Problem Solving with Maple
A Handbook for Calculus Students

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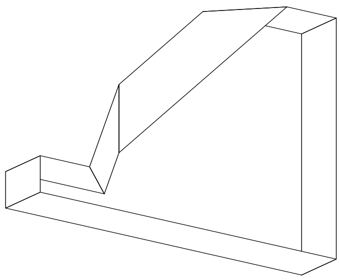
Carl Eberhart, Department of Mathematics
University of Kentucky



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Carl Eberhart, Department of Mathematics, University of Kentucky carl@ms.uky.edu

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Chapter 1

The Maple Language

Here we intend to provide you with just enough information about the Maple language to give a headstart at using it productively in the problem solving process. The particular version of Maple that we are using as we describe it is Maple V, Release 3, running either in Windows 3.1 on DOS or X on Unix. Maple has at least four properties which make it a very useful problem solving tool.

It is an immediate mode language: That means that you can type in commands and execute them, just like in the languages Basic and Logo. A Maple command is simply a string of characters ending in a semicolon ; or colon : . For example, the command

```
>  2+5, factor(x^2+5*x +6),expand((x+y)^2);
```

$$7, (x+3)(x+2), x^2+2xy+y^2$$

tells Maple to do a sequence of three things: add 2 and 5, `factor` the quadratic, and `expand` the binomial. The strings `factor` and `expand` are called Maple 'words'. These are names of procedures which have been defined for performing the action (sometimes) suggested by the name on the stuff enclosed in parentheses just after the word. That stuff is usually called the 'input' of the word. The result of performing the procedure on the input is naturally called the 'output' of the word. If the command is terminated with a semicolon (colon), the output is displayed (not displayed).

It is a symbolic and graphical manipulator: That means that it is built to work with algebraic expressions and draw pictures. There is a large vocabulary of Maple words, such as `factor` , `simplify` , and `expand` which are used to 'symbolically manipulate' expressions in the manner you are used to doing with pencil and paper. There are also a number of plotting words, which are used to draw graphs of functions of one or two variables, curves and surfaces. These drawings can be animated (ie displayed in sequence) to study change. Two of the plotting words most often used, `plot` and `plot3d` ,are part of the regular Maple vocabulary. The rest, and there are several, are found in the the `plots` package, a sort of specialized vocabulary of words which is loaded separately.

It is an extensible language. What this means is that you can define additional words and add them to the vocabulary. Initially, there will be little need to do this, except to define functions. The existing vocabulary is large enough to carry out the solution to many problems. After awhile, it becomes very useful to be able to add new words to the vocabulary. If you develop some words to work on a specialized class of problems, these can be put into a package of words for easy access between worksheets. Maple comes equipped with several such packages already, including `plots` - a package of drawing words, `linalg` - a very useful package of vector and matrix manipulation words, `combinat` - a package of words from combinatorics, and `networks` - a graph theory package.

It comes with a worksheet environment – see below. We want to learn to use Maple to solve problems. In order make good use of the language for this purpose, we need to become familiar with the worksheet environnment, get to know the language, and foster our experimental urges.

1.1 The Worksheet: A handy place to solve problems.

When you click on the Maple Icon in Windows or run xmaple in Unix, an untitled worksheet opens up. Think of it as a clean sheet of paper. Typically, after awhile, the worksheet will contain a record of the work done to date on the problem or problems you have been working on. Very often, you might be in a problem-solving team working on the problems. The worksheet can be given a name and saved onto a disk for later working or for handing in. The worksheet file consists of a number of cells of three different types: Input cells, Output cells, and Text cells.

Input Cells: These are started with a right angle bracket > . Here are a couple. An input cell is where you type in one or more Maple statements (aka commands) to be executed. The nice thing about these cells in a worksheet is that they can be modified and reexecuted over and over again. This enables you to correct typing mistakes with relative ease. For example, suppose I wanted Maple to calculate `3*(4+5*3)*(7+6);` but left out a parenthesis.

```
> 3*(4+5*3)*(7+6 ;

syntax error:
3*(4+5*3)*(7+6 ;
               ^
```

A syntax error is generated which may help you find your typing mistake. So you can make a change in that input cell and reexecute it.

You can also use this ability to change and reexecute input cells to change the numbers in whatever problem you have worked out a solution to and see how the solution is affected.

Output Cells: Almost every input cell, when executed, gives an output cell containing the results of the calculations. It is appended to the input cell which produced it. For example, let's add 3 to the 23rd and 4 to the 12th – then let's factor the result into prime factors.

```
>  s := 3^23 + 4^12;
```

$$s := 94159956043$$

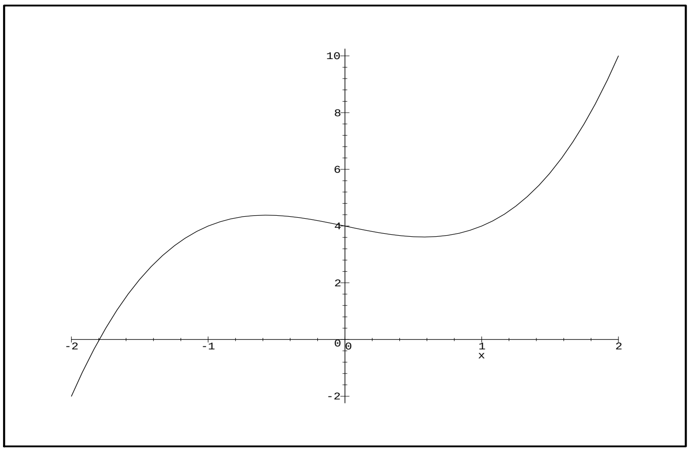
```
>  ifactor(s);
```

$$(727)\,(129518509)$$

You inspect the output cell to see if it is what you want. If it is not, then go back and change the input cell, and reexecute it.

Certain Maple words such as `plot` generate a separate window containing a picture or a page of text (see factor in the next section). You can copy and paste these items into an output cell of your worksheet if the need arises.

```
> plot(x^3-x+4, x=-2..2);
```



Problem :

Execute the following `plot` command. Then `copy` the graph from the plot window (use the `Edit Menu` in that window) and `paste` it into your worksheet.

```
> plot(x^3-x+4,x=-2..2);
```

Text or Comment Cells: A Text Cell is to record remarks and explanations of the solution to the problem you are working on. All of the comments here are typed into text cells. It is a good idea to be liberal with text cells. Undocumented calculations are for the most part worthless to anyone but the person who made the calculations, and even then the value is a rapidly decreasing function of time.

You have very good control over the cells in your worksheet with the Menus at the top of this worksheet. Depending on what platform you are using (Windows, X, or Mac), the menus may vary in their titles, but the effects are the same with minor differences:

You change an input cell to a text cell and back again.

You can split a cell (input or text) and join cells.

You open up new cells (input or text) between existing cells.

You can erase cells (input, text, or output) at your discretion.

You can copy cells (input or text) and paste them into another location.

These controls make it relatively easy to work away on a worksheet, doing computations, recording observations, explanations, etc and then go back and polish the worksheet up into a public document. —

The worksheet does not replace pencil and paper or the chalk board in the problem solving process. When you are in the middle of analyzing a problem and deciding how to solve it, these tools are extremely useful. It is easier to draw a rough diagram by hand than by Maple. After you have decided on a plan and need to do some numerical calculation, some symbolic manipulation, or some plotting to carry out the plan, then a knowledge of the Maple language becomes useful.

1.2 Get to know the language

Maple has a large built in vocabulary of words especially defined to carry out many of the algorithms you have learned in your previous math classes. There are Maple words like `factor` , `expand` , `simplify` , etc. You can learn about them with the on-line Help facility. There is a Browser available in both X and Windows Maple which has the Maple vocabulary nicely indexed by category. Alternatively, you can ask for help in an input cell.

For example, to find out about `factor` just type

```
> ?factor
```

Knowing the word is one thing, but you also need to know the syntax of the word. What is syntax? Every algorithm requires certain input information in order to be performed. After it has been performed, certain output information is produced. To know the syntax of a Maple word is to know the input information needed and the output information produced by the word. The help screen gives you the syntax of the word, and thus tells you how to use it. For example, the help screen for `factor` tells us (in CALLING SEQUENCES) that there are one and possibly two inputs needed and one output produced by factor. The most useful part of the help screen is the bottom part, where there are examples of the usage of the word in question. These examples can be copied into a worksheet and tested out, which gives you a chance to develop a feel for the word by experimenting with its use. In fact, there is a Maple word `example` which brings up a page of examples of the usage of the word, in many cases.

Problems: 1. Discover the difference between `factor` and `ifactor` using `?ifactor`.

2. Use `ifactor` to factor your social security number.

3. Use `nextprime` to find the first 3 primes after 5000.

4. How do `collect` , `coeff` and `expand` work? Use them expand `(x+2*y)^4`, collect the result into a polynomial in x and find the coefficient of the `x^3` term of that.

1.3 Experiment!

Experimentation is a prime source of learning. We are constantly conducting little experiments, learning from them, and using that knowledge in some way. The same holds true when you are learning Maple and/or working on some math problem. The help facility is a great aid to experimentation. Say you are working on a math problem and you need to carry out some algorithm, like solve an equation you have set up. You know the Maple word `solve` will carry out the algorithm, but you have forgotten the syntax. You can use the word `example` to list some examples of its usage, rather than bring up the entire help sheet on `solve` .

```
> example(ifactor);
```

Keep in mind that the help sheets are written for general use by both novices and experts, so don't be intimidated by unfamiliar terminology. Often the examples at the bottom of the sheet suffice to tell you what you need to know. Keep using help!

Problems: 1. Spend a few minutes with the online help deciding what the terms list and set mean in Maple. In particular, what is the main difference between a list and a set?

```
> example(list);
> example(set);
```

2. If you haven't already, find out the usage of the terms `seq` , `NULL` , `subsop` as used in the

example sheet for set and list.

Experiment is used at the beginning of the problem solving process to generate data to conjecture a solution. Towards the end of the process, experiment is used to test the solution. Maple makes the act of experimentation easier to carry out and easier to modify. A Maple Worksheet makes it easier to carry on work from one session to the next, and to prepare public documents, ie, homework assignments, reseach papers, etc for public consumption.

1.4 A Short Introduction to the Maple Language

This section contains an introduction to the Maple vocabulary used for solving problems. It is not meant to cover everything, just some of the basics. In most instances, only the Maple input is described; I am assuming that you are 'typing along', so that you can see the Maple output as we go.

First, there is arithmetic: addition, subtraction, multiplication, division and exponentiation. These can be combined, just as on a calculator. The order of precedence is the the usual one: exponentiation first, then multiplication and division, then addition and subtraction. So entering

```
> 2-3+4/5*6^7;
```

$$\frac{1119739}{5}$$

is the same as entering

```
> (2-3)+(4/5)*(6^7);
```

$$\frac{1119739}{5}$$

Maple works with fractions whenever possible, changing to decimal numbers only on demand. So typing

```
> 1/3 + 1/2;
```

$$\frac{5}{6}$$

will get a return of 5/6. Putting a decimal point in one of the numbers forces Maple to return a decimal answer. Also, you can use the maple word `evalf` to convert a result to decimal form.

```
> evalf(5/6);
```

$$.8333333333$$

Maple does arithmetic with complex numbers too. `I` is a Maple constant standing for `sqrt(-1)`. So entering

```
> (3+2*I)*(2-I);
```

$$8 + I$$

will produce an output of `8+I`.

The name for pi, the area of the circle of radius 1, in Maplese is `Pi` . So to calculate the area of a circle of radius 3, you would enter

```
> Pi*3^2;
```

$$9\,\pi$$

1.4.1 Expressions, Names, Statements, and Assignments

Quantities to be computed like 1/2+1/3 are called expressions . A name is a string of characters which can be used to store the result of a computation. A statement in Maple is a string of names and expressions terminated with a semicolon, which when entered will produce some action. The assignment statement is one of the most common statements. It is of the form `name := expression;` For example, the assignment

```
> area := Pi*3^2;
```

$$area := 9\,\pi$$

stores 9*Pi in a location marked by the name area. A more useful assignment for the area of a circle is

```
> area := Pi*r^2;
```

$$area := \pi\, r^2$$

In this case, the expression Pi*r^2 is stored in area and with this assignment, the area of a circle of any given radius can be computed using the Maple word subs. So to calculate the area when r is 3, we enter

```
> subs(r=3,area);
```

$$9\,\pi$$

Here, it is convenient to think of the assignment as defining area as a function of the radius r.

1.4.2 Functions

We are thinking here of the notion of function as a rule f (possibly very complicated) for assigning to each argument x in a given set of numbers a unique number f(x) called the value of f at x.

Functions can be defined in several useful ways in Maple.

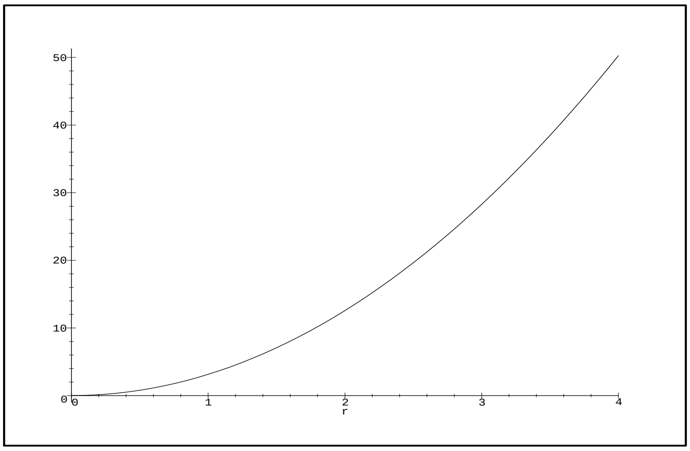
As an expression: The assignment

```
> area := Pi*r^2;
```

$$area := \pi\, r^2$$

defines the area of a circle as a function of it's radius. The area function defined as an expression is evaluated with subs. Since this function assigns real numbers to real numbers, its values can be plotted on a graph with the Maple word plot. So the statement

```
> plot(area,r=0..4);
```



will produce in a separate plot window, the graph of the area function over the interval from r=0 ..4. With the arrow operator the assignment:

```
> area := r -> Pi*r^2;
```

$$area := r \to \pi\, r^2$$

defines the area function also. Now to find the area of a circle of radius 3, we simply enter the statement

```
> area(3);
```

$$9\,\pi$$

To plot this function over the domain r=0..4 , type

```
> plot(area,0..4);
```

Note that the variable r is omitted here. Use `unapply` . This ugly little word transforms expressions of one or more variables into functions defined by an arrow operator. For example, if we had a polynomial defined by the assignment

```
> pol := x^2 + 4*x -1;
```

$$pol := x^2 + 4\,x - 1$$

then the assignment

```
> pol := unapply(pol,x);
```

$$pol := x \rightarrow x^2 + 4\,x - 1$$

turns po l into a function defined by an arrow operator. As a procedure: the assignment

```
> area := proc(r) Pi*r^2 end;

area := proc(r) Pi*r^2 end
```

defines the area function too. It is evaluated and plotted as in the arrow operator definition. One advantage of this way of defining a function is that the domain can be specified. For example, the domain of the area function for a circle is all positive real numbers. This can be inserted into the procedure, with the Maple word ERROR . The message must be enclosed in back-quotes ` , which is on the key with the tilde

```
> area := proc(r)
>   if r <= 0 then ERROR(`radius must be positive`)  else Pi*r^2 fi
> end;

area := proc(r)
            if r <= 0 then ERROR(`radius must be positive`)
            else Pi*r^2
            fi
        end

> area(3);
```

$$9\,\pi$$

```
> area(-3);

Error, (in area) radius must be positive
```

Note the if ... then ... fi; control statement here. Use `?if` to learn more about this command. Functions of two variables can be defined and plotted just as easily in Maple as functions of one variable.

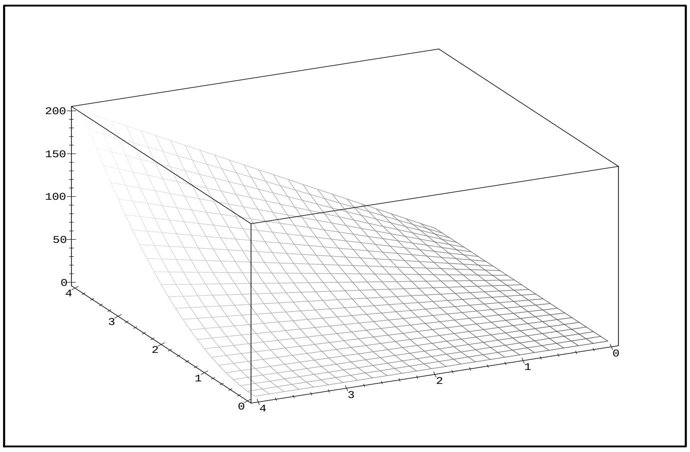
For example, the volume V of a cylinder of height h and radius r is defined by

```
> V := (r,h) -> Pi*r^2*h;
```

$$V := (\,r, h\,) \rightarrow \pi\,r^2\,h$$

To see what the graph of V looks like, use `plot3d` :

```
> plot3d(V,0..4,0..4);
```



Which way of defining a function is the preferred way? That really depends on the situation. The expression method works well for functions which have only one rule of evaluation, but eventually you cannot avoid using an -> or `proc` definition. You will find yourself using arrow or proc definitions more and more as time goes by.

