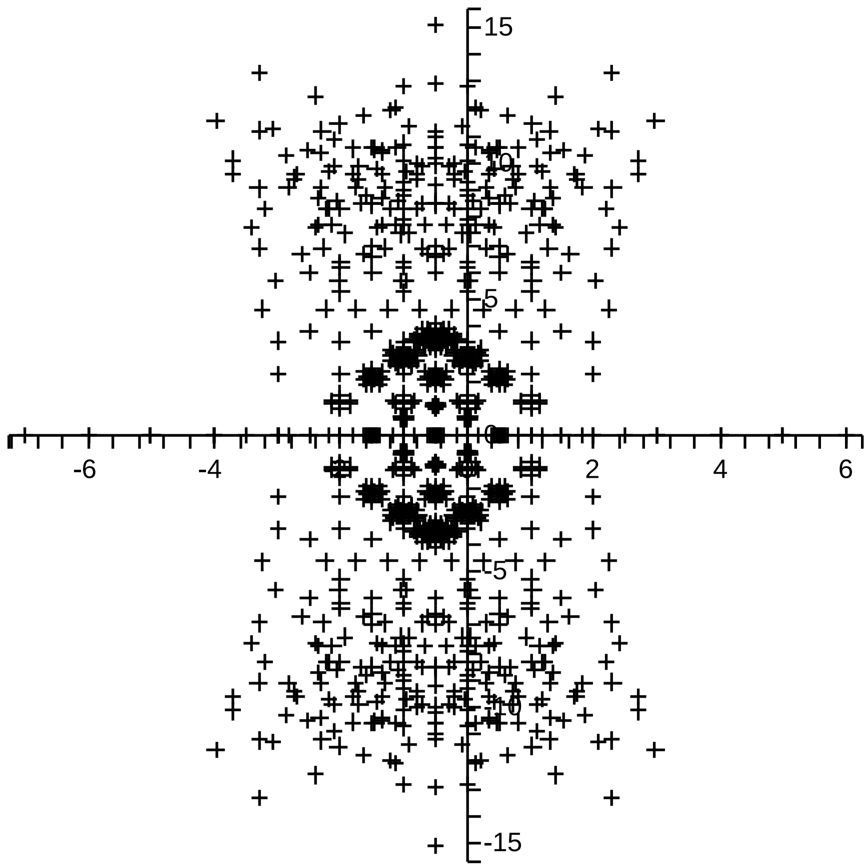


Figure 3.2: The roots of all non-zero degree 7 SNN polynomials with $h_j^* \in \{0, 1\}$ for all j.

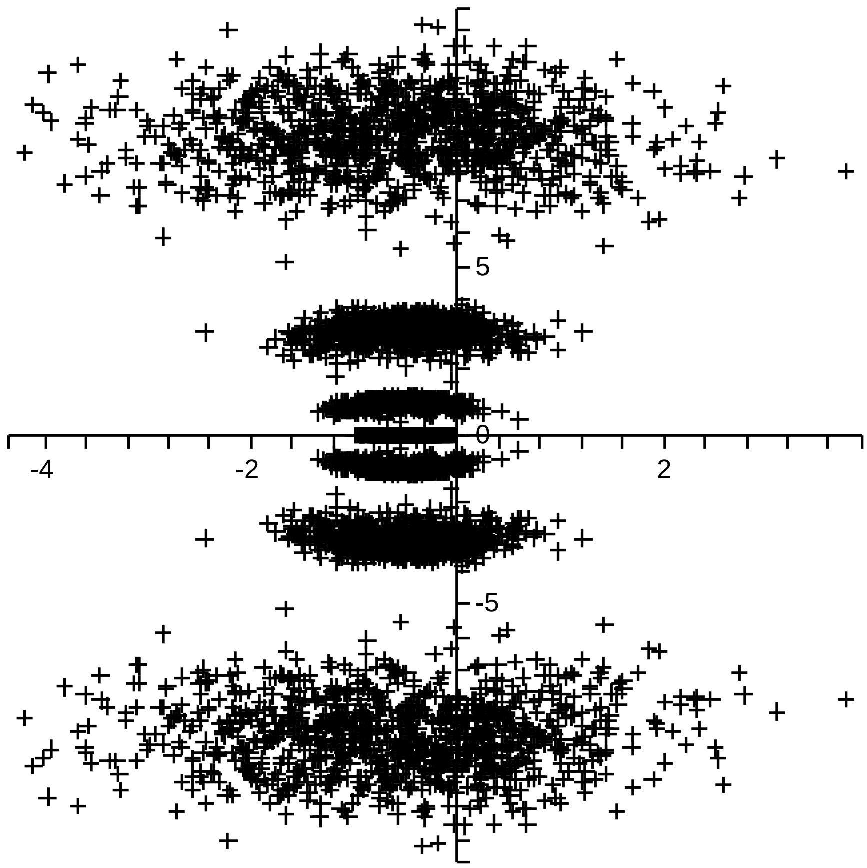


Figure 3.3: The roots of 1000 random degree 7 SNN polynomials.

