1 Exploring Desargues Theorem.

1.1 Points and Lines at Infinity

The principal "synthetic" tool for determining when one plane figure in space is the shadow of another is Desargue's Theorem which is most properly stated as a theorem in Projective Geometry rather than the cartesian or affine geometry in which we are working. This would be a good place to begin an enjoyable study of projective geometry, however one of the prinicpal ideas in problem solving is to learn how to collect, understand, and employ mathematical tools to solve the problem at hand, usually postponing their full development. This often entails extracting a tool, theorem, or technique from its "natural" milieu and phrasing it and its proof or explanation in the language of the problem at hand. This often results in much faster progress on the study of the current problem purchased at the cost of more lengthy and less elegant statements and arguments. With regard to the Desargues Theorem we need the notion of "point at infinity" of a line.

1 Definition: *If L is a line in any number of dimensions) and L is written in parametric form $P + t\,D$ then we call "D" the direction of the parameterization. A line has many parameterizations and different parameterizations may have different directions. , e.g. the parameterization $P + t\,2\,D$ has the 2D as direction. However the directions of any two parameterizations of the same line are proportional (i.e. one is a numerical multiple of the other).*

We define the direction of a line L as the totality of all of the directions of its parameterizations. For any line this is an infinite number of things but any two of them are proportional so if we know one of them we know them all. We use any of its members as a name for the direction of the line. Thus for exampe we say that " [1,0,0] is the direction of the x-axis". We also say "[27,0,0] is the direction of the x-axis". What we we would say to be perfectly precise is "[1.0,0 is a name for the direction of the x-axis". However that is too wordy and experience shows that there is no confusion in the "shorthand" version provided everyone understands the convention. Lest one feel that this is something new we note that this is exactly the same idea as saying " $\frac{3}{2}$ is the average of 1 and 2" while " $\frac{6}{4}$ is the average of 1 and two" is mathematically just as correct. The symbols $\frac{1}{2}$ and $\frac{6}{4}$ are but two of the infinitely many names (numerals) for a particular number. With this introduction we can now give the following:

2 Definition: *If L is a line then "the point at infinity on L" is the direction of L.*

Example: The point at infinity on the x-axis is the direction [1,0,0]. Note that the "finite" point [1,0,0] is also on the x-axis but this is different from the direction [1,0,0] which happens to be repersented by the same name. Note that the point at infinity on the x-axis is also the direction [-193,0,0].

Example:What is the point at infinity on the line K which contains the points [1,2,3] and [8,-3,7].

Solution: A parametric equation for ths line is [1,2,3] + t([8,-3,7] - [1,2,3]). Thus the direction of this parameterization of the line is [7,-5,4] so this is one of the names for the direction of the line, i.e. one of the names for the point at infinity on the line K. [$1,\ -\frac{5}{7},\ \frac{4}{7}$] is another name for this direction. Note that the "finite point" [7,-5,4] does not lie on the line K, In the exercises we find that the "finite point" with coordinates given by one of the names for the point at infinity on a line will also be on the line if and only if the line goes through the origin.

Example: Determine if the line L which contains [1,2,3] and [5,1,2]

and the line M which contains [6,0,1] and [3,2,7]] have a point in common (including points at infinity)

Solution: We can use the Maple solve function to find such points from the parametric forms
.

```
L:=expand([1,2,3]+t*([5,1,2]-[1,2,3]));
M:=expand([6,0,1]+s*([3,2,7]-[6,0,1]));
eq1:= L[1]-M[1]=0;
eq2:=L[2]-M[2]=0;
eq3:=L[3]-M[3]=0;
solve({eq1,eq2,eq3},{s,t});
```

The absence of a solution to these equations tells us that there is no point of intersection "at finite distance". To determine if there is a common point at infinity we calculate the point of infinity of each and see if they are the same. The point of infinity of L is its direction $[5,\ 1,\ 2] - [1,\ 2,\ 3] = [4,\ -1,\ -1]$ and that of M is $[3,\ 2,\ 7] - [6,\ 0,\ 1] = [-3,\ 2,\ 6]$. These are not proportional since there is no number k such that $k\,[4,\ -1,\ -1] = [-3,\ 2,\ 6]$. Thus these lines in 3-space do not meet, even at infinity.