1.4.3 Built in Maple functions and Operations with Functions

All of the standard scientific functions are built into Maple. For example, `sqrt` is the square root function, `abs` is the absolute value function, the trig and inverse trig functions are `sin` , `arcsin` , `cos` , etc., the natural logarithm and exponential functions are `ln` and `exp` . For a complete list of built in functions, type

```
> ?inifcns.
```

New functions can be obtained from old functions by use of the arithmetic operations of addition, subtraction, multiplication, and division together with the operation of composition, which is denoted by `@` . Thus the function defined by the assignment

```
> y := sin(cos(x^2+3));
```

$$y := \sin(\cos(x^2+3))$$

and evaluated at `x=3` by

```
> subs(x=3.,y);
```

$$\sin(\cos(12.))$$

could also be defined by the assignment

```
> y := sin@cos@(x->x^2+3);
```

$$y := \sin@\cos@(x \to x^2+3)$$

and evaluated at `x=3` by

```
> y(3.);
```

$$.7472099577$$

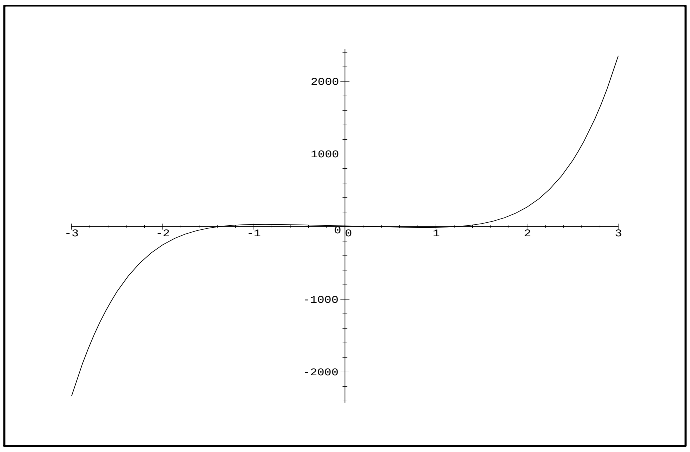
1.4.4 Using Maple as a fancy graphing calculator.

It is convenient to think of Maple as a fancy graphing calculator for many purposes. For example, suppose you want to find the real solutions of the equation `x^5 - 30*x - 2 = 0` in the interval `-3..3`. Then we can just `plot` the right hand side of the equation.

```
> f := x -> 10*x^5 - 30*x +10 ;
```

$$f := x \to 10\,x^5 - 30\,x + 10$$

```
> plot(f,-3..3);
```



By inspection, the graph crosses near 0. We can look closer.

```
>  plot(f,-1.5..1.5);
```

We see that the graph crosses 3 times, the largest solution being between 1 and 1.5. If we wanted the largest solution more accurately, we could use `fsolve` .

```
> fsolve(f(x)=0,x,1..1.5);
```

$$1.214648043$$

1.4.5 Data types, Expression Sequences, Lists, Sets, Arrays, Tables:

Maple expressions are classified into various data types . For example, arithmetic expressions are classified by whether they are sums type '+' , products type '*' , etc. The Maple word `whattype` will tell what type a particular expression is.

```
> whattype(1/2);
```

$$fraction$$

```
> whattype(a + b);
```

$$+$$

```
> whattype(x^2 + x = 2*x - 1);
```

$$=$$

```
> whattype(a,b,3);
```

$$exprseq$$

Expression Sequence. An `exprseq` , expression sequence , is any sequence of expressions separated by commas. For example,

```
> expseq := 1,2, w*r+m, a=b+c, 1/2, (x+y)/z,`hello`;
```

$$expseq := 1, 2, w\,r + m, a = b + c, \frac{1}{2}, \frac{x + \sin@\cos@(\,x \to x^2 + 3\,)}{z}, hello$$

is an assignment to `expseq` of an expression sequence of 7 expressions. To refer to the sixth expression in this sequence, use the expression `expseq[6];`

```
> expseq[6];
```

$$\frac{x + \sin@\cos@(\,x \to x^2 + 3\,)}{z}$$

```
> explist:= [expseq];
```

List. A list is an expression sequence enclosed by square brackets. So

$$explist :=$$
$$\left[1, 2, w\,r + m, a = b + c, \frac{1}{2}, \frac{x + \sin@\cos@(\,x \to x^2 + 3\,)}{z}, hello\right]$$

makes a list whose terms are those in `expseq` . As with expression sequences, we can refer to particular terms of a list by appending to its name the number of the term enclosed in square brackets. Thus to get the fifth term of `explist` , type the expression

```
> explist[3];
```

$$w\,r + m$$

You can also reference the fifth term in this list by typing

```
> op(3,explist);
```

$$w\,r + m$$

In general, `op(n,explist);` returns the nth term in the list `explist` . To count how many terms are in a list, use the word `nops` . So for example,

```
>  nops(explist);
```

$$7$$

tells us that there are 7 terms in the list `explist` . `nops` comes in handy when you don't want to (or can't) count the terms in a list by hand (this is almost always). You can't directly use the word `nops` to count the number of terms in an expression sequence. But you can put square brackets around the expression sequence and count the terms in the resulting list. This device is used again and again.

```
>  nops(3,4,a);

Error, wrong number (or type) of parameters in function nops

> nops([3,4,a]);
```

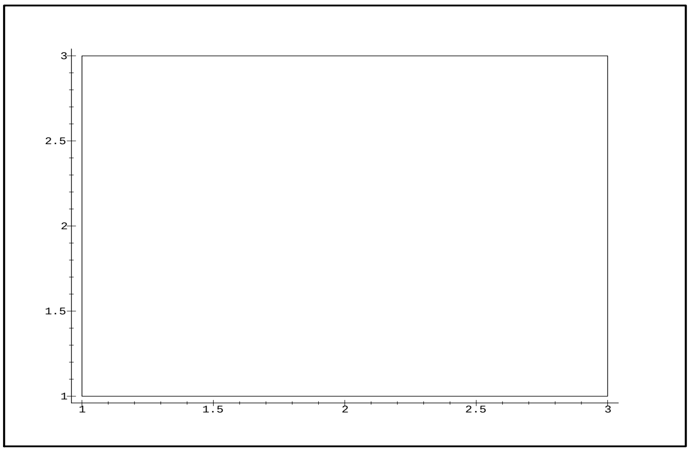
$$3$$

One important use of lists is to make lists of points to plot. For example, to draw a picture of the square with vertices (1,1), (3,1), (3,3), (1,3), make a list and then plot it.

```
> ab := [1,1,3,1,3,3,1,3,1,1];
```

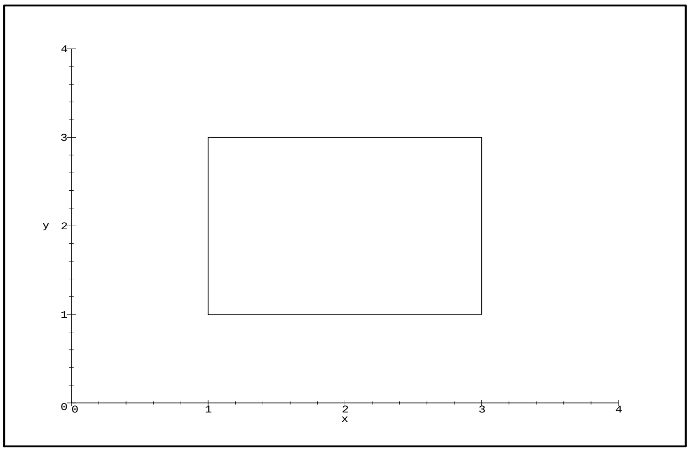
$$ab := [1, 1, 3, 1, 3, 3, 1, 3, 1, 1]$$

```
> plot(ab);
```



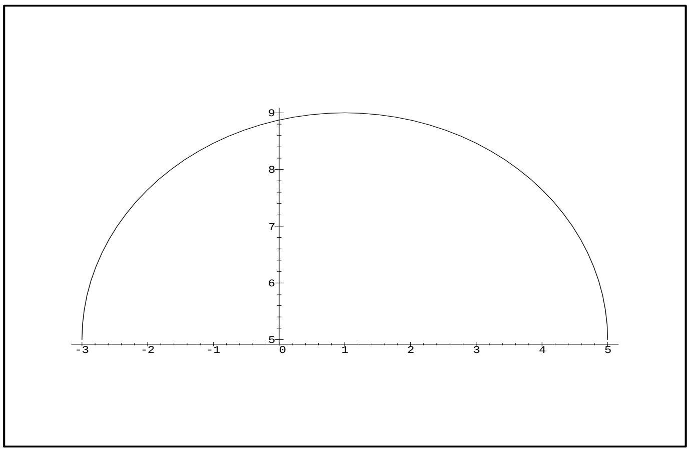
Notice in the graph that the origin is not included in the field of view. We can specify that by restricting the x and y coordinates. We can also choose to have no axes displayed.

```
> plot(ab,x=0..4,y=0..4);
```



Another use of lists is with parametric plots . If you have a curve in the plane described parametrically with x = f(t), y = g(t), as the parameter t runs from a to b, then you can draw it by making up a 3 term list to give to plot. Say you wanted to draw the upper half of the circle of radius 4 centered at (1,5). Then the list consists of the expressions for the x and y coordinates followed by an equation giving the range of the parameter.

```
>  plot( [1+4*cos(t),5+4*sin(t),t=0..Pi],scaling=constrained);
```



If you had to draw several pieces of circles, you might define a function `circ` to simplify things.

```
> circ := (h,k,r,f,l) -> [h+r*cos(t),k+r*sin(t),t=f..l];
```

$$circ := (\,h, k, r, f, l\,) \to [\,h + r\cos(\,t\,), k + r\sin(\,t\,), t = f..l\,]$$

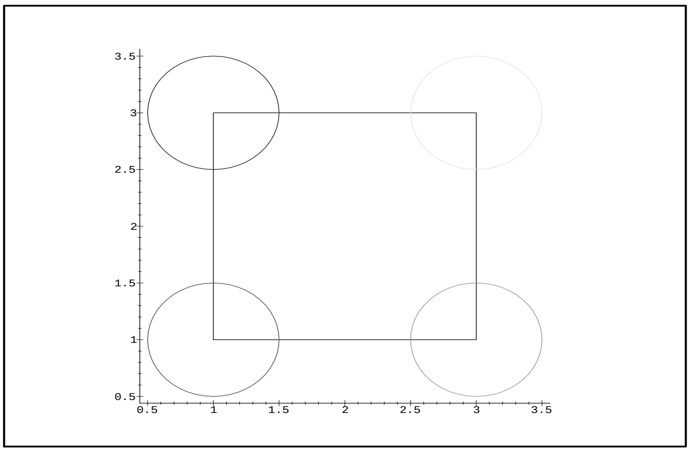
So if we wanted circles of radius 1/2 centered at the corners of the square `ab` we can construct the sequence of lists

```
> circs := seq(circ(ab[2*i-1],ab[2*i],1/2,0,2*Pi),i=1..4);
```

$$circs := \left[1 + \frac{1}{2}\cos(\,t\,), 1 + \frac{1}{2}\sin(\,t\,), t = 0..2\,\pi\right],$$
$$\left[3 + \frac{1}{2}\cos(\,t\,), 1 + \frac{1}{2}\sin(\,t\,), t = 0..2\,\pi\right],$$
$$\left[3 + \frac{1}{2}\cos(\,t\,), 3 + \frac{1}{2}\sin(\,t\,), t = 0..2\,\pi\right],$$
$$\left[1 + \frac{1}{2}\cos(\,t\,), 3 + \frac{1}{2}\sin(\,t\,), t = 0..2\,\pi\right]$$

In order to plot these circles, you need to enclose them in curly brackets to make a set of the sequence before you give them to `plot` . See below for a discussion of sets.

```
> plot({circs,ab},scaling=constrained);
```



Sometime you might want to split a list of points to plot into a list of x-coordinates and another list of ycoordinates. The Maple word seq is very handy for this and many other operations. So to split off from ab the odd and even terms–

```
> xdat := [ seq(ab[2*i-1],i=1..nops(ab)/2)];
```

$$xdat := [\,\mathbf{1, 3, 3, 1, 1}\,]$$

```
> ydat := [seq(ab[2*i],i=1..nops(ab)/2)];
```

$$ydat := [\,\mathbf{1, 1, 3, 3, 1}\,]$$

What about the converse problem? Building up a list of points to plot from two lists can also be done. The first thing you might think of doesn't work, however.

```
>  seq(xdat[i],ydat[i],i=1..nops(xdat));

Error, wrong number (or type) of parameters in function seq
```

Seq doesn't work well with a pure expression sequence as input. However, with some coaxing we can get it to do what we want.

```
>  newab :=[seq([xdat[i],ydat[i]],i=1..nops(xdat))];
```

$$newab := [\,[\,\mathbf{1, 1}\,], [\,\mathbf{3, 1}\,], [\,\mathbf{3, 3}\,], [\,\mathbf{1, 3}\,], [\,\mathbf{1, 1}\,]\,]$$

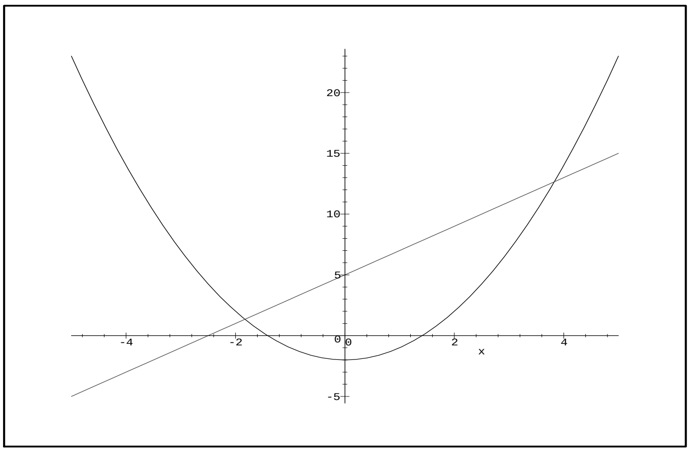
What did we do to change the input to seq ? We enclosed it in square brackets. If you feed such a list of points to plot, it knows what to do. If you wanted to strip out the inside brackets, that can be done too.

```
> newab := [seq(op([xdat[i],ydat[i]]),i=1..nops(xdat))];
```

$$newab := [\,\mathbf{1, 1, 3, 1, 3, 3, 1, 3, 1, 1}\,]$$

Sets. A set is an expression sequence enclosed by curly brackets. Order of appearance is not important in sets. Sets are important when plotting more than one function at at time. For example,

```
> plot({x^2-2,2*x+5},x=-5..5);
```



plots the parabola y=x^2-2 and the line y=2x+5 over the domain x=-5..5 on the same graph.

If you have a very complicated drawing to make, you can use plots[display] from the plots package.

Tables and Arrays A table is a special kind of data structure which is very flexible. The packages of special vocabularies are really tables whose indices of the package are the names of the procedures and whose entries are the bodies of the procedures.

An array is a special kind of table whose indices are numerical. The most useful arrays are matrices (2 dimensional arrays) and vectors (1 dimensional arrays). Matrix operations are made using Maple word evalm together with the symbol for matrix multiplication &*.

```
> a := array(1..2,1..2);
```

$$a := \mathbf{array}(\,\mathbf{1..2}, \mathbf{1..2}, [\;]\,)$$

creates a 2 by 2 matrix, whose entries are accessed as a[1,1] etc.

So to rotate the square ab := [1,1,3,1,3,3,1,3,1,1] through an angle of 31 degrees counter clockwise about the origin and display it, we could proceed as follows.

```
> rot := array([[cos,-sin],[sin,cos]]);  # set up the
```

$$rot := \begin{bmatrix} \mathbf{cos} & -\mathbf{sin} \\ \mathbf{sin} & \mathbf{cos} \end{bmatrix}$$

```
> ang := evalf(Pi/180*31);
```

$$ang := \mathbf{.5410520681}$$

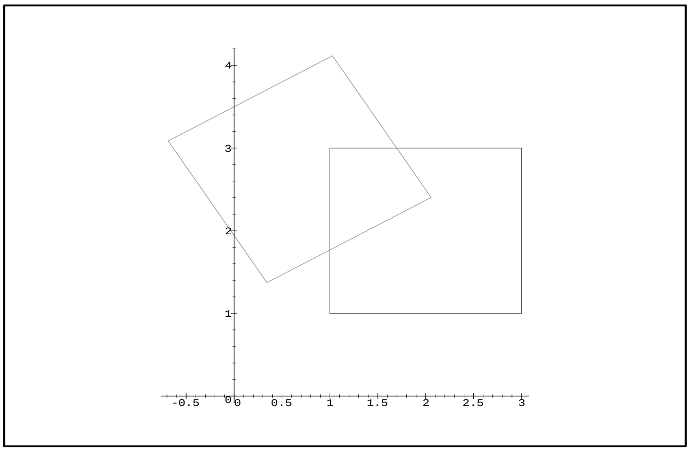
```
> ab :=  [1,1,3,1,3,3,1,3,1,1];
```

$$ab := [\,\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{3}, \mathbf{3}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}\,]$$

```
> rotab := [seq(evalm(rot(ang)&*[ab[2*i-1..2*i]]),i=1..nops(ab)/2)];
```

$$\begin{aligned} rotab := [[\,&\mathbf{.3421292258}\;\mathbf{1.372205376}\,], \\ &[\,\mathbf{2.056463827}\;\mathbf{2.402281526}\,], [\,\mathbf{1.026387677}\;\mathbf{4.116616127}\,], \\ &[\,-\mathbf{.6879469243}\;\mathbf{3.086539977}\,], [\,\mathbf{.3421292258}\;\mathbf{1.372205376}\,]] \end{aligned}$$

```
> plot({[0,0],ab,rotab},scaling=constrained);
```



1.4.6 Maple control statements

There are two especially important control statements . One is the repetition loop, and the other is the conditional execution statement. The repetition loop is

for .. from .. by .. to .. while .. do .. od;

This statement can be used interactively or in a procedure to perform repetitive tasks or to do an iterative algorithm.