While these results and data sets are promising, surprisingly the vertical strip conjecture is not generally satisfied by SNN polynomials. To produce examples illustrating this we fix d and pick a desired root z for which the numbers $\binom{z+d-j}{d}$ do not lie in a half-plane with boundary through the origin. We then produce a positive linear combination of these numbers which is equal to zero, and the polynomial encoded by these coefficients will have a root equal to z. For d large enough, we can find such z which lie outside the vertical strip, as the following examples demonstrate.

Example 3.14. The polynomial

$$f(t) = \binom{t+5}{5} + 33\binom{t}{5}$$

satisfies $f(z) = 0$ for $z \approx 4.00019 + 3.00963i$.

We obtain this polynomial by noting that for $d = 5$ and $z = 4 + 3i$, the sum of the $A(j)$ defined in the proof of Theorem 3.13 exceeds π. Therefore, by taking a convex combination of the values $\left\{\binom{z+5-j}{5}, 0 \leq j \leq 5\right\}$ equaling zero, we obtain a SNN polynomial with root $4 + 3i$. By perturbing the coefficients to their nearest integral value, we obtain a new SNN polynomial with integral h^*-vector having a root of real part strictly greater than 4. However, this polynomial is not an Ehrhart polynomial. In particular, $h_5^* = 33$ but $h_1^* = 0$, which is impossible for Ehrhart polynomials by Theorem 2.26. By bumping up the degree and using a similar approach, we can find candidate Ehrhart polynomials that may or may not be actual Ehrhart polynomials of polytopes.

Example 3.15. Consider the polynomial $g(t)$ of degree 26 with

$$(h_0^*, \ldots, h_{26}^*) = (1, 2, 3, 4, 6, 10, 16, 27, 43, 69, 112, 181, 293, 473, 762, 0, \ldots, 0)\,.$$

Numerical approximation produces

$$26.47331467 - 28.51231239i$$

as a root of $g(t)$.

This leads us to the following, currently unresolved, question:

Question 3.16. *Is the polynomial $g(t)$ from Example 3.15 an Ehrhart polynomial?*

This question illustrates the limits of our understanding of Ehrhart polynomials and series. Computing an Ehrhart series or polynomial from a given lattice polytope is a hard computational problem, and only recently has software been developed to do so. At the moment, there are no known methods for constructing a lattice polytope with a given h^*-vector. In general, our ability to compute h^*-vectors for polytopes of high dimensions is very limited. The next chapter will provide one way to partially overcome this limitation.

Chapter 4

h^*-vectors of Lattice Polytopes

In this chapter we discuss h^*-vectors of lattice polytopes. Our main goal is to prove that Ehrhart series behave well with respect to the free sum operation when a reflexive polytope is involved. The result is a method for producing new h^*-vectors from certain pairs of known h^*-vectors. Much of the material in this chapter is adapted from [7], where the results of Section 4.3 appeared.

4.1 h-vectors for Cohen-Macaulay algebras

We obtained as a corollary to Theorem 2.46 that if $A = \oplus_{t\in\mathbb{Z}_{\geq 0}} A_t$ is a finitely generated $\mathbb{N}$-graded Cohen-Macaulay $\mathbb{C}$-algebra of Krull dimension d with a regular system of parameters all of degree one, then the Hilbert series $H(A;x)$ is of the form

$$H(A;x) = \frac{h_0 + h_1 x + h_2 x^2 + \cdots + h_s x^s}{(1-x)^d},$$

with each $0 \leq h_j \in \mathbb{Z}$. It is a well known fact in algebraic combinatorics that one can construct such algebras with h_j arbitrary for $j > 0$ and $h_0 = 1$, see Exercise 32.10 in [13].

It therefore makes sense to restrict our algebraic considerations as far as possible beyond the Cohen-Macaulay condition if we want to obtain interesting results about h^*-vectors of lattice polytopes via commutative algebra. For example, Ehrhart algebras are clearly integral domains; this observation leads to the following due to R. Stanley.

