Visual Problem Solving with Maple

A Text for Teachers

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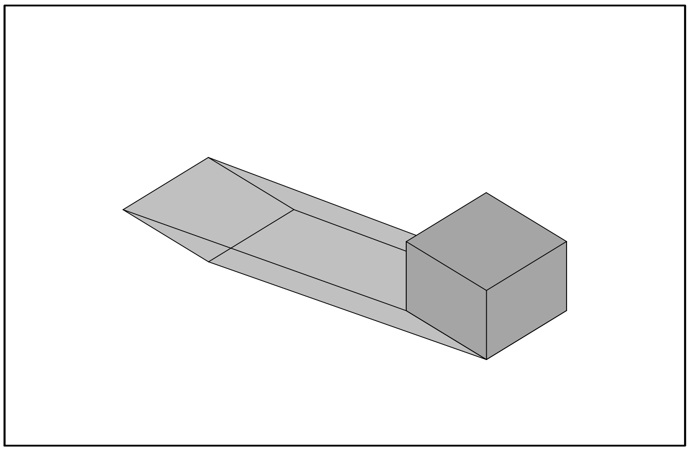

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1 Introduction

1.1 A Lament

Teacher: "William, you are not doing those additions correctly."
William: "Thats the way I do them."

Conversation reported by Dr. Don Coleman when relating his experience as an elementary school teacher in 1980

Our Mathematics has been over five millenia in develoment and is unrivaled among human constructions for its unity, consistency, and utility. Nevertheless it is percieved by many as a disconnected collection of arcane formulas, symbols, and processes whose principal use is the production of numbers, formulas, or symbols called 'answers'. A 'mathematics problem' is typically expected to be stated in a telegraphic style, and produce a single correct answer. The object of mathematical study seems to be learning which steps to apply in a particular situation to produce the answer. Significant weight is attached to the speed and accuracy with which one can produce the answer. 'Problems' tend to be viewed as things found at the end of a chapter or section in a text book or on a test. 'Working a problem' is a task that takes at most a few minutes of locating a similar exercise in the text and then matching the steps employed to solve that problem to the current problem. The test of whether one's answer is correct is that it agrees with the table of answers in the back of the book, or the teacher's answer. Getting the same answer as these oracles 75% of the time is as much as could be expected from an 'average' student.

Viewed this way William's approach is altogether reasonable. His algorithm gave the correct sum of three, three digit numbers most of the time. Since mathematics is simply the application of algorithms to symbols there was no obvious reason why the official algorithm is preferable to the one he devined and could apply quickly, consistently, and accurately.

There is no report on whether Dr. Coleman's admonition had an influence on William's view of mathematics but as measured by the percentage of college freshmen with the same outlook, over a decade of exposure to the same message, delivered by highly qualified, motivated teachers has not resonated with many. These students, many of whom have made good grades in algorithmic, 'plug and chug' courses emerge from a dozen year's study of mathematics with problem solving skills not even remotely commensurate with the investment made in their education. Further, they leave with a view of the discipline and the level to which the average person can master it not so far removed from that of ancient Egyptians peasants for whom priests reconstructed the boundaries of their fields after the floods of the Nile. While obviously imposing career limits on adults who hold them, such attitudes are readily passed on to the next generation by many parents who are unable to assist and encourage their children in studies of mathematics beyond elementary arithmetic or algebra.

1.2 Part of the problem and a remedy

William had no objective criterion for assessing whether his solution was better or worse than any other, other than by comparing it with the 'official' answer. He could work so fast that he got more correct answers than most of his peers so reinforcement from them was unlikely to cause him to re-assess his approach. Nor, one suspects, was input from parents. One may note, however, that if William had invented a new fingering for a musical instrument with the same properties as his mathematical algorithm (80 percent correct) he and his peers would have very quickly determined that the approach was in need of further development. Their common understanding of how some things are supposed to sound provides the basis for an objective assessment. Our object here is to bring something akin to that type of reality to the exercise of elementary mathematics, by providing experience in creating and analyzing problems whose statements and/or solutions are extrinsic to formal mathematics and are phrased in terms which can be appreciated and even assessed independently of the mathematics required to model or solve them. The extrinsic character allows students to guage the effacy of their understanding progress toward solutions and allows peers and parents to provide encouragement and criticism in the venaculor of the context, even when of they have no facility with, or even understanding of the mathematics.

An admittedly incomplete, though useful metaphor is music. Parents and colleagues to whom musical notation and theory is as impenetrable as any page of mathematics are nevertheless able to appreciate, encourage, and to some extent criticize student music. Student musicians themselves have a reasonable feeling for their progress toward 'getting it right'. An analog of this in mathematics occurs when a visual object, such as a picture or animation is produced which can be viewed, appreciated, and critiqued by an 'audience'. This is one of the goals of our visual approach in this text which we call Visual Problem Solving.

In Visual Problem Solving, we study the development and analysis of problems within a context in which a spectrum of explorations are not only feasible but natural. Results of explorations must be expressible in a visual format which can be appreciated independently of the underlying mathematics and through which a 'lay' person can assess in a meaningful way the progress it represents toward the exploration's stated objective. While emphasis on the visual serves the objective of introducing what for lack of a better term we forthrightly call a 'performance' dimension, it's more fundamental purpose is the linking of geometric intuition, the most powerful ananlytic tool most people possess, to systematic problem solving.

In this text, we will introduce and develop a number of problem contexts, each of which satisfies the following:

- The context can be modeled and is amenable to serious exploration using elementary arithmetic, algebra, trigonometry, and plane and analytic geometry. We eschew calculus here but are perfectly willing to use formulas for the volume of a cone and such basic notions as position, velocity, acceleration from calculus.

- The contexts, or at least solutions to explorations they engender must be amenable to highly visual explanation or representation which can be appreciated with no understanding of the mathematical details. Such an observer should be able on the basis of the visual presentation to make an informed decision on whether the work presented reprsents progress toward a solution to the problem under investigation.

- The explorations suggested by the context should lie in a broad spectrum from elementary descriptive projects to high quality student research areas.

In general our approach is to introduce a context and develop a mathematical model which contains or is extensible to a family of explorations. Typically, we initiate a primary investigation and ask students to complete or extend parts of it over a fixed period of time. Except for projects done for individual