Example : Add up the first 100 numbers.

```
>  s := 0:  for i from 1 to 100 do  s := s+i od:
>  s;
```

$$5050$$

Example : Compute the cubes of the first five positive integers and store them in a list. Then do it again, storing them in an array.

Solution with lists:

```
> locube := []:  # start with the empty list
>  for i from 1 to  5 do
>      locube := [op(locube),i^3] od:
> locube ;  # look at the list;
```

$$[1, 8, 27, 64, 125]$$

Note the way the list is built up from nothing. Each time through the loop, one more term is added onto the end of the list. With arrays, one can be more direct.

Solution with arrays:

```
> aocube := array(1..5):  # initialize the array.
> for i from 1 to  5 do aocube[i]:= i^3 od;
```

$$aocube_1 := 1$$

$$aocube_2 := 8$$

$$aocube_3 := \mathbf{27}$$

$$aocube_4 := \mathbf{64}$$

$$aocube_5 := \mathbf{125}$$

```
> op(aocube); # to see the array
```

$$[\,\mathbf{1\;8\;27\;64\;125}\,]$$

Now the array aocube has the numbers stored in it. To refer to the third element of aocube , we would enter aocube[3] just as if it were a list, rather than an array. Why have arrays at all? Well, for one thing, the terms in an array can be more easily modified. For example, to change the third term in aocube to 0 just enter aocube[3] := 0; . To change the third term in locube to 0, you have to make an entirely new list whose terms are all the same as locube except for the third one.

```
> aocube[3]:=0;
```

$$aocube_3 := \mathbf{0}$$

```
> print(aocube);
```

$$[\,\mathbf{1\;8\;0\;64\;125}\,]$$

```
> locube := [locube[1],locube[2],0,locube[4],locube[5]];
```

$$locube := [\,\mathbf{1, 8, 0, 64, 125}\,]$$

Problem .

There is a much less cumbersome way of doing the list solution. Start with the empty expression sequence `locube := NULL;` and just tack the cubes onto the end of the sequence as they are generated. `for i from 1 to 5 do locube := locube,i^3 od:` Now you have the first 5 cubes in a sequence. How do we turn that into a list? By putting square brackets around it! `locube := [locube];` Test this out.

1.4.7 Conditional execution if .. then .. elif .. else .. fi;

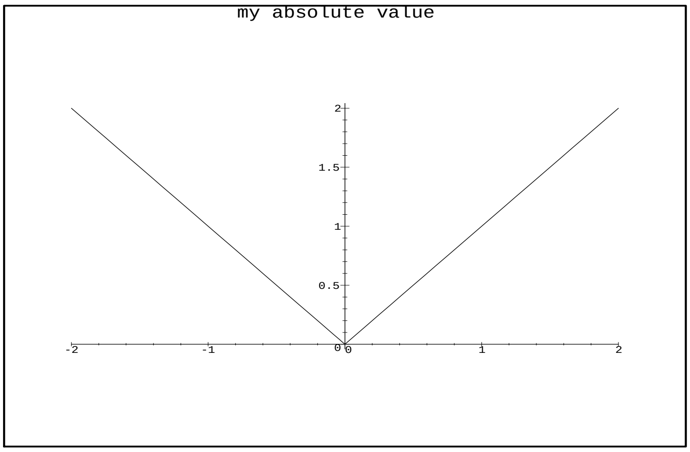
There are lots of times when you need to consider cases, and they can all be handled with the if .. then .. elif .. else .. fi; statement. For example, many functions are defined piecewise. The absolute value function abs is such a function.

Problem : Define your own version of the absolute value function.

```
> myabs := proc(x) if x > 0 then x else -x fi end;

myabs := proc(x) if 0 < x then x else -x fi end

> plot(myabs,-2..2,scaling=constrained,title=`my absolute value`); # to see what it l
> looks like.
```



1.4.8 A Brief Vocabulary of Maple Words

We have gathered here are some Maple words useful in problem solving, together with examples of their usage. For more information on these words and others, look at the help sheets and use the help browser.

```
> y := (x+3)/tan(x^2-1);  # use ´colon-equal´ to make assignments.
```

$$y := \frac{x+3}{\tan(\,x^2-1\,)}$$

```
> collect(x*2 + 4*x,x);  # collects like powers of x.
```

$$6\,x$$

```
> diff(cos(x),x);  # calculates the derivative
```

$$-\sin(\,x\,)$$

```
> D(cos);  # the differential operator
```

$$\mathrm{D}(\,\cos\,)$$

```
>  y := denom((a+b)/(e+f));  # assigns e+f to y.
```

$$y := e+f$$

```
> y := ´y´;  #  makes y a variable again.
```

$$y := y$$

```
> evalc((2+3*I)^3);  # performs complex arithmetic
```

$$-46+9\,I$$

```
> evalf(1/2^9);  #evaluates 1/2^9 to a decimal number
```

$$.001953125000$$

```
> expand((x+b)^7);  # expands the product
```

$$x^7 + 7\,x^6\,b + 21\,x^5\,b^2 + 35\,x^4\,b^3 + 35\,x^3\,b^4 + 21\,x^2\,b^5 + 7\,x\,b^6 + b^7$$

```
> p := x^2+5*x+6;  # assigns the quadratic to p.
```

$$p := x^2 + 5\,x + 6$$

```
> factor(p);  # factors the polynomial
```

$$(\,x+3\,)\,(\,x+2\,)$$

```
> fsolve(x^5-3*x=1,x,0..2);  # solve eqn for x in 0..2
```

$$1.388791984$$

```
> int(x*exp(x),x);   # returns an antiderivative.
```

$$x\,\mathrm{e}^x - \mathrm{e}^x$$

```
> Int(x*exp(x),x=0..1);  # A passive integral.
```

$$\int_0^1 x\,\mathrm{e}^x\,dx$$

```
> map(x->x^2,[1,3,2,5]); # returns a list of squares.
```

$$[\,1, 9, 4, 25\,]$$

```
> nops([3,4,x,1]);  # returns the number of terms  in the list.
```

$$4$$

```
> numer((a+b)/c);  # gives numerator, here a+b
```

$$a + b$$

```
> op([3,4,1,x]);  # strips the brackets off the list
```

$$3, 4, 1, x$$

```
> plot(x^2+x, x=-3..3);  # plots x^2+x as x goes  from -3 to 3.
> plot3d(x^2+y,x=-2..2,y=0..2);  # plots a surface
> f := x -> x^2;  # defines the squaring function.
```

$$f := x \to x^2$$

```
> f(3);  # then returns 9.
```

$$9$$

```
> quo((x^4-4),(x^2-2),x);  # divides polynomials
```

$$x^2 + 2$$

```
> iquo(23,2) ;  # divides the integers
```

$$11$$

```
> rem((x^4-4*x+3),(x^2-2),x);  # gives the remainder
```

$$-4\,x+7$$

```
> irem(23,2) ; # gives the integer remainder
```

$$1$$

```
> restart;  # very handy. This word resets all assignments.
> eq1 := x^2 + 3*x -1 = a;  # assigns the equation
```

$$eq1 := x^2+3\,x-1=a$$

```
> rhs(eq1);  # yields the righthand side of eq1.  There is also an lhs.
```

$$a$$

```
> simplify(a/x+b/y);  # sometimes simplifies expr.
```

$$\frac{a\,y+b\,x}{x\,y}$$

```
> solve(a*x+4*y=0,x);  # solve the equation  for x.
```

$$-4\,\frac{y}{a}$$

```
> subs(x=5,x^2+x);  # substitute 5 for x where it  occurs in x^2+x.
```

$$30$$

```
> i := 'i';   # makes i a variable again
```

$$i := i$$

```
> sum((i^2,i=2..9));  # add up the 2nd thru 9th squares
```

$$284$$

1.4.9 Troubleshooting Notes

Learning to use Maple can be an extremely frustrating experience, if you let it. There are some types of errors which occur from the beginning that can be spotted and corrected easily by a person fluent in Maple, so if you have access to such a person, use them. Here are a few suggestions that may be of use when you're stuck with a worksheet that's not working.

1. Use the help facility: There is a help sheet with examples for every Maple word. A quick read thru will often clear up syntax problems. One very common early mistake is to leave out the

parentheses around the inputs of a word. For example, typing

```
> plot x^2;

syntax error:
plot x^2;
      ^
```

will get you a syntax error, because you left out the parentheses.

2. The maple prompt is > . You can begin entering input after it. Make sure you are typing into an input cell, if you are expecting output.

3. End maple statements with a semicolon ; . Maple does nothing until it finds a semicolon. If you are getting no output when you should be, try feeding in a semicolon. This often works. For example,

4. When in doubt, put in parentheses. For example, `(x+3)/(x-3)` is very different from `x+3 / x-3`.

5. Make sure your variables are variable. You may have assigned a value, say 3, to x in a previous problem. To make x a variable again, type `x := ´x´:` . Use the forward quote ´ key, just below the double quote " here. If you forget this, strange things can happen. One way to handle this is to keep an input cell of variables used. Another way is to use `restart` or `unassign`
.

6. Are you using the correct quote symbol? In Maple, the forward quote ´ is used to suppress evaluation. The back quote ` is used to enclose ascii strings. The double quote " is used to reference the last computation.

7. Do not forget to end loops, conditional statements, and procedures with `od` , `fi` , and `end` respectively. If you start a loop with `do` , Maple does not begin processing until it finds the end of the loop, which is signaled by the word `od;` The same applies to the if .. then ... fi; and proc ... end; constructions.

If you are getting no output when you should be, try feeding an `od;`, `fi;`, or `end;`. This often works.

8. Unwanted output?: Is there output you need but don't want to see? Use a colon : instead of a semicolon to end the Maple statement which generates the output.

Chapter 2

Setting Up and Solving Problems

What is a problem? For our purposes, a problem is anything that can be formulated as a question whose answer involves some mathematics. The main use of mathematics is to solve problems, and the best way to learn mathematics is to solve problems. This idea of a problem includes the 'skill' exercises that are always at the end of the sections in mathematics textbooks. For example, early in beginning algebra there is a section on solving linear equations, and at the end of that section there is a set of exercises consisting of lots of problems like 'solve `4x+3 = 2x -6` for x'.

It also includes 'word problems' which can be solved using the methods mastered in the skill exercises. The word problems often are stated at the beginning of the section as motivation for the methods which are developed in the section. **Problem** s usually arise in a context. Once the context is well understood, the problem can be formulated or posed and a method of solution worked out. From this solution, other problems may arise which require solving. We want to consider this problem identification and formulation as part of problem solving also.

The process of solving a problem is an active process, but can get bogged down for lack of knowing what to do next. So it is helpful to have a list of things to do. Here is one list of steps to carry out when you are solving a problem.

2.1 Setup – Solve – Interpret

SETUP the problem. This involves several steps. 1. Pose/Read the problem carefully, getting straight the meaning of all the terms used in the statement.

2. Draw a picture or diagram. This is a good way to focus thoughts, and gives you a place to put your labels.

3. Label or list the dimensions (or variables) important to the problem. Among these are the given dimensions, that is, the dimensions whose values are stated in the problem, the requested dimensions, that is, the dimensions whose values are requested in theproblem, and the intermediate dimensions, that is, the dimensions which arise in the process of trying to determine the requested dimensions from the given dimensions.

4. List or derive the equations relating the labeled dimensions.

SOLVE the equations we have set up for the required dimensions in terms of the given dimensions. This is one of places where Maple comes in very handy. It is so easy to get bogged down in the calculations that you lose all interest in solving the problem.

INTERPRET the obtained solutions. What are the realistic solutions given the context of the problem? Which should be ignored? Have all solutions been obtained?

Where does Maple come in handy in this process? In the Solve phase, mostly. The actual setting up of the problem as a mathematical problem has to be done by you. Regard Maple as a tool to carry out and record the solution you imagine. A very important word in the Maple vocabulary is solve. This word is used to solve a system of one or more equations which come up.

2.2 A Swimming Pool Problem:

Problem : A swimming pool is three times as long as it is wide. It is also 40 feet longer than it is wide. Find its dimensions.

Solution: Let l and w be the length and width of the pool. Then the first statement of the problem translates to the equation `l = 3w`, and the second statement to `l = w + 40`. We need to solve these two equations simultaneously for l and w. First, set up the equations.

```
>       eq1 := l = 3*w;
```

$$eq1 := l = 3\,w$$

```
>       eq2 := l = w + 40;
```

$$eq2 := l = w + 40$$

Then solve the system for l and w. We can do this by subtracting eq2 from eq1 and solving for w, getting `w = 20`, then substituting that value for w into eq1 getting `l = 60`.

```
>       eq3 := eq1 - eq2;
```

$$eq3 := 0 = 2\,w - 40$$

```
>       eq4 := lhs(eq3) - 2*w = rhs(eq3) - 2*w;
```

$$eq4 := -2\,w = -40$$

```
>       eq5 := -(1/2)*eq4;
```

$$eq5 := w = 20$$

```
>       eq6 := subs(eq5,eq1);
```

$$eq6 := l = 60$$

```
>       solution := {eq5,eq6};
```

$$solution := \{l = 60, w = 20\}$$

The word solve carries out this algorithm automatically.

```
>       solve({eq1,eq2},{l,w});
```

$$\{l = 60, w = 20\}$$

The pool is 20 feet wide and 60 feet long.

The word solve is a very good to know, but it is not infallible. It uses some methods of solving equations which you know and some which you probably don't know. Don't give up just because solve doesn't give you a solution.

2.3 Four methods of solving equations

So it is important to also be able to solve equations by various means.

Guess and check. One such method is to guestimate and check your accuracy. For example, suppose you had guestimated that the dimensions of the pool are about 25 by 55 feet and wanted to check that. Use the word subs.

```
>       subs({w=25,l=75},[eq1,eq2]);
```

$$[\,75 = 75, 75 = 65\,]$$

This line says to substitute 25 and 75 for w and l in eq1 and eq2. (Notice the use of braces and brackets here. Braces are used to enclose the members of a set and brackets are used to enclose the members of a list. In a set, the order is not important and repetitions are not counted; in a list, the order is important and repetitions are counted.) As we can clearly see, our guestimate is off. We have satisfied the first equation, but not the second. If we decrease the value of w by 1, then we have to decrease the value of l by 3 in order to continue to satisfy the first equation. Will we come closer to satisfying the second? Let's see.

```
>        subs({w=24,l=72},[eq1,eq2]);
```

$$[\,72 = 72, 72 = 64\,]$$

Well, yes, if ever so slightly. The left-hand side of the second equation has decreased by 2 towards the right-hand side. We could pursue this method of guessing and then trying to improve our guess, but let's put that off until later.

By Hand, as with pencil and paper. You can choose to try to solve your equations by hand by which I mean to manipulate the equations the same way you would do using pencil and paper. This can be done without using solve at all. This is what we did in our original solution to the swimming pool problem. On the other hand, we could simplfy the equations and then use the word solve. For example, let's solve the two equations, eq1 and eq2, by hand. Using the Maple word solve we need only solve one equation for one unknown. Here is a possible way to proceed:

```
>     eq3 := subs(l=solve(eq1,l),eq2);
```

$$eq3 := 3\,w = w + 40$$

```
>     sol1 := w = solve(eq3,w);
```

$$sol1 := w = 20$$

```
>     sol2 := l = solve(subs(sol1,eq1),l);
```

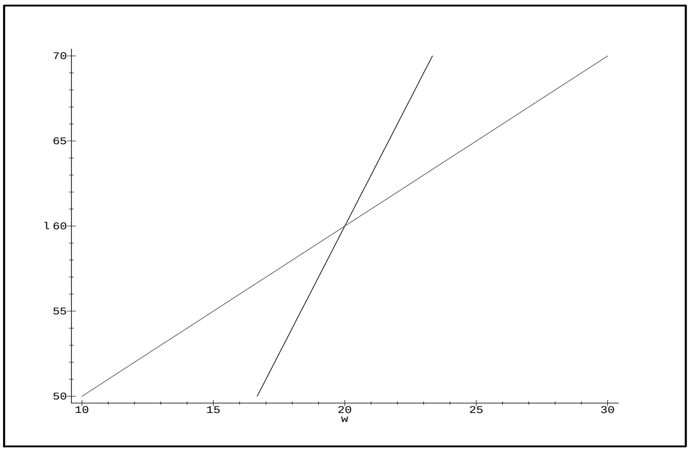
$$sol2 := l = 60$$

```
>     sol := {sol1, sol2};
```

$$sol := \{\,l = 60, w = 20\,\}$$

Graphical solution. Another way to solve equations is graphically. Here we can plot each equation, using plot or implicitplot and use the pointer to locate the approximate solutions. These solution(s) are the located where the graphs of the equations coincide. Note: the word implicitplot can only be found in Maple V Release 2. Earlier versions of Maple only plot functions. This method also works very well on a graphing calculator.

```
> with(plots):
> implicitplot({eq1,eq2},w=10..30,l=50..70);
```



Using fsolve. Another important way to solve equations is with `fsolve`. This word employs methods of calculus to find floating point approximations to the equation you are trying to solve. If you do not supply fsolve with an interval in which to search for a solution, then it returns the first solution it finds. This word works well in conjunction with `plot`. You can use plot to narrow down the search interval, and then use fsolve to get the 'exact' answer.