Theorem 4.1. ([22]) *Given a lattice polytope P of dimension d and any $r \in \mathbb{N}$ such that $0 \leq r \leq \left\lfloor \frac{d}{2} \right\rfloor$, the h^*-vector of P satisfies*

$$h_0^* + h_1^* + \cdots + h_r^* \leq h_d^* + h_{d-1}^* \cdots + h_{d-r}^*.$$

Theorem 4.1 follows from a general result about h-vectors of semi-standard Cohen-Macaulay domains, see [22] for details.

Theorem 4.1 is one of many indicators that identifying vectors of non-negative integers that are h^*-vectors of lattice polytopes is a non-trivial problem. The polynomial violating the Vertical Strip Conjecture given in Question 3.16 is of degree 26, a much larger polynomial than that given in Example 3.14. One reason that our potential counterexample has a comparatively large degree is that our algorithm for finding SNN polynomials violating the Vertical Strip Conjecture does not immediately produce polynomials with h^*-vectors satisfying the known linear constraints on h^*-vectors for Ehrhart polynomials such as that given in Theorem 4.1. From all of these observations, a central question arises: how might one construct large vectors of non-negative integers which are known to be h^*-vectors for lattice polytopes? Further, can we do so in a way which leads to constructions for lattice polytopes corresponding to those h^*-vectors? One answer to these questions comes from previously known constructions involving polytopes and duality.

4.2 Products, Sums, and Duality

Given two convex lattice polytopes $P \subset \mathbb{R}^{d_P}$ and $Q \subset \mathbb{R}^{d_Q}$ of dimensions d_P and d_Q, respectively, we can form the product polytope $P \times Q$. It is easy to show that the product polytope is also a lattice polytope, as illustrated by Figure 4.1 where the cartesian product of $[-1, 1]^2$ and $[-1, 1]$ is shown.

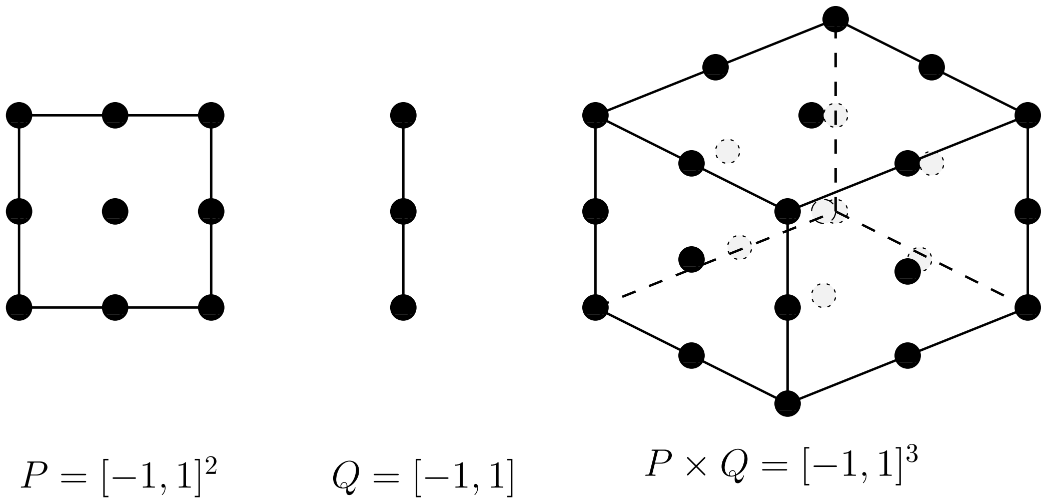


Figure 4.1: A product polytope.