It is true, however that any two lines in space which lie in a plane will either meet at finite distance or will have a common point at infinity. Indeed if the two lines do not meet they are parallel in that plane and thus have the same direction. This fact is checked in the exercises.

Now we can state a the following "obvious" fact:

1 *Theorem: Two lines are parallel if and only if they have the same point at infinity.*

The proof is worked out in the exercises.

Now we have some new ways to express familiar ideas

3 *Definition: The line L which contains the point A and the point at infinity P is simply the line through A with direction P. For instance one parameterization of L would be* $A + t\,P$. *The line through two points at infinity is the collection of all directions of lines in a plane which contains two lines having those points at infinity The projection of a point W onto a plane L from a point at infinity, P, is the point of intersection with L of the line through W having direction P. The projection of an object H onto a plane L from a point at infinity, P, is the set of projections of all points of H onto L from P.*

1.2 Exercises On Points and Lines at Infinity

1.2.1 *EXERCISE: If two lines lie in a plane and do not have a point at finite distance in common then they have the same direction and thus have a common point at infinity.*

Hint: Suppose L and M are lines in a plane P which do not meet at finite distance. This means that if A and B are on L and C and D are on M then the equation A+t*(B-A) = C+s*(D-C) has no solution for any s an t. That is there are no s,t such that A-C= t*(B-A) + s*(D-C). However if aX+bY+cZ+d=0 is the equation for P then you can check that A-C, B-A, and D-C are all in the plane Q whose equation is aX+bY+cZ=0. If, say, c is not zero then this means that the Z-coordinate of any element of Q is completely determined by its X and Y coordinates. This means that one can't find s and t so that the first two coordinates of t*(B-A) + s*(D-C) are euqal to the first two coordinates of

A-C because if you could then the third coordinates would automatically agree. However solving for the first two coordinates amounts to solving two linear equations in two unknowns which can always be done unless the rows of coefficients are proportional. This means that the first two coordinates of B-A and D-C are proportional but the third coordinates will then also be proportional by the fact that we can solve for the third coordinate (This ability to solve for one of the coordinates in terms of the other two is how the fact that the lines are in a plane comes into play).

1.2.2 *EXERCISE: Show that if L is a line in space then the directions of any two parameterizations of L are proportional*

1.2.3 *EXERCISE: If K is a line and W=[a,b,c] is the point at infinity on K then the "finite point [a,b,c]" is on K if and only if K goes through the origin.*

1.2.4 *EXERCISE: Prove that two lines in the plane are parallel if and only of they have the same point at infinity.*

Hint: First suppose the lines have the same point at infinity. If they are different lines we need to show that they do not meet. One way to do this is to show that if you could find a point of intersection then the lines are identical. Let Dir be the common point at infinity then parameterizations for K and L are of the form $\boldsymbol{A} + \boldsymbol{t}\,\boldsymbol{Dir}$ and $\boldsymbol{B} + \boldsymbol{s}\,\boldsymbol{Dir}$. If A and B are the same then the lines K and L are the same. If there were a common point we could find s_0 and t_0 such that $\boldsymbol{A} + t_0\,\boldsymbol{Dir} = \boldsymbol{B} + s_0\,\boldsymbol{Dir}$, i.e. $\boldsymbol{A} - \boldsymbol{B} = (s_0 - t_0)\,\boldsymbol{Dir}$. Show that if there are such numbers then the points A and B would be on both lines.

For the converse one must show that if the lines are parallel then any two paramaterizations of these lines have proportional directions. We may assume that the lines are distinct. Suppose E+tF is a parameterization of L and $\boldsymbol{G} + \boldsymbol{s}\,\boldsymbol{H}$ is a parameterization for K. We can attempt to find values of s and t which give common points on K and L. Since the lines are distinct and parallel the following must fail to find such and s and t. Find the condition on E and H which this implies and explain why this means that the two directions are proportional.