Problems A. Solve the following systems of equations.

1. `x^2 + y^2 = 10`, y=3*x
2. a+3*b+c=10, 2*a-b+72*c=20, 7*a-3*b+21*c=30.
3. `x^5 - 5*x^2 + 2 = 0` use plot and fsolve here.
4. `x-3*y+z=a`, 2*x+y-3*z=b, 9*x +3*y+5*z=c for x, y, and z.

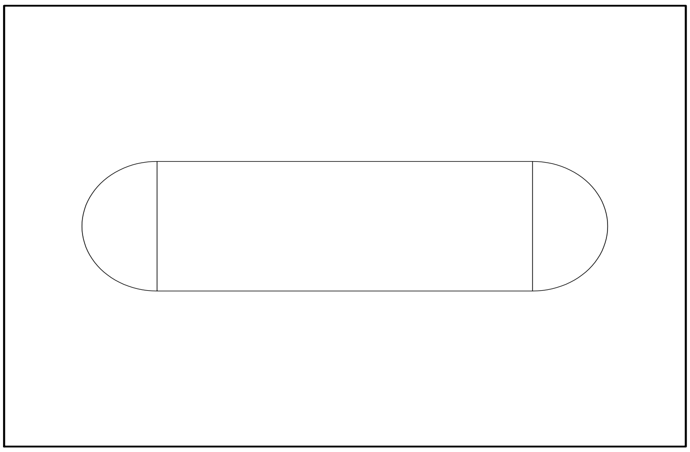
B. Solve the following word problems by setting up and solving a system of equations.

1. The height of the Eiffel Tower in Paris is 125 feet less than twice the height of the Washington Monument. The latter is 75 higher than the Great Pyramid of Cheops and 105 feet higher than the dome of St. Peter's Church in Rome. If the sum of the heights of these four edifices is 2510 feet, find the height of each to the nearest foot.

2. A large conference table is to be constructed in the shape of a rectangle with two semicircles at the ends (see figure). Find the dimensions of the table, given that the area of the rectangular portion is to be twice the sum of the areas of the circular ends, and the perimeter of the whole table is to be 40 feet.

```
> with(plots): # Press Enter twice to see Figure!
> F:=plot([-cos(t)+1,sin(t)+1,t=Pi/2..-Pi/2]):
> G:=plot([cos(t)+6,sin(t)+1,t=Pi/2..-Pi/2]):
```

```
> H:=plot([1,0,6,0,6,2,1,2,1,0],style=LINE):
> display({F,G,H},axes=none,scaling=constrained);
```



Notice that we can cut and paste graphs into a Maple worksheet. This can be helpful to you when the time comes to make your own worksheets.

3. A local hardware store worker is making up a fertilizer mix from some left over fertilizer from the summer. There are three types, with 30 percent, 20 percent and 15 percent nitrogen content respectively. When he mixed all the fertilizers together and tested the nitrogen content, he found that the mix weighed 600 pounds and contained 25 percent nitrogen content. How many pounds of each type did he have left over? How many solutions does this problem have?

4. A 30 foot ladder and a 40 foot ladder are positioned so as to cross each other in an alley. That is, they are leaning up against opposite walls with their bases snug up against the base of the opposite walls. Given that they cross at a point 10 feet above the floor of the alley, determine the width of the alley.

5. Make up your own algebra problem to set up and solve using Maple. It can be a variation on one of the problems above, or it can be something entirely different.

2.4 More About Plotting

We have already used a few words from the `with(plots);` package. When you want to learn more about any package (in this case with(plots);) start by looking at the commands in the package. Do this by typing

```
> with(plots);
```

$$[animate, animate3d, conformal, contourplot, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, loglogplot, logplot, matrixplot, odeplot, pointplot, polarplot, polygonplot, polygonplot3d, polyhedraplot, replot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]$$

We have already used `plot` , `implicitplot` , and `display` (in problem 2 of part B). To learn more about these very useful commands simply type

```
> ?plot
> ?implicitplot
> ?display
> ?textplot
> ?animate
```

or anything else that might interest you. Look at the bottom of the help files for examples. Sometimes, you are your best teacher. Learn by experimenting with a few (or all) of these commands. Also, you will learn more about these words in worksheets to come.

2.5 Putting in a parameter.

One of the nice features of solving problems in a Maple worksheet is that it gives the capability of going back and changing the numbers in the problem to study how the solution changes. In fact, you are led to the practice of putting a parameter into the problem. For example, to put a parameter in the swimming pool problem, we have two natural choices: Replace the number 'three' or the number 40 by a parameter. Let's replace three with p.

A Parameterized Swimming Pool Problem **Problem** : swimming pool is p times as long as it is wide. It is also 40 feet longer than it is wide. What are its dimensions in terms of p? Describe how the dimensions vary with p.

Solution: We could just go back to the cell containing the original equations and put in a p for 3 in eq1.

```
>    restart;
```

The word `restart` clears all variables by rebooting Maple.

```
>    p := ´p´;                # put in the parameter.
```

$$p := p$$

```
>    eq1 := l = p*w;
```

$$eq1 := l = p\,w$$

```
>    eq2 := l = w + 40;
```

$$eq2 := l = w + 40$$

Now we would solve for l and w, just as before, except now the solution is given in terms of the parameter p.

```
>      sol := solve({eq1,eq2},{l,w});
```

$$sol := \left\{ w = 40\,\frac{1}{-1+p},\, l = 40\,\frac{p}{-1+p} \right\}$$

Now by inspection, we can see that as p gets large, both w and l get small. Also, as p approaches 0 from the left, l and w get large. In more complicated situations, we could use plot to study how the solution changes as the parameter changes.

```
> with(plots):
> animate({p*w,w+40},w=0..80,p=0..2);
```

The word `animate` , which is used out of the with(plots); package, is very useful in seeing the range of possible solutions for this problem. Press your pointer on play to see the animation.

Problem s

1. What happens to the solution to the swimming pool problem as the difference of the width and length is allowed to vary? (Keep the ratio of the width to the length constant.)

2. A large conference table is to be constructed in the shape of a rectangle with two semicircles at the ends (see figure). Find the dimensions of the table, given that the area of the rectangular portion is to be p times the sum of the areas of the circular ends, and the perimeter of the whole table is to be 40 feet.

```
> with(plots):  # Press Enter twice to see Figure!
> F:=plot([-cos(t)+1,sin(t)+1,t=Pi/2..-Pi/2]):
> G:=plot([cos(t)+6,sin(t)+1,t=Pi/2..-Pi/2]):
> H:=plot([1,0,6,0,6,2,1,2,1,0],style=LINE):
> display({F,G,H},axes=none,scaling=constrained);
```

3. Study the solution to the mixture problem, no.3, in the previous problem set. Vary a parameter of your choice.

4. A 30 foot ladder and a 40 foot ladder are positioned so as to cross each other in an alley. That is, they are leaning up against opposite walls with their bases snug up against the base of the opposite walls. Given that they cross at a point h feet above the floor of the alley, determine the width of the alley.

2.6 Defining your own Maple words.

This is a good place to learn how to develop and define Maple words in a worksheet. The idea is very simple:

A. Name the given quantities – As you are solving a problem or developing an algorithm, assign the given quantities to appropriate names which you have chosen. Put these assignments into a single input cell, which we will call the parameter cell.

B. Compute the desired quantity – Use the names assigned in your maple statements which you make in your algorithm. This part will change and grow as you develop the definition, but try to keep all the statements in one input cell, which we will call the definition cell. Once you get the desired output,

C. Make the Definition – This involves mostly choosing a name for the procedure, inserting it at the top of the definition cell in the line

```
name := proc(p1,p2,...,pn)
```

which begins the definition by assigning the name and declaring the input parameters p1, p2, etc.

At the bottom of the definition cell is the word `end;` which signals the end of the definition.

Now, when you execute the definition cell, you should get a nicely formatted version of the definition as output. This is not the only set of steps you could follow when developing a definition, but it is very natural one and works well for small definitions.

By way of example, suppose we wanted to define a word `swim` which would return the solution to the swimming pool problem.

So our worksheet looks like this. We have joined input cells if necessary to get all the lines in one cell. p := 'p'; eq1 := l = p*w; eq2 := l = w + 40; sol := solve(eq1,eq2,l,w); All we need to do to define the word is to insert a proc line at the top and an end at the bottom, like so.

```
>      swim := proc(p)         # no semicolon on this line.
>     eq1 := l = p*w;
>     eq2 := l = w + 40;
>     sol := solve({eq1,eq2},{l,w});
>    end;
```

```
Warning, `eq1` is implicitly declared local
Warning, `eq2` is implicitly declared local
Warning, `sol` is implicitly declared local

swim := proc(p)
        local eq1,eq2,sol;
            eq1 := l = p*w;
            eq2 := l = w+40;
            sol := solve({eq1,eq2},{w,l})
        end
```

When you execute the input cell containing the definition, the word `swim` has been added to the vocabulary and can be used like any other Maple word.

```
>     swim(3);
```

$$\{l = 60, w = 20\}$$

```
>     swim(-4);
```

$$\{w = -8, l = 32\}$$

You can redefine the word if there is something that needs changed. For example, the swimming pool problem doesn't really have a solution if p is not positive, so we could insert an error trap here.

```
>      swim := proc(p)
>     if not type(p,name) and not p > 0 then ERROR(`oops`)
>     else       eq1 := l = p*w;
>                  eq2 := l = w + 40;
>                  sol := solve({eq1,eq2},{l,w});
>     fi
>    end;
```

```
Warning, `eq1` is implicitly declared local
Warning, `eq2` is implicitly declared local
Warning, `sol` is implicitly declared local

swim := proc(p)
        local eq1,eq2,sol;
            if not (type(p,name) or 0 < p) then ERROR(oops)
            else
                eq1 := l = p*w;
                eq2 := l = w+40;
                sol := solve({eq1,eq2},{w,l})
            fi
        end
```

Now redefine swim and check –

```
>     swim(-4);

Error, (in swim) oops
```

Problems 1. Define a word `quadform` to take three numbers a, b, c and return the roots of the quadratic equation `a*x^2 + b*x + c = 0`, if they are real, and the message 'no real roots' if they are complex. Don't use `solve` .

2. Take one of the word problems in the previous sections and develop a word to return its solution.

3. Define a word which takes a function and tabulates its value over a given interval with a given increment between x values.

2.7 More Problems.

In this section, we have gathered some more problems, their solutions, and followup questions for your inspection. Perhaps you can get some ideas from these.

A billiard ball problem: The cue ball is 2 feet from the edge of the table. The player wants to bounce the cue ball off the edge and hit the red ball which is 3 feet from the same edge. If the distance between the vertical projections of the cue ball and the red ball to the edge is 5 feet, find the angle at which the player should bounce the ball off the edge.

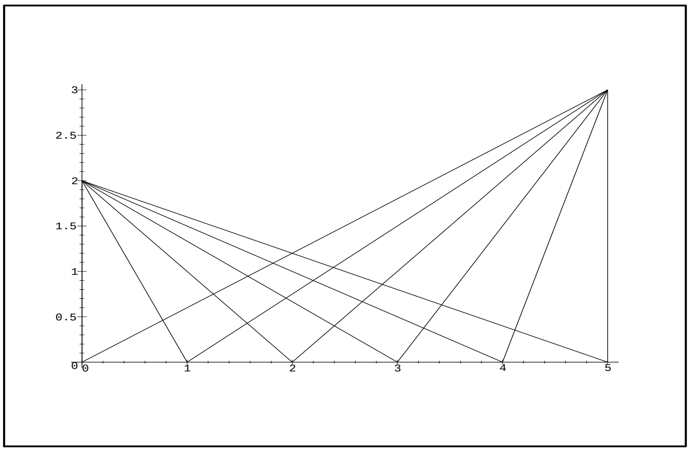
Solution .

First draw a picture and label the unknowns. We can think of the edge as the x-axis and the cue and red balls being located at (0,2) and (5,3) respectively. Then let (x,0) be the point on the x-axis where the ball should bounce. The angle (0,2),(x,0),(0,0) of incidence is going to be the same as the angle (5,0),(x,0),(5,3) of reflection.

```
> cue := [0,2]: red := [5,3]:
> path := x -> plot([cue,[x,0],red]) ;
```

$$path := x \to \mathbf{plot}(\,[\,cue, [\,x, 0\,], red\,]\,)$$

```
> plots[display]([seq(path(i),i=0..5)],scaling=constrained);
```



So we can express the tangent of the angle in two ways.

```
> eq1 := tan(a)=2/x;
```

$$eq1 := \tan(\,a\,) = 2\,\frac{1}{x}$$

```
> eq2 := tan(a)=3/(5-x);
```

$$eq2 := \tan(\,a\,) = 3\,\frac{1}{5-x}$$

```
> sol := solve({eq1,eq2},{a,x});
```

$$sol := \left\{x = 2, a = \frac{1}{4}\,\pi\right\}$$

So the player should bounce the ball off at a 45 degree angle. Used `plot` , `solve` , and `seq` here. We can parameterize this problem in several ways. For example, replace the distance of the cue ball from the edge with a parameter h and rework the problem.

```
> eq1 := tan(a)=h/x;
```

$$eq1 := \tan(a) = \frac{h}{x}$$

```
> eq2 := tan(a)=3/(5-x);
```

$$eq2 := \tan(a) = 3\,\frac{1}{5-x}$$

```
> sol := solve({eq1,eq2},{a,x});
```

$$sol := \left\{x = 5\,\frac{h}{h+3}, a = \arctan\left(\frac{1}{5}\,h + \frac{3}{5}\right)\right\}$$

Now we can investigate questions such as:

Question :

What happens to the angle as the cue ball moves away from the edge? towards the edge?

A variation on the billiard ball problem. Suppose that the edge of the billiard table opposite the x-axis is 4 feet wide and the player wants to bounce the cue ball twice before it hits the red ball. What angle should he bounce it off the edge?

Solution .

The second edge has equation y=4, and the second bounce point will be (x2,4). Notice that the angle of reflection of the first bounce equals the angle of incidence of the second bounce, since the two edges are parallel. So we have three ways two express the angle a in terms of x and x2.

```
> eq1 := tan(a)=2/x;
```

$$eq1 := \tan(a) = 2\,\frac{1}{x}$$

```
> eq2 := tan(a)=1/(5-x2);
```

$$eq2 := \tan(a) = \frac{1}{5 - x2}$$

```
> eq3 := tan(a)=4/(x2-x);
```

$$eq3 := \tan(a) = 4\,\frac{1}{x2 - x}$$

Now solve the equations for x,x2, and a.

```
> solve({eq1,eq2,eq3},{x,x2,a});
```

$$\left\{x2 = \frac{30}{7}, x = \frac{10}{7}, a = \arctan\left(\frac{7}{5}\right)\right\}$$

```
> evalf(convert(arctan(7/5),degrees));
```

$$54.46232221\ degrees$$

So the player should bounce the ball off at about a 54.5 degree angle. Comparing this with the first problem, we note that the bounce angle should be larger than 45 degrees since the player must bounce the ball twice. Used `evalf` , `convert` , and `degrees` here.

Problem . What is the angle if the cue bounces off the edge y=4 first?

Water tank problem. Suppose you are standing at the point (0,0,4) on the z-axis, looking towards the x-axis. A spherical water tank 2 units in diameter is centered on the origin (0,0,0). You can't 'see' the points on the x-axis close to (0,0,0) because the water tank is in the way, but you can see points far out on the x-axis. So, for example, the line connecting the point (10,0,0) to (0,0,4) doesn't touch the water tank (does it?), so you can 'see' that point. The question is, what is the smallest point on the positive x-axis you can see from (0,0,4).

Solution .

Let (c,0,d) be the point on the tank such that the line through (0,0,4) and (c,0,d) is tangent to the tank. So the triangle with vertices (0,0,0), (c,0,d), and (0,0,4) is a right triangle. This means that

```
> eq1 := c^2 + d^2 + c^2 + (d-4)^2 = 4^2;
```

$$eq1 := 2\,c^2 + d^2 + (\,d - 4\,)^2 = 16$$

But also

```
> eq2 := c^2+d^2=1;
```

$$eq2 := c^2 + d^2 = 1$$

Now since the legs of the triangle above are perpendicular,

```
> eq3 := (d-0)/(c-a)=4/(-a);
```

$$eq3 := \frac{d}{c-a} = -4\,\frac{1}{a}$$

```
> solve({eq1,eq2,eq3},{c,d,a});
```

$$\left\{a = \mathrm{RootOf}(\,15\,\_Z^2 - 16\,), c = \frac{15}{16}\,\mathrm{RootOf}(\,15\,\_Z^2 - 16\,), d = \frac{1}{4}\right\}$$

By inspection, we see that `a = 4/sqrt(15)` is the smallest positive real number the observer can see from (0,4).

Problem : Suppose you are at the point (0,0,h) on the z-axis looking toward the positive x-axis. What is the smallest point you can see on the positive x-axis? Where are you if the smallest point you can see is (10,0,0)?

Question : What happens if we replace the spherical tank with a parabolic cylinder, say `z = 1 - x^2`?—

A ladder problem: The foot of a ladder is placed at (3,0) and leaned up against a parabolic hill `y = 3 - x^2`. Where does the ladder cross the y-axis?