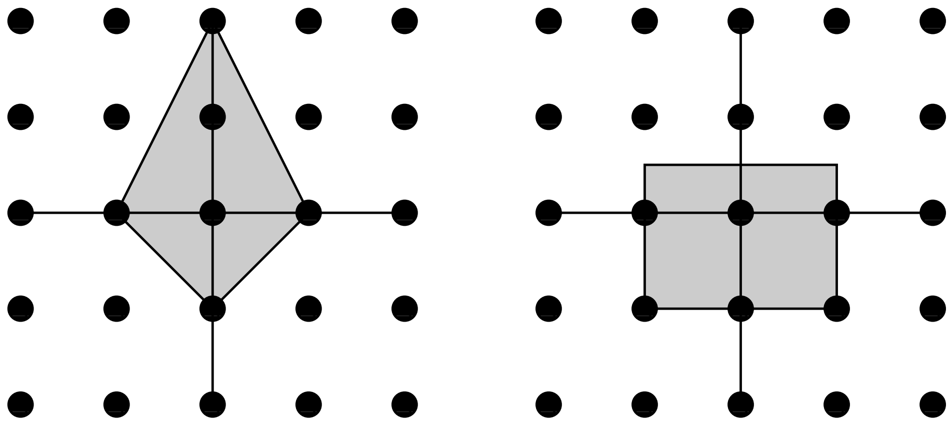
For a product polytope $P \times Q$, the Ehrhart polynomial of $P \times Q$ satisfies $L_{P \times Q}(t) = L_P(t)L_Q(t)$. Thus, Ehrhart polynomials are multiplicative with respect to cartesian products, which provides an easy way to obtain Ehrhart polynomials of higher degree from those of lower degree. This does not, however, translate into a simple formula for Ehrhart series, i.e. h^*-vectors.

Another important concept is that of a dual polytope.

Definition 4.2. *For a convex lattice polytope $P \subset \mathbb{R}^{d_P}$ of dimension d_P, the* dual *of P is*

$$P^\Delta := \left\{ x \in \mathbb{R}^{d_P} : x \cdot p \leq 1 \text{ for all } p \in P \right\}.$$

This idea of duality is the usual one, in the sense that P^Δ is the set of all points



P $\qquad$ P^{Δ}

Figure 4.2: A polytope and its dual.

in $\mathbb{R}^{d_P}$, considered as a space of linear functionals, that evaluate to less than or equal to 1 on P. If $0 \in P^{\circ}$, where P° denotes the interior of P, then P^{Δ} is also a convex polytope and $(P^{\Delta})^{\Delta} = P$, see [25] for details. If P is a lattice polytope, it is not necessarily the case that P^{Δ} is a lattice polytope, as illustrated by Figure 4.2.

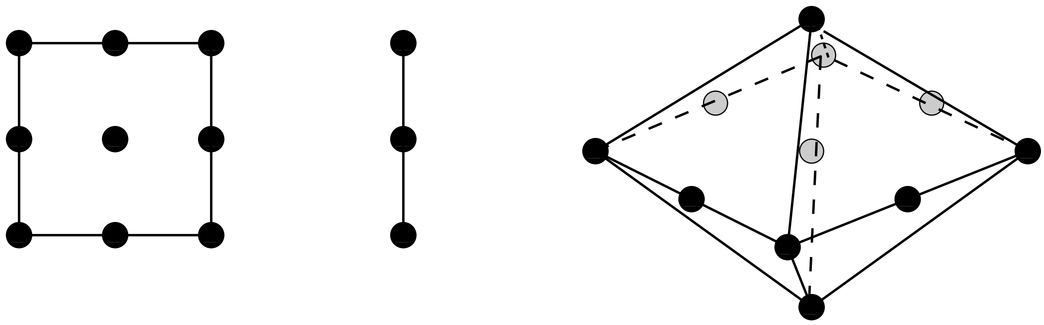
A natural question to ask is how the product operation and the dualizing operation interact. This leads us to the notion of a free sum.

Definition 4.3. *For two polytopes $P \subseteq \mathbb{R}^{d_P}$ and $Q \subseteq \mathbb{R}^{d_Q}$ of dimension d_P and d_Q, the free sum of P and Q is $P \oplus Q = \operatorname{conv}\{(0_P \times Q) \cup (P \times 0_Q)\} \subseteq \mathbb{R}^{d_P+d_Q}$.*

The free sum of $[-1,1]^2$ and $[-1,1]$ is shown in Figure 4.3. The free sum is important because if $0 \in P^{\circ}$ and $0 \in Q^{\circ}$, then the free sum operation is dual to the product operation, i.e.

$$(P \times Q)^{\Delta} = (P^{\Delta}) \oplus (Q^{\Delta}),$$

see [12]. A basic example of this duality is given by the d-dimensional crosspolytope and its dual the d-dimensional cube. These are the free sum of d copies of the interval $[-1,1]$ and the product of d copies of $[-1,1]$, respectively. The free sum of two lattice



$P = [-1, 1]^2$ $\qquad$ $Q = [-1, 1]$ $\qquad$ $P \oplus Q$

Figure 4.3: A free sum.

polytopes is also a lattice polytope, with vertices in one to one correspondence with the vertices of the summands.