E:=[e1,e2]; F:=[f1,f2]; G:=[g1,g2]; H:=[h1,h2];
DIFF:=expand(E+t*F - (G+s*H));
solve({DIFF[1]=0,DIFF[2]=0},{s,t});

1.2.5 *EXERCISE: Show that if C is not equal to C' then the points from which the triangles ABC and ABC' are in perspective are an entire line (including its point at infinity).*

1.3 Desargues's Theroem

We want to explore a very useful result, discovered in the mid 1600's by Girard Desargues, a french geometer who pioneered the development of projective geometry. First, we will just provide a couple of words to draw some pictures and then use them to draw a few pictures.

The first word projxy takes two points X and P and returns the point Q on the line thru X and P which is on the xy plane. The assumption is the line XP strikes the xy plane in only one point. The point Q is called the projection of X from P onto the xy plane.

```
projxy:=proc(X,P)
```

```
local Line,t,T,Q:
Line:=expand(t*P+(1-t)*X):
T:=solve(Line[3]=0,t):
Q := subs(t=T,Line):
end:
```

The second word takes a triangle ABC (think of it as lying above the xy plane) and a point P (think of it as lying above the triangle), computes the triangle obtained by the projection of ABC from P , call it triangle A1B1C1. Then line sements and triangles are plotted and displayed in one big picture.

```
desarguexy:= proc (A, B, C,P)
local triABC, triABCprime,A1,B1,C1, AA1, BB1, CC1, AB, abpoint, BC,
bcpoint, CA, capoint, abline, abprimeline, bcline, bcprimeline,
caprimeline, caline, axis, DEAS;
A1:=projxy(A,P);
B1:=projxy(B,P);
C1:=projxy(C,P);
triABC := plots[polygonplot3d]([A, B, C],style = patch,thickness =
3,color = red);
triABCprime := plots[polygonplot3d]([A1, B1, C1],style =
patch,thickness = 3,color = blue);
AA1 := plots[polygonplot3d]([P, A1],style = wireframe,color = tan,
thickness = 3);
BB1 := plots[polygonplot3d]([P, B1],style = wireframe,color = tan,
thickness = 3);
CC1 := plots[polygonplot3d]([P, C1],style = wireframe,color = tan,
thickness = 3);
AB := expand(s*A+(1-s)*B-t*A1-(1-t)*B1);
abpoint := expand(subs(solve({AB[1] = 0, AB[2] = 0, AB[3] = 0},
{s, t}),s*A+(1-s)*B));
BC := expand(s*C+(1-s)*B-t*C1-(1-t)*B1);
bcpoint := expand(subs(solve({BC[3] = 0, BC[2] = 0,BC[1]=0},
{s, t}),s*C+(1-s)*B));
CA := expand(s*A+(1-s)*C-t*A1-(1-t)*C1);
capoint := expand(subs(solve({CA[1] = 0, CA[3] = 0, CA[2] = 0},{s,
t}),s*A+(1-s)*C));
abline := plots[polygonplot3d]([B, abpoint],style = wireframe,color =
red,
thickness = 2);
abprimeline := plots[polygonplot3d]([B1, abpoint],style =
wireframe,color = blue,thickness = 2);
bcline := plots[polygonplot3d]([C, bcpoint],style = wireframe,color =
red,thickness = 2);
bcprimeline := plots[polygonplot3d]([C1, bcpoint],style =
wireframe,color = blue,thickness = 2);
caprimeline := plots[polygonplot3d]([A1, capoint],style =
wireframe,color = blue,thickness = 2);
caline := plots[polygonplot3d]([A, capoint],style = wireframe,color =
red,thickness = 2);
axis := plots[polygonplot3d]([abpoint, bcpoint, capoint],style =
wireframe,
color = green,thickness = 2);
DEAS := [triABC, triABCprime, AA1, BB1, CC1, abline, abprimeline,
```