Solution :

Draw a picture, showing the hill and the ladder. Mark the point where the ladder crosses the y-axis (0,a). Mark the point where the ladder touches the hill, `(b, 3-b^2)`. The ladder is tangent to the hill at `(b,3-b^2)`. We could figure out a if we knew the slope of the ladder. So take a point close to `(b,3-b^2)` on the hill. Say the point is `(b+h, 3 - (b+h)^2)` where h is a small number (but not zero). The approximate slope of the ladder is the slope of the line through these two points. Calculate this slope, simplifying as much as possible. Then find the limiting value of this slope as h approaches 0 (that is, as the point `(b+h, 3-(b+h)^2)` moves toward `(b,3-b^2)` on the parabola.

To find the slope using maple,

```
> f := x -> 3 - x^2;
```

$$f := x \to 3 - x^2$$

```
> approxslope := (f(b+h) - f(b)) / ((b+h)  - b);
```

$$approxslope := \frac{-(b+h)^2 + b^2}{h}$$

```
> approxslope := simplify(approxslope);
```

$$approxslope := -2\,b - h$$

```
> slope1 := limit(approxslope,h=0);
```

$$slope1 := -2\,b$$

But the slope of the ladder is also the slope of the line through (3,0) and `(b,3-b^2)`.

```
>  slope2 := (3-b^2)/(b-3);
```

$$slope2 := \frac{3 - b^2}{b - 3}$$

Now set these slopes equal and solve for b.

```
>  sol := solve(slope1=slope2,b) ;
```

$$sol := 3 - \sqrt{6}, 3 + \sqrt{6}$$

Which solution do we want? It will have to be the first term in sol. (why?) After having decided that, find a by finding where the ladder crosses the y-axis.

```
> b := sol[1];    # assign b the first number in the expression sequence sol.
```

$$b := 3 - \sqrt{6}$$

```
> ladder :=  x -> slope2*(x-b) + f(b);   # define the ladder function.
```

$$ladder := x \to slope2\,(x - b) + \mathrm{f}(b)$$

```
> a := ladder(0);   #  find where the ladder crosses the y-axis.
```

$$a := -\frac{1}{6}\left(3 - \left(3 - \sqrt{6}\right)^2\right)\sqrt{6}\left(-3 + \sqrt{6}\right) + 3 - \left(3 - \sqrt{6}\right)^2$$

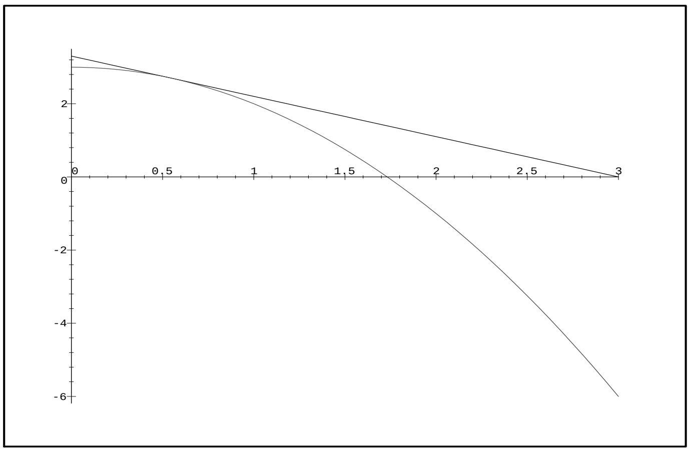
```
> simplify(a);    # simplify a
```

$$-6\sqrt{6} + 18$$

```
> evalf(a,4);      # convert a to  decimal approximation
```

$$3.303$$

```
> plot({f,ladder},0..3);  # draw a picture.
```



Used simplify , limit , arrow , evalf , and solve here.

A ladder problem. A 5 foot ladder is leaning up against a parabolic hill `y = 3 - x^2`. The foot of the ladder is on the positive x-axis, and the top of the ladder is on the y-axis. Describe the position of the ladder, by locating the coordinates of the foot of the ladder and the top of the ladder. Note: This problem may have two different solutions. Draw a picture and show how two positions may be possible.

Solution :

First define the hill. The ladder leans up against the hill at the point (b,f(b)), and will be tangent to the hill there. By our calculations in the previous problem, the slope of the ladder will be -2*b at that point.

```
> restart;
> f := x->3-x^2;
```

$$f := x \to 3 - x^2$$

```
> slope := -2*b;
```

$$slope := -2\,b$$

So we could define the tangent line function at (b,f(b)). That will be the ladder.

```
> tang := x -> -2*b*(x-b)+ f(b);
```

$$tang := x \to -2\,b\,(\,x - b\,) + \mathrm{f}(\,b\,)$$

Then get the x and y intercepts of the ladder, call them xb and yb.

```
> xb := solve(tang(x)=0,x);
```

$$xb := \frac{1}{2}\,\frac{b^2+3}{b}$$

```
> yb := tang(0);
```

$$yb := b^2 + 3$$

The ladder is 5 feet long. This gives us the equation we need to determine the value of b.

```
> eq1 := xb^2+yb^2=5^2;
```

$$eq1 := \frac{1}{4}\,\frac{(\,b^2+3\,)^2}{b^2} + (\,b^2+3\,)^2 = 25$$

Let's manipulate this equation.

```
> eq2 := expand(4*b^2*simplify(eq1));
```

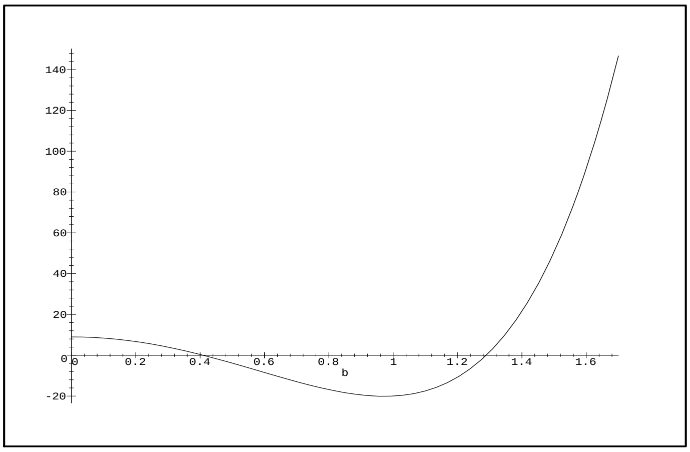
$$eq2 := 25\,b^4 + 4\,b^6 + 42\,b^2 + 9 = 100\,b^2$$

```
> eq3 := lhs(eq2)-rhs(eq2);
```

$$eq3 := 25\,b^4 + 4\,b^6 - 58\,b^2 + 9$$

Now we are interested in finding the roots of eq3, where the graph of eq3 crosses the b axis. So plot the graph. (First we graphed it over the interval 0..3, but then narrowed it down to 0..1.7.

```
> plot(eq3,b=0..1.7);
```



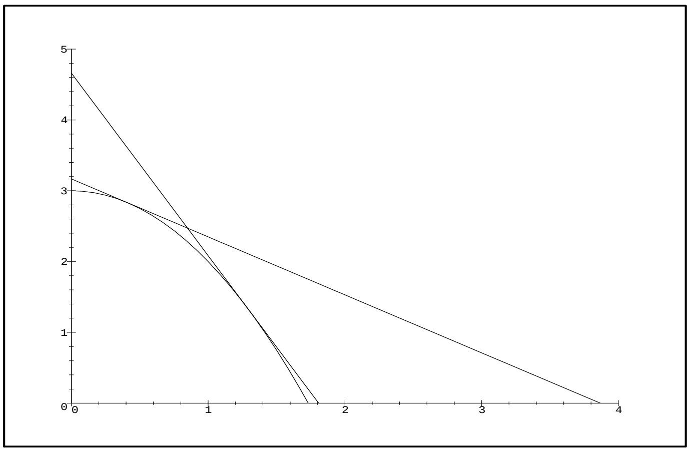
Indeed, we can see by looking that there are two positions for the ladder, a high position and a low position.

```
> sol := fsolve(eq3,b,0..1.5);
```

$$sol := .4093966141, 1.289029267$$

Now, to draw a picture, showing the hill and both positions of the ladder.

```
> plot({f,subs(b=sol[1],op(tang)),
>               subs(b=sol[2],op(tang))},0..4,0..5,color=black);
```



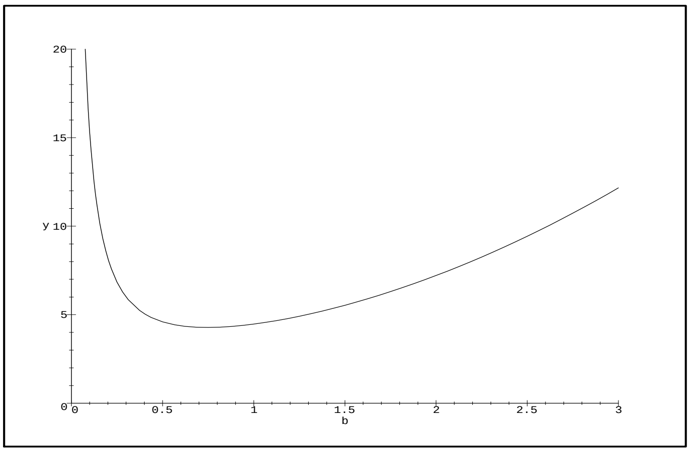
Used `subs` , `expand` , `evalf` , `restart` , `solve` , `rhs` , `lhs` , and `restart` here.

Variation on the ladder problem. In the ladder problem above, suppose we shorten the ladder. We can imagine that as the ladder gets shorter, the two positions the ladder can occupy get closer to each other, until at last they coincide. What is the shortest ladder that can lean against the hill so that its foot is on the x-axis and its top is on the y-axis?

Solution :

Go back to the ladder and look at the expression `xb^2 + yb^2`. This represents the square of the length of the ladder. What we can do is plot the square root of this expression over the interval 0..3 and find the low point on the graph.

```
> plot(sqrt(xb^2+yb^2),b=0..3,y=0..20);
```



The shortest ladder is around 4.3 feet long.

Chapter 3

Differentiation and its uses.

The notion of rate of change has application in almost any area of thought. In this chapter, we begin to make serious use of differentiation in solving problems.

3.1 Defining Derivatives

The Maple word `limit` can be used to calculate derivatives from the definition. Say `f(x)=x^2 + x - 3` and you want to compute f ' by the definition. So enter

```
> f := x -> x^2 + x -3;
```

$$f := x \rightarrow x^2 + x - 3$$

```
> fp := limit((f(x+h)-f(x))/h,h=0);
```

$$fp := 2\,x + 1$$

You would get the same thing by entering

```
> fp := diff(f(x),x);
```

$$fp := 2\,x + 1$$

The main Maple words here are `diff` and `limit` , but in practice you will use `diff` much more. Derivatives of functions defined by expressions, such as

```
> y := x^2 + x;
```

$$y := x^2 + x$$

would be computed with diff, as follows:

```
> yp := diff(y,x);
```

$$yp := 2\,x + 1$$

Now evaluated the derivative of y at x=3 using subs,

```
> subs(x=3,yp);
```

$$7$$

to get the derivative at x=3. Another way to do this is with `unapply` ,

```
> yp:=unapply(yp,x);
```

$$yp := x \rightarrow 2\,x + 1$$

```
> yp(3);
```

$$7$$

3.2 The student package

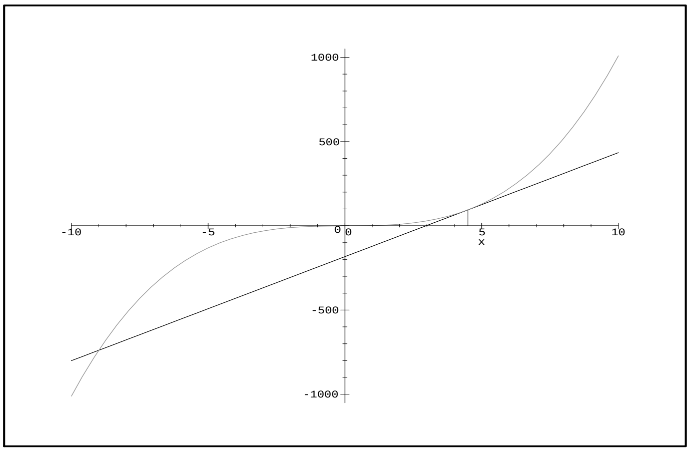
Maple contains a package of words, called the student package, that pertain to the study of calculus. Most of them are dispensible, but this is a good spot to examine some of them. (IMPORTANT: You must type in the package name using the Maple word with in order to use the words in the package). In order to load the words for the student package, enter

```
> with(student);
```

$$[D, \mathit{Doubleint}, \mathit{Int}, \mathit{Limit}, \mathit{Lineint}, \mathit{Sum}, \mathit{Tripleint}, \mathit{changevar}, \mathit{combine}, \mathit{completesquare}, \mathit{distance}, \mathit{equate}, \mathit{extrema}, \mathit{integrand}, \mathit{intercept}, \mathit{intparts}, \mathit{isolate}, \mathit{leftbox}, \mathit{leftsum}, \mathit{makeproc}, \mathit{maximize}, \mathit{middlebox}, \mathit{middlesum}, \mathit{midpoint}, \mathit{minimize}, \mathit{powsubs}, \mathit{rightbox}, \mathit{rightsum}, \mathit{showtangent}, \mathit{simpson}, \mathit{slope}, \mathit{trapezoid}, \mathit{value}]$$

One interesting word is `showtangent` . Give it a function and a point where the derivative of the function is defined. The word showtangent will return a picture of the function and the tangent line to the function at the given point. So to see the graph of `y = x^3 +x -1` and its tangent line at `x = 4.5`, enter

```
> showtangent(x^3+x-1,x=4.5);
```



Problem : Let's say we want find the derivative of a function f(x) and turn it into a function using Maple. For example, suppose your function is

`f(x)=x^(1/3) + sin(ax).`

Then define f by entering

```
> f := x-> x^(1/3) + sin(a*x);
```

$$f := x \to x^{1/3} + \sin(a\,x)$$

Now the derivative of f can not be defined in the obvious way:

```
> fp := x -> diff(f(x),x);
```

$$fp := x \to \mathbf{diff}(\mathbf{f}(x), x)$$

```
> fp(4);
Error, (in fp)
wrong number (or type) of parameters in function diff
```

it simply doesn't work. What will work is the following:

```
> fp := unapply(diff(f(x),x),x);
```

$$fp := x \to \frac{1}{3}\frac{1}{x^{2/3}} + \cos(a\,x)\,a$$

```
> fp(4);
```

$$\frac{1}{12}4^{1/3} + \cos(4\,a)\,a$$

We have two different ways to do the same thing here, but wait, here is another way to define the derivative of a function. Use the operator D .

```
> fp:=D(f);
```

$$fp := x \to \frac{1}{3}\frac{1}{x^{2/3}} + \cos(a\,x)\,a$$

```
> fp(4);
```

$$\frac{1}{12}4^{1/3} + \cos(4\,a)\,a$$

The word D takes a function as its input and yields a function for the derivative of the function as it's output. A good suggestion would be to look at the help files for the Maple words D, unapply, and showtangent so that you can become more comfortable with how they work. That is, look at the differences and similarities between their uses. Other good words are `slope` and `intercept` . Find out what they do. Do not be shy when experimenting with words in a package. (Reminder: Use ? before the name of the word to bring up the help file.)

Problems 1. Use each word `unapply` and `D` to define the derivative of
`f(x) = x^4 - 2*x^3 + x -3.`

2. Use 'for x from -1 by .25 to 2 do' to tabulate the values of f and its derivative at increment of .25 over the interval from -1 to 2.

3. Plot f and its derivative over this interval. Then use `showtangent` to plot f and the tangent line to the graph at `x=-5`.

4. Where does f cross the x-axis? Find the zeros of f '(x). That is, find the extreme values of f.

3.3 Newton's Method.

You have used `fsolve` to solve an equation `f(x) = 0` for x. How does that work? It possibly might use Newton's method, which is a very fast method for solving such equations when f is differentiable.

The idea is simple. By graphing or guessing, you come up with a first estimate of the solution, call it x0. Now the point (x0,f(x0)) is probably not on the x-axis (if it were then f(x0) is 0 and x0 is a solution), so we go along the tangent line to the graph of f until we come to the x-axis, at the point (x1,0) say. We find that we can express x1 in terms of x0, f(x0), and f'(x0).

```
>   eq :=  diff(f(x0),x0) = (f(x0)-0)/(x0-x1);
```

$$eq := \mathbf{fp}(\ x0\ ) = \frac{\mathbf{f}(\ x0\ )}{x0 - x1}$$

```
> expand(solve(eq,{x1}));
```

$$\left\{ x1 = x0 - \frac{\mathbf{f}(\ x0\ )}{\mathbf{fp}(\ x0\ )} \right\}$$

This equation is called Newton's method. It has the property that you can iterate it, that is, after computing x1 you rename it x0 and apply the method again. You can continue this iteration until sequence of numbers converges.