Another motivating question for this chapter is whether or not there is a nice formula for the Ehrhart polynomial or series of a free sum, in the way that Ehrhart polynomials are multiplicative across cartesian products. To answer this question, thereby providing one answer to our questions about h^*-vectors as a corollary, we must introduce the special class of reflexive polytopes. In Figure 4.2, we saw that the dual of a lattice polytope is not necessarily also a lattice polytope. When both P and P^Δ are lattice polytopes, there are lots of interesting consequences.

Definition 4.4. *Say $P \subset \mathbb{R}^{d_P}$ is a lattice polytope of dimension d_P with $0 \in P^\circ$. If P^Δ is also a lattice polytope, we say P is* reflexive.

Reflexivity is preserved by the operations of free sum and cartesian product, as these operations preserve the lattice polytope property and are linked via duality. Reflexive polytopes were first investigated in the early 1990's in separate investigations by V. Batyrev and T. Hibi, and exhibit the following remarkable properties.

Theorem 4.5. ([14]) *P is reflexive if and only if P is a lattice polytope with $0 \in P^\circ$ that satisfies one of the following (equivalent) conditions:*

1. *P^{Δ} is a lattice polytope.*

2. *$L_{P^\circ}(t+1) = L_P(t)$ for all $t \in \mathbb{N}$, i.e. all lattice points in $\mathbb{R}^{d_P}$ sit on the boundary of some non-negative integral dilate of P.*

3. *$h_i^* = h_{d_P - i}^*$ for all i, where h_i^* is the i^{th} coefficient in the numerator of the Ehrhart series for P.*

The equivalence of the first and third conditions of Theorem 4.5 is often referred to as Hibi's Palindromicity Theorem, as it implies that the numerator of the Ehrhart series for any reflexive P is a palindromic polynomial, i.e. exhibits the indicated symmetry in its coefficients. Hibi's proof of this characterization of reflexive polytopes relies heavily on Ehrhart-Macdonald reciprocity, which is not surprising given that the second condition above relates the number of lattice points in a dilation of P° to the number of lattice points in a smaller dilation of P. From this characterization, it would appear that reflexive polytopes are a very restricted class of lattice polytopes. In some ways, this is true; for example, it is known that there are only finitely many reflexive polytopes in each dimension up to lattice preserving affine transformations. However, reflexive polytopes are also very general, in the sense of the following theorem due to C. Haase and I. Melnikov; for the definition of lattice equivalent and a proof, see [11].

Theorem 4.6. ([11]) *Every lattice polytope is lattice equivalent to a face of some reflexive polytope.*

Reflexive polytopes are a rich source of open problems and conjectures. They are related to the Mirror Symmetry Conjecture in theoretical physics, see [1], to various questions in toric geometry, and to questions about error-correcting codes and random walks. For more about the last topic see [3], where free sums of reflexive polytopes play an important role. Reflexive polytopes are also exactly what we need to construct large h^*-vectors.

4.3 h^*-vectors for Free Sums

The following is the main result of this chapter.

Theorem 4.7. *If P is a d_P-dimensional reflexive polytope in $\mathbb{R}^{d_P}$ and Q is a d_Q-dimensional lattice polytope in $\mathbb{R}^{d_Q}$ with $0 \in Q^\circ$, then*

$$\mathrm{Ehr}_{P\oplus Q}(x) = (1-x)\mathrm{Ehr}_P(x)\mathrm{Ehr}_Q(x). \tag{4.1}$$

The key point in the following proof is that the $\mathbb{R}^{d_P}$ and $\mathbb{R}^{d_Q}$ components of lattice points in $t(P \oplus Q)$ cannot simultaneously be far from the origin. For what follows, consider vectors in P and Q as actually being in $P \oplus 0_Q$ and $0_P \oplus Q$, respectively.

Proof. Note that (4.1) is equivalent to

$$L_{P\oplus Q}(t) = L_Q(t) + \sum_{k=1}^{t} L_Q(t-k)(L_P(k) - L_P(k-1)) \tag{4.2}$$

for every $t \in \mathbb{N}$. This equivalence is seen by expanding the product on the right hand side of (4.1) as follows:

$$(1-x)\mathrm{Ehr}_P(x)\mathrm{Ehr}_Q(x) = (1-x)(\sum_{r\geq 0} L_Q(r)x^r)(\sum_{s\geq 0} L_P(s)x^s)$$

$$= (\sum_{r\geq 0} L_Q(r)x^r)(\sum_{s\geq 0} L_P(s)x^s - \sum_{s\geq 1} L_P(s-1)x^s)$$

$$= (\sum_{r\geq 0} L_Q(r)x^r)(1 + \sum_{s\geq 1}(L_P(s) - L_P(s-1))x^s)$$

$$= \sum_{t\geq 0}[L_Q(t) + \sum_{k=1}^{t} L_Q(t-k)(L_P(k) - L_P(k-1))]x^t.$$

We will therefore show that (4.2) holds for every $t \in \mathbb{N}$.