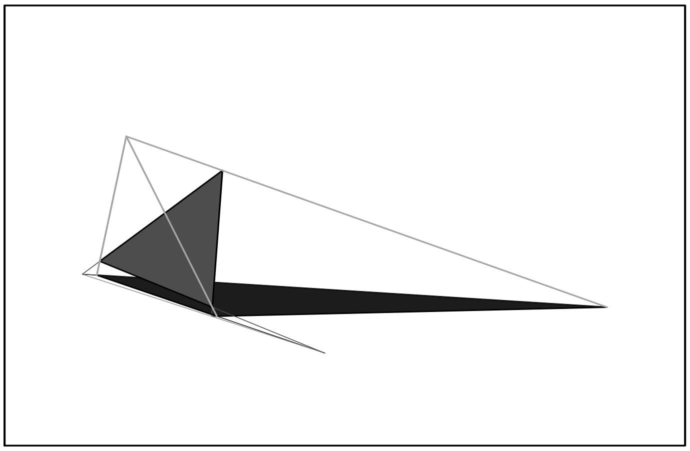
```
bcline,
bcprimeline, caline, caprimeline, axis];
plots[display](DEAS)
end;
```

Here is a short movie made with desarguexy.

```
movie:=seq(
desarguexy([1,0,1],[2,3,4],[3,1,2],[1,-2+i/2,5]),i=0..10):

plots[display]([movie],orientation=[-36,41],i
nsequence=true,axes=normal);


plots[display](desarguexy([1,0,1/2],[2,3,4],[
3,1,1/4],[1,1,5]), orientation=[-36,41]);
```



4 *Definition: Two triangles ABC and A'B'C' are said to be in perspective if there is a point O such that OAA', OBB' and OCC' are all straight lines. In this case they are said to be in perspective from 0.*

Note that the point O can be a point at infinity (which says that the lines OAA', OBB' and OCC' are parallel. Further, O need not be unique, for instance if A=A',B=B',C=C' then 0 can be any point.

Desargue's Theorem gives necessary and sufficient conditions for two triangles to be in perspective. It can be easily stated in that form. However for convenience in understanding why it is true we break it into two theorems. Our treatment follows that of Hodge and Pedoe in *Methods of Algebraic Geometry*, Cambridge University Press, Cambridge University Press 1968

2 *Theorem 1: If two triangles ABC and A'B'C' are in perspective then there exists a line l passing through every point of intersection of (extensions of) corresponding pairs of edges of the two triangles. That is there is a line which contains: the intersection of AA' and BB', the intersection of AA' and CC', and the intersection of BB' and CC'.*

3 *Theorem 2: If ABC and A'B'C' are two triangles such that there is a line l passing through a point common to BC and B'C', a point common to AC and A'C', and a point common to AB and A'B' then the triangles are in perspective.*

1.4 Proof of Desargue's Theorems

The theorems include the cases where the points of intersection and the line which contain them may lie at infinity. The case where the line is at infinity is simply that where the two planes defined by the triangles are parallel.

CASE 1: The two triangles do not lie in a plane.

Theorem 1: Suppose, as in the diagram below that the two triangles are in perspective and that their planes are distinct. Then the line of intersection of the two planes must contain all points of intersection of corresponding sides since each such point is the intersection of a line in one of the planes with a line in the other. Thus if triangles in different planes are in perspective then the points of intersection of corresponding sides lie on a line.

Theorem 2: Suppose the two triangles ABC and A'B'C' are arranged so that the corresponding extensions of edges meet along a line - as shown in the diagram.

Plot: desar1.eps

We claim that there is a point of perspective which is the same as saying that the three lines AA' BB', and CC' are concurrent, i.e have a point in common. Note that since BC and B'C' meet, the points B,B', C. C' are all in a plane, hence the lines BB' and CC' must meet or be parallel (i.e. meet at infinity). If these lines have two points in common or are parallel and have a point in common then they are the same line and since each lies in either the plane of ABC or that of A'B'C' they would have to lie in both and hence be the line of intersection (the green line). In such a case we would have C=C' and B=B' in which case projection from any point on the line AA' would project ABC onto A'B'C'.

If the points A and A' coincide then projection from the point of intersection of BB' and CC' (or from infinity if they are parallel) places the triangles in perspective. Thus the only remaining case is when none of the corresponding vertices are identical.