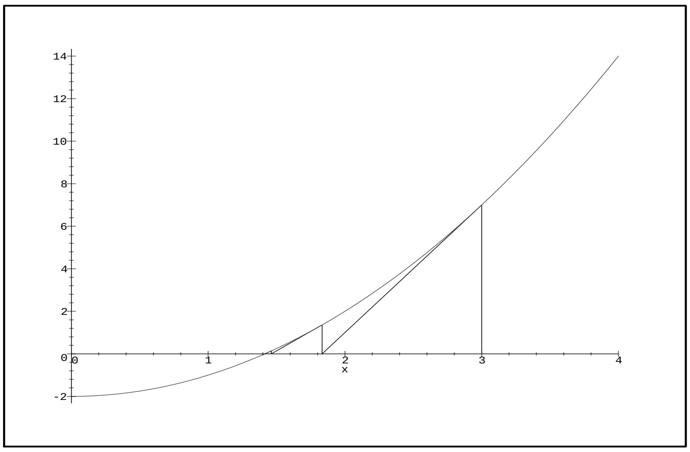
Here is a version of Newton's method which draws a so-called stairstep picture to go with the calculations.

```
> vnewt := proc(f,start,a,b)          #  give a name and the inputs
>   local  i, x0, fp, p;                  # declare your local variables
>   x0 :=  evalf(start);                ##  initialize x0;
>   fp := D(f);                             ##  get the derivative of f
>   p := x0,0;                             ## start the picture  at x0, 0 on x-axis
>   for i from 1 to 10 do               #  begin the loop
>          p := p, x0, f(x0);           #  tack the point x0, f(x0) to the picture
>          x0 := x0 - f(x0)/fp(x0);  #  update x0 with newtons method
>         p := p,x0,0;                   #  tack another point onto to the picture
>   od;                                     # end the loop.
> plot({f(x),[p]},x=a..b,style=LINE,);  # draw the picture
> end;
```

```
vnewt :=
    proc(f,start,a,b)
    local i,x0,fp,p;
        x0 := evalf(start);
        fp := D(f);
        p := x0,0;
        for i to 10 do
            p := p,x0,f(x0); x0 := x0-f(x0)/fp(x0); p := p,x0,0
        od;
        plot({[p],f(x)},x = a .. b,style = LINE)
    end
```

Now test `vnewt` to solve `x^2 = 2` by starting starting at 3 and using a plot interval of 0 to 4..

```
> vnewt(x -> x^2 - 2, 3, 0,  4);
```



Exercises. 1. Use `vnewt` to solve `cos(x) - x = 0` for x for various starting points from 2 to 3.05. (Hint: type it in. Then save it before you experiment much. First, do `vnewt(x->cos(x)-x, 2, -1,Pi)`; then increase 2 gradually to 3.05 Describe what happens as the starting point changes.

2. Use `vnewt` to solve `x^2 + .01 = 0`. Try start at .2, a at -.3, b at .3. What happens?

3. Modify the definition of `vnewt` so that you can control n, the number of times Newtons method gets used. (In vnewt, n is 10). Then test it out on `cos(x) + 2` for start at .3, .7, 1.1 and n at 5, 10, 20. Describe what happens.

3.4 Use of the derivatives in plotting

There is a close connection between the first and second derivative of a (differentiable) function f and what its graph looks like, as you have learned. The local extreme points of f occur amongst the critical points (the zeros of f '), and the inflection points occur amongst the zeros of f ". Also f is increasing (decreasing) on intervals where f ' is positive (negative), and f is concave up (down) on intervals where f " is positive (negative).

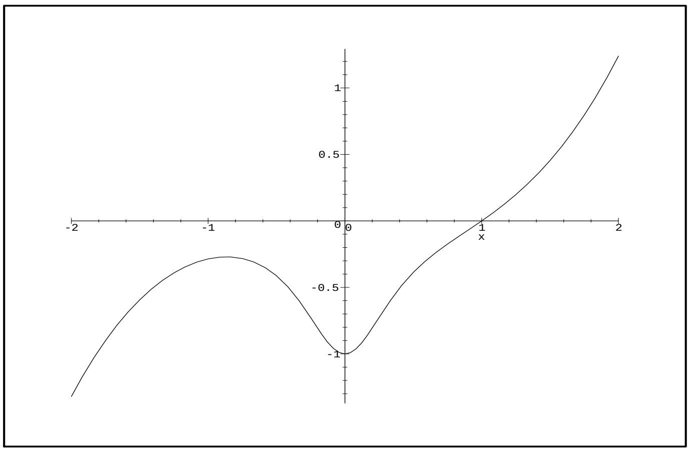
For example, let's look at the function

```
> y := (x^5-1)/(6*x^2+1);
```

$$y := \frac{x^5 - 1}{6\,x^2 + 1}$$

First plot it over an interval, say from -2 to 2.

```
> plot(y,x=-2..2);
```



Look at the graph carefully and estimate the coordinates of local extrema and inflection points. Now lets get these points exactly using some calculus. First compute the derivative of y with respect to x.

```
> yp := diff(y,x);
```

$$yp := 5\,\frac{x^4}{6\,x^2+1} - 12\,\frac{(\,x^5-1\,)\,x}{(\,6\,x^2+1\,)^2}$$

```
> yp := simplify(yp);
```

$$yp := \frac{x\,(\,18\,x^5+5\,x^3+12\,)}{(\,6\,x^2+1\,)^2}$$

Get the critical points by solving yp = 0. Actually, we get better results by finding when the numerator of yp is 0.

```
> cp := fsolve(numer(yp),x);
```

$$cp := 0, -.8657650194$$

cp is an expression sequence of two critical points, cp[1] and cp[2]. By inspecting the graph, we can see that cp[1] is a local maximum and cp[2] is a local min.

Let's evaluate y at `x = cp[1]`, and at `x = cp[2]`.

```
> subs(x = cp[1],y);
```

$$-1$$

```
> subs(x = cp[2],y);
```

$$-.2703889018$$

Now we know the coordinates of the extreme points on the graph. Next let's get the coordinates of the inflection points.

```
> ypp := diff(yp,x);
```

$$ypp := \frac{18\,x^5+5\,x^3+12}{(\,6\,x^2+1\,)^2} + \frac{x\,(\,90\,x^4+15\,x^2\,)}{(\,6\,x^2+1\,)^2} - 24\,\frac{x^2\,(\,18\,x^5+5\,x^3+12\,)}{(\,6\,x^2+1\,)^3}$$

```
> simplify(ypp);
```

$$4\,\frac{54\,x^7+27\,x^5+5\,x^3-54\,x^2+3}{(\,6\,x^2+1\,)^3}$$

```
> pip := fsolve(numer(ypp),x);
```

$$pip := -.2324164053, .2392938071, .8746375912$$

```
> for i from 1 to 3 do print(pip[i], subs(x=pip[i],y)) od;
```

$$-.2324164053, -.7557396749$$

$$.2392938071, -.7437022342$$

$$.8746375912, -.08732685718$$

So we see there are three inflection points.

Question .

Is there always an inflection point between two extreme points? Is there an extreme point between two inflection points?

3.5 Max-min Problems

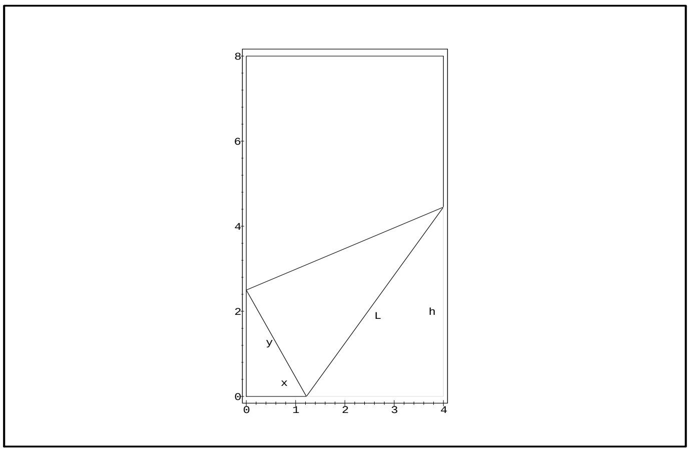
A vast number of problems fall into the category of the so-called max-min problems. These are problems in which some quantity, Q, is to be maximized or minimized as needed. The quantity Q is a function of one or more other variables which are subject to some constraints. We attempt to use these constraints to eliminate all but one of the variables in the computation of Q. If we are successful in doing this, we can then use calculus to locate the maximum or minimum if it exists.

Let's take an example.

A Paper folding problem: A sheet of paper 4 inches wide by 8 inches high is folded so that the bottom right corner of the sheet touches the left hand edge of the sheet. The tip of the corner is no more than 4 inches above the bottom edge of the paper. Then the paper is creased (see figure).

Find the length L of the crease, and find how to fold the paper so that L is minimum. Let h be the length from the top right edge to the crease.

```
> restart;
> with(plots): # Press Enter twice to see Figure!
> A1:=textplot({[2.6,1.9,'L'],[3.7,2,'h'],[.4,1.25,`y`],[.7,.3,`x`]},align=RIGHT):
> A3:=plot({[0,0,0,8,4,8,4,4.45,1.219,0,0,0],[4,4.45,0,2.5,1.219,0]},color=blue):
> A4:=plot([4,4.45,4,0,1.219,0],color=yellow):
> display([A1,A3,A4],axes=boxed,scaling=constrained);
```



Remark : this is an arbitrary figure of the setup and not the actual solution.

Now let's set up equations involving x, y, L, and h, and try to get one equation involving only y and L.

```
> eq1 := y^2+x^2=(4-x)^2;
```

$$eq1 := y^2 + x^2 = (4 - x)^2$$

```
> eq2 := L^2 = (4-x)^2 + h^2;
```

$$eq2 := L^2 = (4 - x)^2 + h^2$$

```
> eq3 := 4/(h-y)=y/x;
```

$$eq3 := 4\,\frac{1}{h - y} = \frac{y}{x}$$

eq1 and eq2 come from Pythagoras' theorem, and eq3 comes from similar triangles.

```
> h := solve(eq3,h);
```

$$h := -\frac{-4\,x - y^2}{y}$$

```
> x := solve(eq1,x);
```

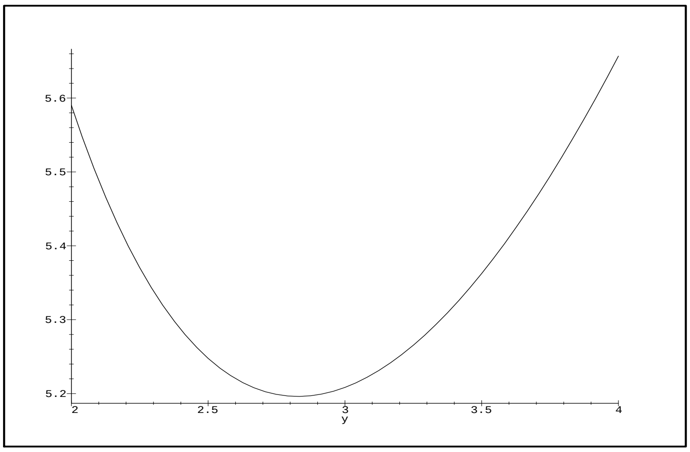
$$x := -\frac{1}{8}\,y^2 + 2$$

```
> L := sqrt(op(2,simplify(eq2)));
```

$$L := \frac{1}{8}\sqrt{\frac{(16 + y^2)^3}{y^2}}$$

So we have L expressed as a function of one variable y. Examining the behavior of L as y varies,

```
> plot(L,y=2..4);
```



we see that there is a minimum length crease of about `L = 5.2` inches at about `y = 2.8` inches. Getting a more precise value,

```
> y1 := fsolve(diff(L,y),y,2.6..3);
```

$$y1 := \mathbf{2.828427125}$$

Now check the value of x and L for this value of y.

```
> minL := subs(y=y1,L);
```

$$minL := \mathbf{5.196152423}$$

```
> minx := subs(y=y1,x);
```

$$minx := \mathbf{1.000000000}$$

Note the nice integer value of `minx` . This gives rise to a simple construction of the crease of minimum length: by folding twice along the bottom edge, we can mark the point 1 inch from the left edge of the paper. Then bring the corner point up to the left edge and crease. Used `subs` , `diff` , `fsolve` , `simplify` , `solve` , `plots` , `display` , `textplot` , and `plot` here.

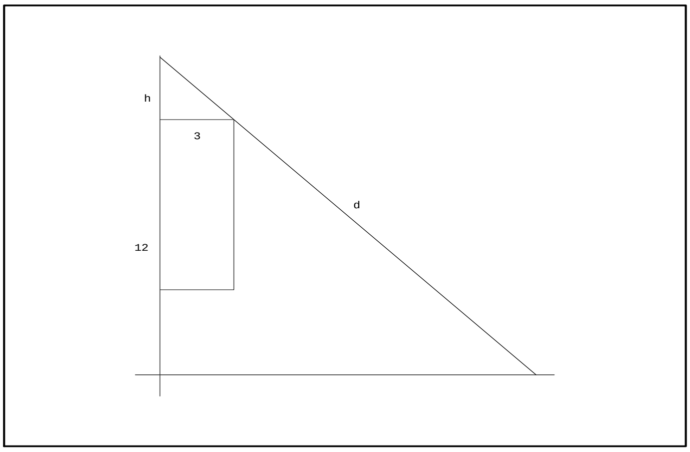
3.6 Stumbling onto max-min Problems

Sometimes, in the process of working on one problem, which may not be a max-min problem at all, one may stumble onto an interesting max-min problem. This can occur naturally in a Maple worksheet. Just take a problem in which you need to calculate a quantity in terms of some given quantities. Then make one of the given quantities a variable quantity and go through the calculation. At the end, you have obtained a formula expressing the desired quantity as a function of the variable quantity. Then you can study the behavior of this function with plot and diff, locating any extreme values it may possess. Here is an example.

A Ladder Problem: A 20 foot ladder is leaning against the top corner of a bay window in such a way that the top of the ladder is just resting against the wall of the house, and the base of the ladder is on the ground. The top corner of the window is 3 feet out from the house and 12 above the ground. How high up on the wall is the top of the ladder touching?

```
> restart;
```

```
> with(plots):
> A1:=plot({[0,15,0,-1],[0,12,3,12,3,4,0,4]},color=navy):
> A2:=plot({[-1,0,16,0],[0,14.937,15.257,0]},color=black):
> A3:=textplot({[-.5,13,´h´],[8,8,´d´],[-.75,6,`12`],[1.5,11.25,`3`]}):
> display({A1,A2,A3},axes=none,scaling=constrained,style=line);
```



Solution : Let's parameterize this problem by changing the length of the ladder to a parameter, say d. If we set up a coordinate system so that the wall is the positive y-axis, the ground is the positive x-axis, and the corner of the bay window is at (3,12), then by similar triangles, `h/sqrt(h^2+3^2) = (h+12)/d`, where h+12 is the height above the ground the ladder touches the wall. Let's let Maple do some work. First set up the equation and simplify it.

```
> restart;
> eq := h^2/(h^2+9) - (h+12)^2/d^2;
```

$$eq := \frac{h^2}{h^2+9} - \frac{(h+12)^2}{d^2}$$

Let's manipulate eq a little.

```
> eq := (collect(numer(normal(eq)),h));
```

$$eq := -h^4 - 24\,h^3 + (-153 + d^2)\,h^2 - 216\,h - 1296$$

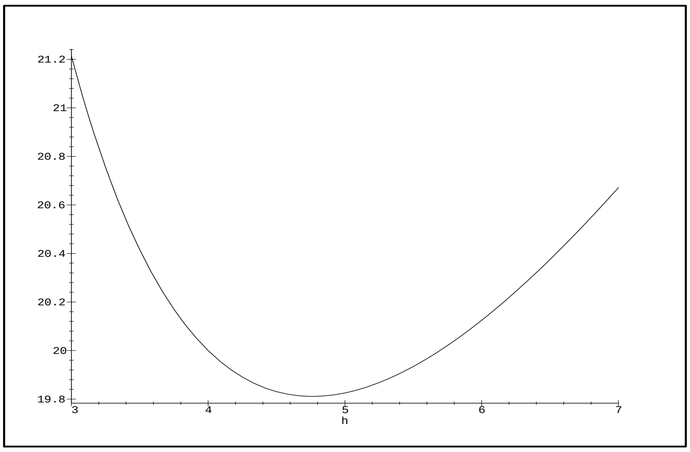
If we solve for the height above the wall, h, in terms of the length of the ladder, d, we get more extraneous solutions than if we solve eq for d in terms of h.

```
> sol := [solve(eq,d)];
```

$$sol := \left[\frac{\sqrt{h^4 + 153\,h^2 + 24\,h^3 + 216\,h + 1296}}{h},\right.$$
$$\left. -\frac{\sqrt{h^4 + 153\,h^2 + 24\,h^3 + 216\,h + 1296}}{h}\right]$$

Clearly, we are interested in the positive values for d, ie, sol[1].

```
> plot(sol[1],h=3..7);
```



Upon examination of the graph, we see that there are two solutions to the original problem. In fact, and this is clear if we reflect on the situation, there will generally be two positions the ladder can occupy if it is long enough, no positions if it is too short, and exactly one position if the ladder exactly the right length. That length is slightly under 20 feet, the y-coordinate of the base of the graph. We can discover this length easily.

```
> hmin := fsolve(diff(sol[1],h),h,4..5);
```

$$hmin := \mathbf{4.762203156}$$

This gives the height above the bay window that the ladder touches.

```
> dmin := subs(h=hmin,sol[1]);
```

$$dmin := \mathbf{19.81098307}$$

So, we have stumbled onto the solution of the problem of finding the shortest ladder such that there is a solution to the problem. Used `fsolve` , `subs` , `diff` , `collect` , `numer` , `normal` , `plot` , `with(plots)`, `textplot` , `display` , and `restart` here.