Suppose first that k is a real number between 0 and t, $t \in \mathbb{N}$. For $p \in kP$, $q \in (t-k)Q$, we have

$$\frac{1}{k}\begin{bmatrix} p \\ 0 \end{bmatrix} \in P, \frac{1}{t-k}\begin{bmatrix} 0 \\ q \end{bmatrix} \in Q,$$

and we see that

$$\left(\frac{k}{t}\right)\left(\frac{t}{k}\right)\begin{bmatrix} p \\ 0 \end{bmatrix} + \left(\frac{t-k}{t}\right)\left(\frac{t}{t-k}\right)\begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \in tP \oplus tQ = t(P \oplus Q).$$

Thus there is a copy of $kP \times (t-k)Q$ in $t(P \oplus Q)$ for any such k.

We will now show that $t(P \oplus Q) = \bigcup_{k \in [0,t]} kP \times (t-k)Q$. Suppose $p \in \partial(kP)$ and $\begin{bmatrix} p \\ q \end{bmatrix} \in t(P \oplus Q)$. We will show that q cannot be outside $(t-k)Q$.

Note that $(tP)^{\Delta} = \frac{1}{t}P^{\Delta}$. So,

$$(t\,(P \oplus Q))^{\Delta} = \frac{1}{t}(P^{\Delta}) \times \frac{1}{t}(Q^{\Delta}).$$

If $\frac{p}{k} \in \partial P$ then there exists some $p^{\Delta} \in P^{\Delta}$ such that $p^{\Delta} \cdot \frac{p}{k} = 1$. Thus

$$p \cdot \frac{1}{t}p^{\Delta} = \frac{k}{t}\left(p^{\Delta} \cdot \frac{p}{k}\right) = \frac{k}{t}.$$

If $\frac{q}{\alpha} \in \partial Q$, where $\alpha > (t-k)$, then similarly there exists $q^{\Delta} \in Q^{\Delta}$ such that

$$q \cdot \frac{1}{t}q^{\Delta} = \frac{\alpha}{t}\left(q^{\Delta} \cdot \frac{q}{\alpha}\right) = \frac{\alpha}{t} > \frac{t-k}{t}.$$

But, we know that

$$(P \oplus Q)^{\Delta} = P^{\Delta} \times Q^{\Delta}$$

and thus

$$\frac{1}{t}\begin{bmatrix} p^{\Delta} \\ q^{\Delta} \end{bmatrix} \in \frac{1}{t}(P^{\Delta} \times Q^{\Delta})$$

has a dot product with $\begin{bmatrix} p \\ q \end{bmatrix} \in t(P \oplus Q)$ of greater than one, a contradiction.

Thus, every lattice point in $t(P \oplus Q)$ can be assigned uniquely to a lattice point in tP by projection onto the $\mathbb{R}^{d_P}$ coordinate. Further, as P is reflexive, each lattice point in $\mathbb{R}^{d_P}$ is contained in the boundary of kP for some $k \in \mathbb{N}$. Therefore,

$$t\,(P \oplus Q) \cap \mathbb{Z}^{d_P + d_Q} =$$
$$\biguplus_{k \in \{0,\dots,t\}} \left\{ \begin{bmatrix} p \\ q \end{bmatrix} : p \in \partial(kP) \cap \mathbb{Z}^{d_P}, q \in (t-k)Q \cap \mathbb{Z}^{d_Q} \right\}. \tag{4.3}$$

It is immediate that (4.2) counts the lattice points in $t(P \oplus Q)$ using this partition.

$\square$

Note that the product on the right hand side of (4.1) is obtained by multiplying the numerators of $\mathrm{Ehr}_P(x)$ and $\mathrm{Ehr}_Q(x)$ and dividing by $(1-x)^{d_P+d_Q+1}$. So, just as the product polytope $P \times Q$ induces a product of Ehrhart polynomials, we see that, subject to some restrictions, the free sum induces a product of the numerators of the Ehrhart series of the summands. It is exactly this observation that allows us to produce large new h^*-vectors from existing ones.