Let O be the point of intersection of BB' and CC' (or their common direction if they are parallel). If O does not lie on AA' (or,in the parallel case, if AA' is not parallel to the other two lines) then the points (A,A', and O) determine a plane L. In the parallel case we take L to be the plane containing line AA' and a line through A parallel to the lines BB' and CC'.

Since AB and A'B' meet, the lines AA' and BB' are in a common plane. Since the point 0 is on BB' that plane is L. Similarly L contains CC' and consequently A,B,C and A',B',C' are all co-planar. In the parallel case, since A,B,A', and B' are co-planar a line through A parallel to BB' would line in the plane of ABA'B' and hence L would be that plane. However the same holds for AA' and CC' so in this case also, L is the plane containing AA' and CC'. Thus in this case also we have that A,A',B,B', C,C' are all co-planar. This contradicts the assumption that ther planes of the two triangles are different and we have proved Case 1 of Desargue's Theorem.

Case2: The triangles lie in the same plane We will do this case in the form of an exercise set.

Theorem 1: (The Planar Case)

1.4.1 *EXERCISE: If A=A', B=B', C=C' then the theorem is trivial. (You should check through this to be sure you understand the statement(s) in the theorem.*

1.4.2 *EXERCISE: If B=B', C=C' then the line BC satisfies the theorem.*

1.4.3 *EXERCISE: If A=A' the line connecting A to any point common to BC and B'C' satisfies the theorem.*

We now show that the if there is a line which contains the edges BC and B'C' then we are done.

1.4.4 *EXERCISE: If no two corresponding vertices are equal and there is a line contining both the edges BC and B'C' then no other line contains an edge of each triangle.*

1.4.5 *EXERCISE: If BC and B'C' lie on a line they are the the only corresponding sides which extend to the same line. We then have that AB meets A'B' in one point and AC meets A'C' in just one point. The line connecting these two points satisfies the theorem.*

1.4.6 *EXERCISE: We now have that the coresponding vertices are all distinct and that no pairs of them lie on the same line. If the point O is one of the vertices, say A then B'C' satisfies all of the requirements of the theorem.*

1.4.7 *EXERCISE: If two non-corresponding vertices, say B and C' are equal but not they are not the point O then the lines BC and B'C' must coincide. Thus this case is contained in one covered in a previous exercise.*

We are thus in the planar case of Theorem 1 where all of the six vertices and the point of projection are distinct and thepairs of lines extending corresponding sides are also distinct.

1.4.8 *EXERCISE: Let L be a line through the point O which is not contained in the plane of the two triangles. Let V and V' be two points other than O on the line L and take the lines VV', and AA' which meet at 0. These lines span a plane PI and in PI the lines VA and V'A' have a unique point, A*, in common. SImilarly VB and V'B' have B* and VC and V'V' have C*.*

1.4.9 *EXERCISE: With the notation of the preceeding exercise, none of A*,B*,C* is in the plane of the two triangles, nor is any of them on the line L.*

1.4.10 *EXERCISE: The points A*, B*,C* do not lie on a line*

hint: Show that if they did then these points and V would lie in a plane which would be different from the plane of the triangles and would contain the points A,B, and C. Show that this would make A.B, and C lie on a line.

1.4.11 *EXERCISE: The triangles ABC and A*B*C* are two triangles in perspective from V and the triangles A'B'C' and A*B*C* are in perspective from V'*

Now that we know that these triangles are in perspective, by Case 1 we know that the lines AB and A*B* meet along the line of interesection of the planes of ABC and the point of intersection must be the point of intersection of the line A*B* with the plane of the triangles. Similarly, the lines AB and A'B' meet along the same line of interesection and the point of intersection must also be the point of intersection of the line A*B* with the plane of ABC. Thus AB and A'B' meet on the line of interesection of the two planes. The same argument holds for the intersections of the lines AC and A'C' and BC and B'C'. Thus line intersection of the two planes is the line promised by the theorem. This completes the proof of Theorem 1.

Theorem 2 (The Planar Case)

Recall that for Theorem 2 we must show that if the extended sides meet along a line then the triangles are in perspective. We proceed as with Theorem 1, reducing to the case where all points and lines are distinct. Let L be the line along which corresponding sides meet.