3.7 Problems:

1. Wandering Critical Points

For the cubic `f(x)= x^3 + ax^2 - x +3`, plot in the same window the graphs of f for several positive values of a. What happens to the local extreme points and the inflection points of f as a increases from 0 towards infinity? Also, try to use the word animate to do the same thing.

```
> plots[animate](x^3+a*x^2-x+3,x=-12..9,a=0..10,color=blue);
```

Used `plots[animate]` here.

2. Asymptotes

Make a sketch of the function `y =(x^3-8)/(x^3+2x^2)`. Find all asymptotes of the function. Also, find the intercepts, the local maxima, local minima, and inflection points of the function.

Two Pigpen problems Pigpen 1 A pigpen (shed for holding pigs) is to be shaped like a box with no floor and no front. The top is to be square, and the volume is to be 18 cubic yards. What dimensions should the pigpen have in order to minimize the amount of material used to construct it?

Pigpen II Instead of requiring the top to be square, suppose we want the top to be b times as wide as it is long, where b is some fixed positive number. Find the minimum amount, A, of material needed to construct such a pigpen as a function of b. Draw a conclusion about the most economical shape for the top of the pigpen.

Big Clock Problems Big Clock I A tower clock has a 4 foot minute hand and a 3 foot hour hand. How fast are the tips of the hands moving apart at 12:15?

Big Clock II At what time are the hands moving apart most quickly?

3.8 Some solutions

Wandering points Define the function and compute its derivatives.

```
> restart;
> f:= x -> x^3 + a*x^2 - x + 3;
```

$$f := x \rightarrow x^3 + a\,x^2 - x + 3$$

```
> fp := unapply(diff(f(x),x),x);
```

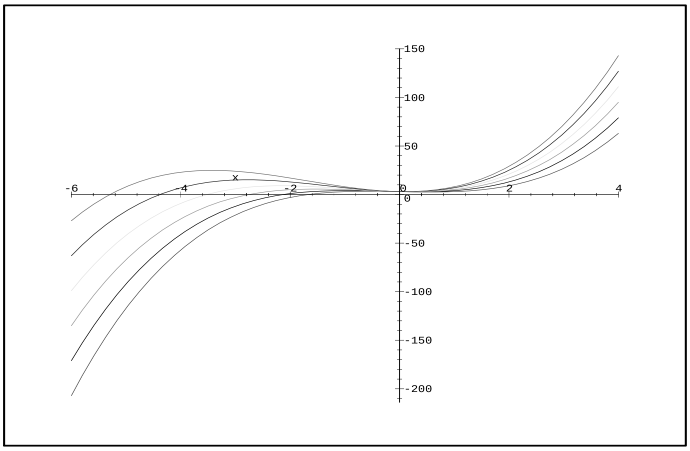
$$fp := x \rightarrow 3\,x^2 + 2\,a\,x - 1$$

```
> fpp := unapply(diff(f(x),x,x),x);
```

$$fpp := x \rightarrow 6\,x + 2\,a$$

Let's plot f over x from -6 to 4 and a from 0 to 5.

```
> plot({seq(f(x),a=0..5)},x=-6..4); a := 'a':
```



Looking at the graphs suggests that the local maximum (the leftmost critical point) wanders off to the left and drifts upward. The other critical point and the inflection point seem to stay stationary, although that may be an illusion. Let's calculate the points and look at their limits as a goes to infinity. First, find the x-coordinates of the critical points and the inflection point.

```
> cp := solve(fp(x),x);
```

$$cp := -\frac{1}{3}\,a + \frac{1}{3}\,\sqrt{a^2+3}, -\frac{1}{3}\,a - \frac{1}{3}\,\sqrt{a^2+3}$$

```
> ip := solve(fpp(x),x);
```

$$ip := -\frac{1}{3}a$$

We can see that as a gets large, cp[1] and cp[2] go to 0 and -infinity, respectively.

```
> limit(cp[1],a=infinity);
```

$$0$$

```
> limit(cp[2],a=infinity);
```

$$-\infty$$

The local minimum of f(x) as a approaches infinity is (0,3), and the local maximum and inflection points head towards (-infinity,-infinity), with the inflection point roughly halfway between the two extrema.

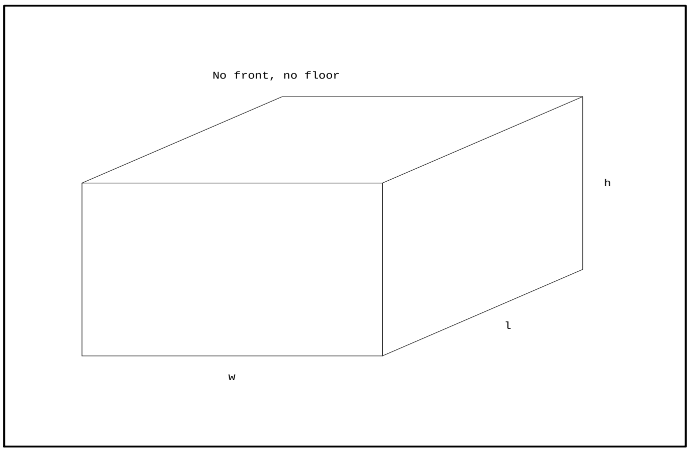
To observe the graph of f(x) as a increases (for example, from a=0 to a=10) is a great problem for using the word animate. Here is a sample of what you might have done. Once you have activated the animate plot press your pointer on the 'once' button to make the animate plot 'loop'.

```
> plots[animate](x^3+a*x^2-x+3,x=-12..9,a=0..10,color=blue);
```

Used `limit` , `infinity` , `unapply` , `plot` , `seq` , and `plots[animate]` here.

Pig Pen II First draw a picture

```
> with(plots):
> A1:=plot({[0,0,0,2,2,3,5,3,5,1,3,0,3,2,0,2],[0,0,3,0,3,2,5,3]},color=navy):
> A2:=textplot({[1.5,-.25,´w´],[4.25,.35,´l´],[5.25,2,´h´]},color=black):
> A3:=textplot([2,3.25,`No front, no floor`],color=black):
> display({A1,A2,A3},axes=none,scaling=constrained);
```



We are given that the length l of the pen is bw, where b is a fixed positive number. The problem is to minimize the area, `A = 2hl + hw + lw`, of the pen subject to the constraint that the volume, V=hlw, of the pen is 18 cubic feet.

```
> restart;
> A := 2*h*l + h*w + l*w;
```

$$A := 2\,h\,l + h\,w + l\,w$$

```
> l :=  b*w;
```

$$l := b\,w$$

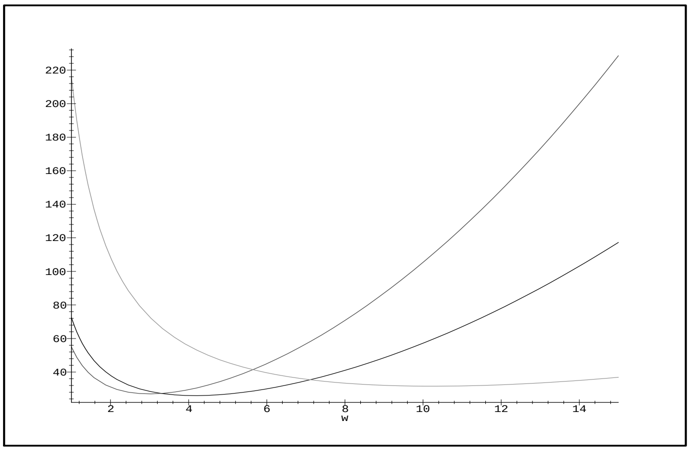
Now the area is expressed as a function of the height and width of the pen. But the height of the pen is h = 18/(lw) from the constraint equation. Let's substitute for h and turn A into a function.

```
> A := unapply(subs(h=18/(l*w),A),(b,w));
```

$$A := (\,b, w\,) \rightarrow 36\,\frac{1}{w} + 18\,\frac{1}{b\,w} + b\,w^2$$

Plotting the area function for various values of b

```
>  plot({A(1,w),A(1/2,w),A(1/10,w)},w=1..15);
```




```
> plot(A(1/2,w),w=1..15);
```

'might' lead one to suspect that the minimum of the minimums A_b occurs at b=1/2. We can express A_b as a function of b. First, find the width w_b of the pen with minimum area. That will occur at the real root of diff(A(b,w),w)=0. Then substitute that in for w in A.

```
> Aw := normal(diff(A(b,w),w));
```

$$Aw := 2\,\frac{-18\,b - 9 + b^2\,w^3}{b\,w^2}$$

```
> w_b := ((9+18*b)/b^2)^(1/3);
```

$$w\_b := \left(\frac{9 + 18\,b}{b^2}\right)^{1/3}$$

```
> A_b := unapply(A(b,w_b),b);
```

$$A\_b := b \rightarrow 36\,\frac{1}{\left(\frac{9 + 18\,b}{b^2}\right)^{1/3}} + 18\,\frac{1}{b\,\left(\frac{9 + 18\,b}{b^2}\right)^{1/3}} + b\,\left(\frac{9 + 18\,b}{b^2}\right)^{2/3}$$

```
> fsolve(diff(A_b(b),b),b,0..1);
```

$$.5000000000$$

Indeed, our suspicion is borne out.

Big Clock II. By the law of cosines, the distance s between the tips of the minute hand and the hour hand is

```
> restart;
> s := sqrt(3^2+4^2 - 2*3*4*cos(theta));
```

$$s := \sqrt{25 - 24\cos(\theta)}$$

where theta is the angle between the hands. This angle is a linear function of time t measured in minutes. At t=0 the angle is 0 radians and after 12/11*60 minutes the angle is 2*Pi radians, so in general,

```
> theta := (11/12)*(2*Pi/60)*t;
```

$$\theta := \frac{11}{360}\pi t$$

To find when the distance between the minute hand and hour hand is increasing most rapidly, lets think through what happens. At 12:00, the hands are together, but the minute hand is moving away from the hour hand. Actually they are both moving to the right, but the minute hand moves 12 times faster. The time at which the distance between the two hands are increasing most rapidly would be when the second derivative of s with respect to time is zero. This will occur sometime before the hands are pointing in opposite directions, which works out to be 1/2(12/11)(60) minutes past 12.

```
> sp := diff(s,t);
```

$$sp := \frac{11}{30}\frac{\sin\left(\frac{11}{360}\pi t\right)\pi}{\sqrt{25 - 24\cos\left(\frac{11}{360}\pi t\right)}}$$

```
> spp := normal(diff(sp,t));
```

$$spp := -\frac{121}{10800}\pi^2 \left(12\sin\left(\frac{11}{360}\pi t\right)^2 - 25\cos\left(\frac{11}{360}\pi t\right) + 24\cos\left(\frac{11}{360}\pi t\right)^2\right) \Big/ \left(25 - 24\cos\left(\frac{11}{360}\pi t\right)\right)^{3/2}$$

```
> ti := fsolve(spp,t,0..30*12/11);
```

$$ti := 7.529022202$$

```
> evalf(subs(t=ti,sp));
```

$$.2879793266$$

```
> evalf(subs(t=ti,s));
```

$$2.645751311$$

So at about 7.529 minutes after 12, the distance between the tips of the minute hand and the hour hand is increasing at the maximum rate of about .288 feet per minute. Also, the tips are about 2.65 feet apart at that time. Used `fsolve` , `evalf` , `subs` , `normal` , `diff` , `restart` , `sqrt` , and `cos` here.

Chapter 4

Early Integration.

At least as many problems can be solved using integration as can be solved using differentiation.

4.1 Learning to use the Maple words Sum and sum .

```
> Sum(i, i=1..100)=sum(i, i=1..100);
```

$$\sum_{i=1}^{100} i = 5050$$

Some sum problems. Find each of the following sums

1. The sum of the arithmetic sequence 4 + 7 + 10 + 13 + ... + 301

```
> (301-4)/3;
```

$$99$$

```
>  sum(4+i*3,i=0..99);
```

$$15250$$

2. The sum of the square roots of i, as i goes from 1 to 100.

```
>  evalf(sum(sqrt(i),i=1..100)) ;
```

$$671.4629471$$

3. Calculate the value of the following sum for n taking values 4, 8, 16 and n. What is the limiting value of the sum as n gets large?

```
>  Sum((1/2)^i,i=0..n);
```

$$\sum_{i=0}^{n} \left(\frac{1}{2}\right)^{i}$$

```
> sum((1/2)^i,i=0..n);
```

$$-2\left(\frac{1}{2}\right)^{(n+1)} + 2$$

As n gets large, 1/2 to the nth goes to 0 so the whole sum goes to 2.

```
> limit(sum((1/2)^i,i=0..n),n=infinity);
```

$$2$$

Used sum , Sum , evalf , infinity , and sqrt here.

4.2 Integration with the student package.

There are some modestly useful words in the student package dealing with integration.

```
> with(student);
```

$$[D, Doubleint, Int, Limit, Lineint, Sum, Tripleint, changevar, combine, completesquare, distance, equate, extrema, integrand, intercept, intparts, isolate, leftbox, leftsum, makeproc, maximize, middlebox, middlesum, midpoint, minimize, powsubs, rightbox, rightsum, showtangent, simpson, slope, trapezoid, value]$$

Learn to use the words middlesum and middlebox from the student package. Describe what these words do. Are there any other words in the package analogous with these?

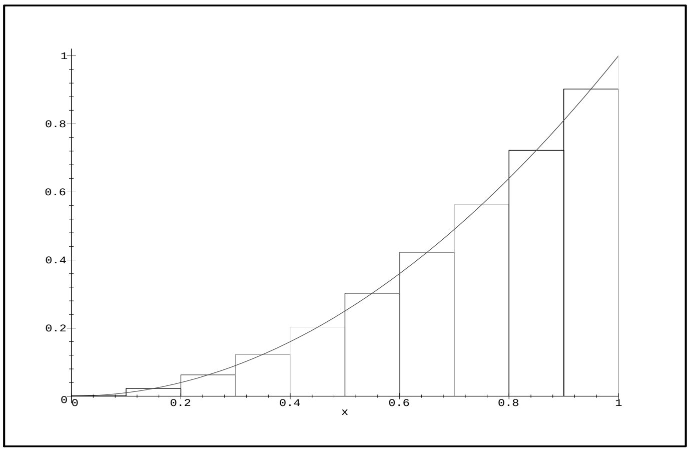
```
> middlesum(x^2,x=0..1,10) = value(middlesum(x^2,x=0..1,10));
```

$$\frac{1}{10}\left(\sum_{i=0}^{9}\left(\frac{1}{10}\,i+\frac{1}{20}\right)^2\right)=\frac{133}{400}$$

```
> evalf(middlesum(x^2,x=0..1,10));
```

$$.3325000000$$

```
> middlebox(x^2,x=0..1,10);
```



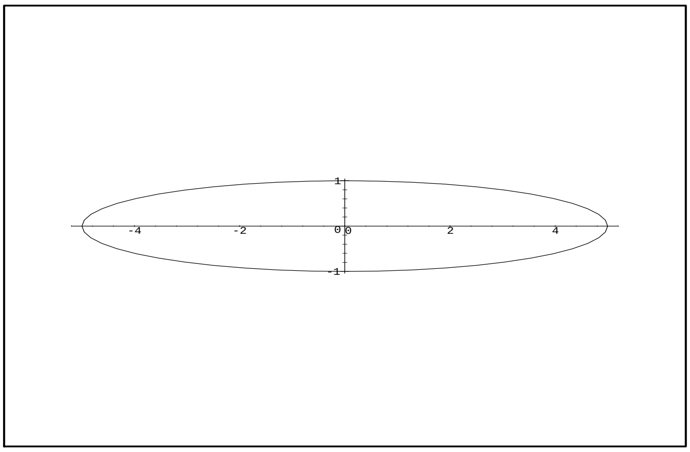
4.3 Two area problems.

1. Plot the ellipse x^2/25 + y^2 = 1 with a parametric plot. Draw the graph.

```
> ellipse := [5*cos(t),sin(t),t=0..2*Pi];
```

$$ellipse := [\,5\cos(\,t\,), \sin(\,t\,), t = 0..2\,\pi\,]$$

```
> plot(ellipse,scaling=CONSTRAINED);
```



Estimate the area of the ellipse by thinking of it as twice the area under the graph of

```
> f := x -> sqrt(1- x^2/25);
```

$$f := x \to \mathrm{sqrt}\left(1 - \frac{1}{25}\,x^2\right)$$

Then use `middlesum` from the student package to estimate the area. Use 10 subintervals in the partition.

2. For the function

```
> f := x -> 10 - x^4 + 10*x^2;
```

$$f := x \to 10 - x^4 + 10\,x^2$$

a) Plot the function and determine the region R where the graph of f lies above the x-axis. Make the sketch of R below, and supply the endpoints of the base of R accurate to 9 digits.

```
> restart;
```

b) Use `middlesum` from the student package to find the area of the region R to within 1/10.