Example 4.8. The polytope $P = \mathrm{conv}\{(-1,-1),(1,0),(0,1)\}$ is reflexive with h^*-vector $(1,1,1)$. Given any other lattice polytope Q with $0 \in Q^{\circ}$, where $\mathrm{Ehr}_Q(x) = \frac{\sum_{j=0}^{d} h_j^* x^j}{(1-x)^{d+1}}$, the h^*-vector of $P \oplus Q$ is

$$(h_0^*, h_1^* + h_0^*, h_2^* + h_1^* + h_0^*, \dots, h_r^* + h_{r-1}^* + h_{r-2}^*, \dots, h_d^* + h_{d-1}^* + h_{d-2}^*, h_d^* + h_{d-1}^*, h_d^*).$$

It is interesting that the Ehrhart polynomial and the Ehrhart series are "dual" to each other in the sense of Theorem 4.7. It would be interesting to find other examples of duality involving polynomials and their associated series where similar patterns arise, in particular to find a general theorem from which Theorem 4.7 follows as a corollary.

A simple example shows that Theorem 4.7 does not hold for arbitrary polytopes. Given $P = [-2, 2]$, we see that

$$\mathrm{Ehr}_P(x) = \frac{1 + 3x}{(1 - x)^2},$$

while

$$\mathrm{Ehr}_{P\oplus P}(x) = \frac{1 + 10x + 5x^2}{(1 - x)^3}.$$

However, reflexivity is not necessary either. If $0 \in Q$, the *pyramid* over Q is defined to be $[0, 1] \oplus Q$. Though $[0, 1]$ is not reflexive, it is well known that

$$\mathrm{Ehr}_{[0,1]\oplus Q}(x) = (1 - x)\mathrm{Ehr}_{[0,1]}(x)\mathrm{Ehr}_Q(x).$$

Despite not being reflexive, $[0, 1]$ shares the property with reflexive polytopes that the lattice point in $t[0, 1] - (t - 1)[0, 1]$ lies on the boundary of $t[0, 1]$. Since $[0, 1]$ is "half" of a reflexive polytope, the lattice points in $t([0, 1] \oplus Q)$ are "filtered" uniquely by the lattice points in $t[0, 1]$ and hence we get our result. This example can be generalized as follows:

Corollary 4.9. *Suppose that P and Q are as in Theorem 1 and that $\{H_i\}_{i=1}^k$ and $\{K_j\}_{j=1}^l$ are halfspaces of the form $H_i = \left\{y \in \mathbb{R}^{d_P} : y \cdot a_i \geq 0\right\}$ for some $a_i \in \mathbb{R}^{d_P}$ and $K_j = \left\{u \in \mathbb{R}^{d_Q} : u \cdot b_j \geq 0\right\}$ for some $b_j \in \mathbb{R}^{d_Q}$, respectively. Set $H = \cap H_i$ and*

$K = \cap K_j$. If $H \cap P$ and $K \cap Q$ are lattice polytopes, then

$$\mathrm{Ehr}_{(H\cap P)\oplus(K\cap Q)}(x) = (1-x)\mathrm{Ehr}_{H\cap P}(x)\mathrm{Ehr}_{K\cap Q}(x). \tag{4.4}$$

Proof. We can extend H_i and K_j to halfspaces $\hat{H}_i \subseteq \mathbb{R}^{d_P+d_Q}$ and $\hat{K}_j \subseteq \mathbb{R}^{d_P+d_Q}$ by setting

$$\hat{H}_i = \left\{ z \in \mathbb{R}^{d_P+d_Q} : \begin{bmatrix} a_i \\ 0_Q \end{bmatrix} \cdot z \geq 0 \right\}$$

and

$$\hat{K}_j = \left\{ z \in \mathbb{R}^{d_P+d_Q} : \begin{bmatrix} 0_P \\ b_i \end{bmatrix} \cdot z \geq 0 \right\}.$$

We will first show that

$$t\left((H\cap P)\oplus(K\cap Q)\right) = \left(\cap_i \hat{H}_i\right) \cap \left(\cap_j \hat{K}_j\right) \cap t\left(P \oplus Q\right). \tag{4.5}$$