1.4.12 *EXERCISE: If A=A', B=B',C=C' then the theorem is trivial.(One should go thorugh this to firm up the statement of the theorem).*

1.4.13 *EXERCISE: If A=A' and B=B' but C and C' are distinct then the two triangles are in perspective from any point on CC'*

1.4.14 *EXERCISE: if A=A' but B and B' and C and C' are distinct then the triangles are in perspective from any point common to BB' and CC'.*

1.4.15 *EXERCISE: If the line BB' is equal to the line CC' then the triangles are in perspective from any point common to this line and AA'*

1.4.16 *EXERCISE: If L coincides with the line BC then C'A' must pass through C and B'A' must pass through B. The triangles are therfore in perspective from A'.*

1.4.17 *EXERCISE: If (the extensions of) two non-corresponding sides are equal, but not equal to L, say BC = C'A' then C=C' and this case reduces to a previous one.*

Hint: Argue that C' must be the point of intersection of BC and L and that C must be the point of intersection of A'C' and L

The previous cases place us in the situation where all of the corresponding vertices are distinct and all of the lines extending edges of the triangles are distinct.

Choose a plane PI which meets the common plane of ABC and A'B'C' at the line L and let P, Q, R be the intersections of be the intersections of BC, CA, and AB respectively with L.

1.4.18 *EXERCISE: Show that the points P, Q, R are distinct.*

hint: Show that if Q=R then A=A'

Draw through P, Q, R three lines in the plane PL which meet to form a triangle which we denote by A*B*C*.

1.4.19 *EXERCISE: Show that the triangles A*B*C* and ABC are in perspective from some point V and similarly A*B*C* and A'B'C' are in perspective from some point V'*

hint: Use the non-planar case of the theorem

1.4.20 *EXERCISE: Show that V and V' are distinct and neither lies in the plane of ABC*

hint: Show that if V=V' then A=A'. Show that if V were in the plane of ABC then A*B* and C* would be in that plane too.

1.4.21 *EXERCISE: Let O be the point where VV' meets the plane of ABC. Show that the lines AA', BB', and CC' all pass through O and hence the triangles ABC and A'B'C' are in perspective from O.*

hint: VA and V'A' meet in A*. Show that this implies that VV' and AA' meet in O.

1.5 Remarks on Using Desargues Theorem To Represent one Object as a Shadow of Another

Remark: To use Desargue's theorem to decide if two figures are in perspective we note that if they are in perspective then any pair of correpsponding triangles must also be in perspective. Assuming they are in different planes this means that all lines through corresponding pairs of lines in the figures must meet at the line of intersection of the two planes. Moreover since the point of perspective (or common direction) is determined by any pair of lines through corresponding points (or their direction if they are parallel), any two pairs of triangles which share an edge and are in perspective with their corresponding triangles, must correspond under projection from the same point (namely the point of intersection or common direction) of the two corresponding lines.

Thus if we want to determine if two figures which are not co-planar are in perspective we check that:

(1) The figures are composed of corresponding triangles such that each pair of points in a figure is contained in two members a chain of triangles with successive members of the chain sharing an edge

(2) Each pair of lines through corresponding vertices either meet along or are both parallel to the line of intersection of the planes of the two figures.

If both of the above conditions are true then the two figrues are in perspective.

Remark 2: If two non-coplanar, planar figures are in perspective then they remain in perspective through any rotation of one of them about the line of intersection of their planes.

The previous remarks provide us with a way to construct plane figures which are shadows of another plane figures which we state for quadrilaterals.

1. Draw the first quadrilateral on a plane surface and draw one of its diagonals

3. Draw a line (the axis) transversal to all of the extended edges and the diagonal but not intersecting the original figure.

4. From the point of intersection of the diagonal with the axis draw line.

5. From the points of intersection of each pair of edges which do not meet in one of the chosen diagonal points, draw a pair of lines which meet along the line in the previous step. These lines form the extended edges of a quadrilateral which is in perspective to the original.

6. "Fold" the plane at the axis so that the two quadrilaterals now lie in different planes which meet along the axis. They remain in perspective from a point which can be found as the intersection of the lines through any pair of corresponding points.