Let $\begin{bmatrix} p \\ q \end{bmatrix} \in t\left((H\cap P)\oplus(K\cap Q)\right)$. Then

$$\begin{bmatrix} p \\ q \end{bmatrix} = \sum \alpha_m p_m + \sum \beta_n q_n$$

where each $p_m \in t\left(H \cap P\right)$, each $q_n \in t\left(K \cap Q\right)$, $\sum \alpha_m + \sum \beta_n = 1$, and $\alpha_m, \beta_n \geq 0$. Thus, for all i,

$$\begin{bmatrix} a_i \\ 0_Q \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} a_i \\ 0_Q \end{bmatrix} \cdot \begin{bmatrix} \sum \alpha_m p_m \\ \sum \beta_n q_n \end{bmatrix} \geq 0$$

and, for all j,

$$\begin{bmatrix} 0_P \\ b_j \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0_P \\ b_j \end{bmatrix} \cdot \begin{bmatrix} \sum \alpha_m p_m \\ \sum \beta_n q_n \end{bmatrix} \geq 0.$$

Hence we see that $\begin{bmatrix} p \\ q \end{bmatrix} \in \left(\cap_i \hat{H}_i\right) \cap \left(\cap_j \hat{K}_j\right) \cap t\,(P \oplus Q)$.

Let $\begin{bmatrix} p \\ q \end{bmatrix} \in \left(\cap_i \hat{H}_i\right) \cap \left(\cap_j \hat{K}_j\right) \cap t\,(P \oplus Q)$. We know that $p \cdot a_i \geq 0$ for all i, $q \cdot b_j \geq 0$ for all j and (from the proof of Theorem 4.7) that $\begin{bmatrix} p \\ q \end{bmatrix} \in kP \times (t-k)Q$ for some real number $0 \leq k \leq t$. Thus, $\frac{p}{k} \in H \cap P$, $\frac{q}{t-k} \in K \cap Q$, and

$$\left(\frac{k}{t}\right)\left(\frac{t}{k}\right)\begin{bmatrix} p \\ 0 \end{bmatrix} + \left(\frac{t-k}{t}\right)\left(\frac{t}{t-k}\right)\begin{bmatrix} 0 \\ q \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \in t\,((H \cap P) \oplus (K \cap Q)),$$

hence we are done with (4.5).

Using (4.3), we see that

$$\begin{gathered} \left(\cap_i \hat{H}_i\right) \cap \left(\cap_j \hat{K}_j\right) \cap t\,(P \oplus Q) \cap \mathbb{Z}^{d_P + d_Q} = \\ \biguplus_{k \in \{0,\ldots,t\}} \left\{ \begin{bmatrix} p \\ q \end{bmatrix} : p \in H \cap \partial\,(kP) \cap \mathbb{Z}^{d_P}, q \in K \cap (t-k)Q \cap \mathbb{Z}^{d_P} \right\}. \end{gathered} \tag{4.6}$$

Since (4.5) shows that

$$L_{((H \cap P) \oplus (K \cap Q))}(t) = L_{\left(\cap_i \hat{H}_i\right) \cap \left(\cap_j \hat{K}_j\right) \cap (P \oplus Q)}(t),$$

our partition above shows that

$$\mathrm{Ehr}_{(H \cap P) \oplus (K \cap Q)}(x) = (1-x)\mathrm{Ehr}_{H \cap P}(x)\mathrm{Ehr}_{K \cap Q}(x), \tag{4.7}$$

as the right hand side of (4.7) enumerates the elements of the disjoint union (4.6). □

We end this chapter by mentioning an application of Theorem 4.7 due to S. Payne. T. Hibi conjectured in [13] that the h^*-vector of any reflexive polytope is unimodal, i.e. $h^*_0 \leq h^*_1 \leq \cdots \leq h^*_{\lfloor \frac{d}{2} \rfloor}$. This conjecture was disproved in a recent paper due to M. Mustaţă and S. Payne, [18], in which counterexamples were found in all even dimensions greater than 6. The following theorem provides a larger class of counterexamples.

Theorem 4.10. (Payne, [19]) *For any positive integers m and n, there exists a reflexive polytope P and indices $i_1 < j_1 < i_2 < j_2 < \cdots < i_m < j_m < i_{m+1}$ such that*

$$h^*_{i_l} - h^*_{j_l} \geq n \text{ and } h^*_{i_{l+1}} - h^*_{j_l} \leq n,$$

for $1 \leq l \leq m$. Furthermore, P can be chosen such that $\dim(P) = O(m \log\log n)$.

The method of proof of this theorem is to use two special families of reflexive polytopes with known h^*-vectors and apply Theorem 4.7 with summands from these families to produce the required h^*-vectors.

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