4.4 Learning to use Int and int.

The fundamental theorem of calculus, tells us that a definite integral of a continuous function f can be evaluated by first finding an antiderivative of f and then calculating the difference of its values at the endpoints: In symbols,

```
> restart; with(student);
> Int(f(x),x=a..b) = F(b) - F(a), ` provided `, diff(F(x),x) = f(x);
```

$$\int_a^b \mathrm{f}(\,x\,)\,dx = \mathrm{F}(\,b\,) - \mathrm{F}(\,a\,),\ \textit{provided}\ ,\frac{\partial}{\partial x}\mathrm{F}(\,x\,) = \mathrm{f}(\,x\,)$$

You can use the Maple word `int` to find antiderivatives, (aka indefinite integrals). For example to find an antiderivative of

```
> x^2 + cos(x/5);
```

$$x^2 + \cos\left(\frac{1}{5}\,x\right)$$

Use int

```
> int(x^2 + cos(x/5),x);
```

$$\frac{1}{3}x^3 + 5\sin\left(\frac{1}{5}x\right)$$

You can also use it to evaluate definite integrals. For example, to plot the region under the graph of `x^2 + cos(x/5)` for x between 0 and 2 and calculate its area

```
> f := x ->x^2 + cos(x/5);
```

$$f := x \to x^2 + \cos\left(\frac{1}{5}x\right)$$

```
>  Int(f(x),x=0..2);
```

$$\int_0^2 x^2 + \cos\left(\frac{1}{5}x\right)\,dx$$

```
>  area :=  int(f(x),x=0..2);
```

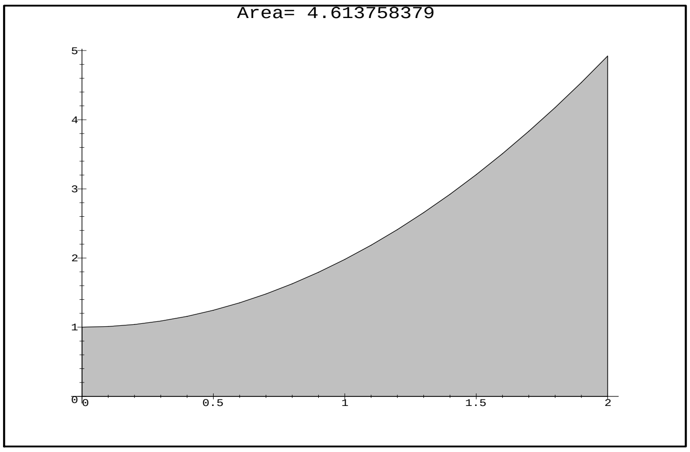
$$area := \frac{8}{3} + 5\sin\left(\frac{2}{5}\right)$$

```
>  area := evalf(area);
```

$$area := 4.613758379$$

We will use `plots[polygonplot]` to shade the region.

```
>  plots[polygonplot]( [ [2,0],[0,0],[0,f(0)],seq([x/10,f(x/10)] , x=0..20) ],color=gray,titl
> e= cat(`Area= `,convert(area,string)) );
```



Exercise s:

For each of the functions below, Sketch the region under the graph, Calculate the area of the region (use `int` if you like).

a) `f :=  x-> (1-x+ x^2)^2   , for  x = -1..2`

b) `f :=  x ->  cos(x)^2 + 3*x,    for  x = 0..1.`

c) `f := proc(x)  if  x < 1 then sqrt(x) else x^2 fi end; for x = 0..2`

4.5 Average value of the square

Suppose you picked at random 1000 numbers between 0 and 1. What would you expect the average of the squares of these numbers to be, about?

Solution .

```
> ?rand
> s := 0 : for i from 1 to 1000 do s := s+(rand()/999999999999.)^2 od: s/1000;
```

$$.3365089741$$

Looks like its around a third. You could estimate the average value by averaging the values at n equally spaced points in the interval. Here is a procedure for doing this. The inputs are

```
>   av := proc(f,a,b,n)
>    local i,dx;
>    dx := (b-a)/n;
>    evalf(sum(f(a+i*dx),i=1..n)/n);
>    end;

av := proc(f,a,b,n)
      local i,dx;
          dx := (b-a)/n; evalf(sum(f(a+i*dx),i = 1 .. n)/n)
      end
```

So to estimate the average value of x^2 as x ranges from 0 to 1, we could generate a sequence of estimates

```
> for i by 10 from 10 to 40 do print(i, av(x->x^2,0,1,i)) od;
```

$$10, .3850000000$$

$$20, .3587500000$$

$$30, .3501851852$$

$$40, .3459375000$$

Problem .

Use av to estimate the average value of each of the functions in question 1over the given interval.

4.6 Modeling the inflow and outflow of air in lungs:

The rate of change of the volume of air in the lungs can be modeled (according to some texts) by the function

```
> f := t -> 1/2*sin(2*Pi*t/5);
```

$$f := t \to \frac{1}{2}\sin\left(\frac{2}{5}\pi\, t\right)$$

where volume is measured in liters and time is measured in seconds. So the actual volume of air in the lungs would obtained by integrating f. Here we are assuming there is no air in the lungs at time 0.

```
>   Int(f(tau),tau=0..t) = int(f(tau),tau=0..t);
```

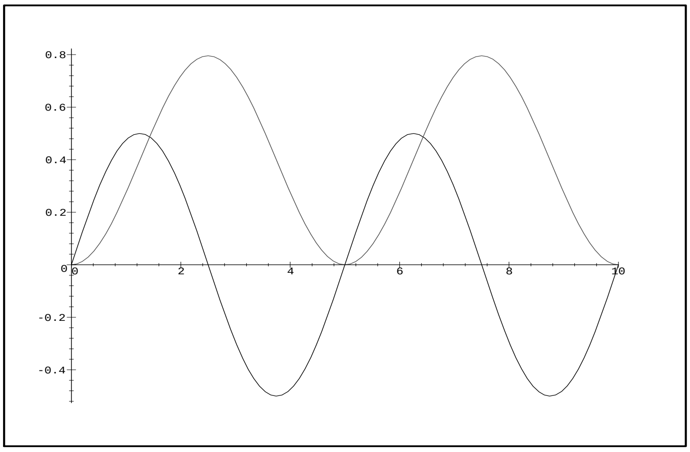
$$\int_0^t \frac{1}{2}\sin\left(\frac{2}{5}\pi\,\tau\right)\,d\tau = -\frac{5}{4}\,\frac{\cos\left(\frac{2}{5}\pi\, t\right)}{\pi} + \frac{5}{4}\,\frac{1}{\pi}$$

```
> F := unapply(int(f(tau),tau=0..t),t);
```

$$F := t \to -\frac{5}{4}\,\frac{\cos\left(\frac{2}{5}\,\pi\,t\right)}{\pi} + \frac{5}{4}\,\frac{1}{\pi}$$

Let's plot the rate of airflow and the volume function to get a feel for their behavior.

```
> plot({f,F},0..10);
```



Questions. 1. Which is the rate of airflow function and which is its integral?

2 What is the maximum volume of air in the lungs in this model? The minimum volume?

3) When are the lungs half-full?

4) What is the average rate of airflow in the lungs over the time interval [0,2.5]? [2.5,5]? [0,5]?

```
> ar := evalf(int(f(t),t=0..2.5)/2.5);
```

$$ar := .3183098861$$

About .32 liters per second on average. 5) What is the average volume of air in the lungs over the time interval [0,2.5]? [0,5]?

```
> av := evalf(int(F(t),t=0..2.5)/2.5);
```

$$av := .3978873575$$

About .4 liter air in the lungs on average.

Actually, there are different, perhaps more realistic models of the rate of airflow into the lungs. For example, consider this one,

```
>  g := t -> (3/8-1/4*cos(4*Pi*t/5))*sin(2*Pi*t/5);
```

$$g := t \to \left(\frac{3}{8} - \frac{1}{4}\cos\left(\frac{4}{5}\,\pi\,t\right)\right)\,\sin\left(\frac{2}{5}\,\pi\,t\right)$$

6 Using the function g to model the rate of airflow in the lungs, find the volume G of air in the lungs at time t. (Assume as before that G(0) = 0.) Plot both g and G over the time interval [0,10]. Describe qualitatively a difference that you note between this model and the model we looked at above.

7 Calculate the maximum rate of airflow in this model.

8 What is the maximum volume of air in the lungs in this model?

9 What is the average volume of air in the tank in the time interval [0,5]

4.7 An area problem:

Find the upside down parabola `y = a - x^2` so that the area between it and the parabola `y = x^2` is 100. Draw a diagram.

Solution .

Set up the functions which bound the region.

```
> restart;
>  f := x -> a-x^2;
```

$$f := x \to a - x^2$$

```
>  g := x-> x^2;
```

$$g := x \to x^2$$

find the points where the functions cross.

```
> sol := solve(f(x)=g(x),x);
```

$$sol := -\frac{1}{2}\sqrt{2}\sqrt{a}, \frac{1}{2}\sqrt{2}\sqrt{a}$$

Get the area of the region.

```
> area := int(a-2*x^2,x=sol[1]..sol[2]);
```

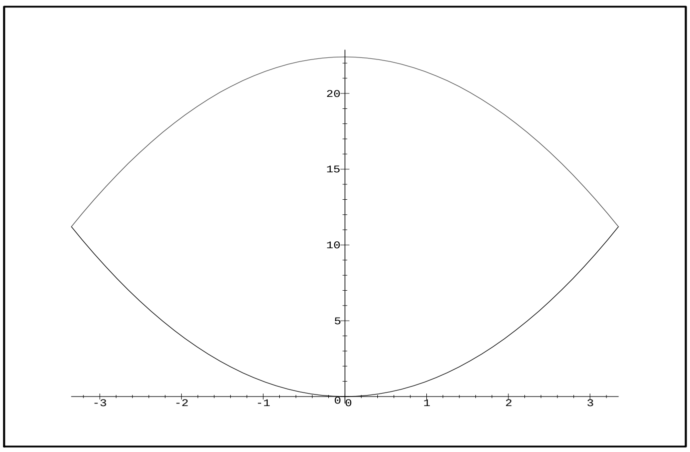
$$area := \frac{2}{3}a^{3/2}\sqrt{2}$$

Solve an equation.

```
> solve(area=100,{a});
```

$$\{a = 75^{2/3}\,2^{1/3}\}$$

```
> assign(");
> plot({f,g},sol[1]..sol[2]);
```



4.8 Another area problem:

The parabola `y = a*x^2 + b*x + c` is tangent to the graph of `y = 2 +abs(x - 3)` at two points and the area of the region bounded by their graphs is 10. Find a, b, and c. Make a sketch.

Solution :

The axis of the parabola is `x = - b/(2*a)`. That is also the axis of `y = 2 +abs(x-3)`, so `-b/(2*a) = 3`, or `b = -6*a`.

The point where the slope of the parabola is 1 is on both graphs. Call the point x0,y0. Then

`y0 = 2 + x0-3 = x0-1 = ax0^2 -6ax0 + c` and `1 = 2*a*x0 - 6*a`.

```
> restart;
```

```
> eq1 := x0-1 = a*x0^2 -6*a*x0 + c;
```

$$eq1 := x0 - 1 = a\,x0^2 - 6\,a\,x0 + c$$

```
>  eq2 := 1 = 2*a*x0 - 6*a;
```

$$eq2 := 1 = 2\,a\,x0 - 6\,a$$

```
> ac := solve({eq1,eq2},{a,c});
```

$$ac := \left\{a = \frac{1}{2}\,\frac{1}{x0 - 3}, c = \frac{1}{2}\,\frac{x0^2 - 2\,x0 + 6}{x0 - 3}\right\}$$

Finally, the area between the curves is 100, so the righthand half is 50.

```
>  eq3 := Int (a*x^2-6*a*x+c-(2+x-3),x=3..x0)=50;
```

$$eq3 := \int_3^{x0} a\,x^2 - 6\,a\,x + c + 1 - x\,dx = 50$$

```
> eq3 := int (a*x^2-6*a*x+c-(2+x-3),x=3..x0)=50;
```

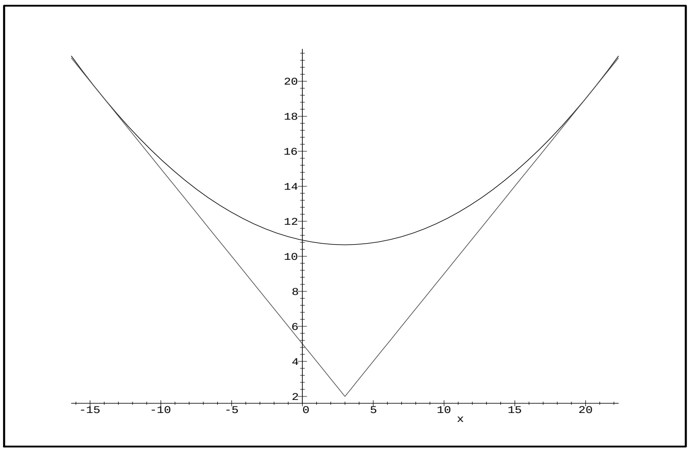
$$eq3 := \frac{1}{3}\,a\,x0^3 + x0 + x0\,c - 3\,a\,x0^2 - \frac{1}{2}\,x0^2 + 18\,a + \frac{3}{2} - 3\,c = 50$$

```
>  sol :=solve(subs(ac,eq3),x0);
```

$$sol := 3 + 10\,\sqrt{3}, 3 - 10\,\sqrt{3}$$

```
> assign({x0=sol[1]}); assign(ac);
```

```
> plot({2+abs(x-3),a*x^2-6*a*x+c},x=sol[2]-2..sol[1]+2);
```


20
18
16
14
12
10
8
6
4
2
-15
-10
-5
0
5
10
15
20
x

Chapter 5

Using Maple in Calculus II

Here is a standard final exam in Calculus I worked with Maple. Observe the output carefully to see how Maple works. Use the Maple word help(..) or ?.. to clarify the usage of a Maple word.

1. Find the indicated quantity.

(a) Find the limit.

$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

Solution :

```
> y := (x-2)/(x^2-4);
```

$$y := \frac{x-2}{x^2-4}$$

```
> limit(y,x=2);
```

$$\frac{1}{4}$$

```
> help(limit);
> ?`:=`;
```

(b) Find the limit.

$$\lim_{x \to 0-} \frac{|x|}{x}$$

Solution :

```
> help(abs);
> limit(abs(x)/x,x=0,left);
```

$$-1$$

(c) Find the limit.

$$\lim_{x \to \infty} \frac{x^2-4}{(x-2)(x+3)}$$

Solution :

```
> y := (x^2-4)/((x-2)*(x+3));
```

$$y := \frac{x^2-4}{(x-2)(x+3)}$$

```
> simplify(y);
```

$$\frac{x+2}{x+3}$$

```
> help(simplify);
> limit(y,x=infinity);
```

(d) Find the derivative of y.

$$y := \frac{3\,x+1}{\sqrt{x}}$$

Solution :

```
> y := (3*x + 1)/sqrt(x);
```

$$y := \frac{3\,x+1}{\sqrt{x}}$$

```
> diff(y,x);
```

$$3\,\frac{1}{\sqrt{x}} - \frac{1}{2}\,\frac{3\,x+1}{x^{3/2}}$$

```
> help(diff);
```

(e) Find the derivative of

$$\cos(\,3\,x\,)^2$$

Solution :

```
> diff(cos(3*x)^2,x);
```

$$-6\cos(\,3\,x\,)\sin(\,3\,x\,)$$

(f) Find the integral.

$$\int \frac{1}{x^2}\,dx$$

Solution :

```
> help(int);
> int(1/x^2,x) + C;
```

$$-\frac{1}{x} + C$$

(g) Find the definite integral.

$$\int_0^{1/2\,\pi} \cos(\,x\,)\,dx$$

Solution :

```
> int(cos(x),x=0..Pi/2);
```

$$1$$

2. Sketch the graph of y = x^4 - 6x^2 + 5. Find the local maxima and local minima. Also, find the x and y intercepts. **Solution** :

```
> y := x^4 -6*x^2 +5;  #  set up the function
```

$$y := x^4 - 6\,x^2 + 5$$

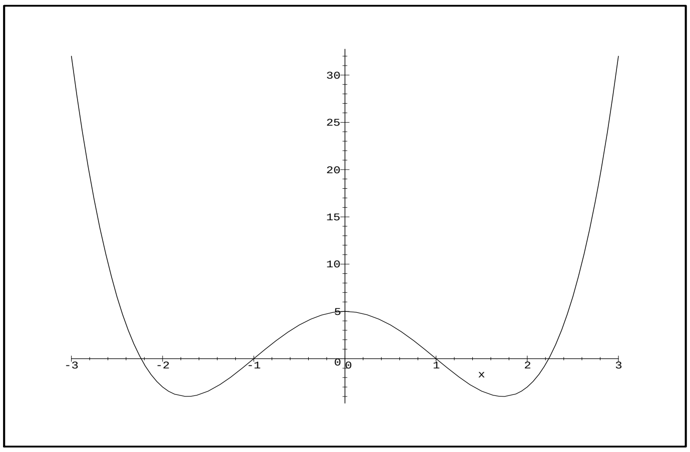
```
> yp := diff(y,x);  #  compute the derivative
```

$$yp := 4\,x^3 - 12\,x$$

```
> crits := fsolve(yp=0,x);  #  get the critical points
```

$$crits := 0, -1.732050808, 1.732050808$$

```
> help(fsolve);
> plot(y,x=-3..3);  #  plot the function
```



Get the y coordinates of the local maxima and minima,

```
> subs(x=crits[1],y); subs(x=crits[2],y); subs(x=crits[3],y);
```

$$5$$

$$-4.000000001$$

$$-4.000000001$$

3. A pot is to be made in the shape of a cylinder with a bottom and no top. The volume of the pot is to be 192*Pi cubic feet. If the material for the bottom costs 3 dollars per square foot and the material for the sides costs 1 dollar per square foot, what should the dimensions of the pot be for minimum cost?

Solution : Let r be the radius and h be the height of the pot. Then the volume is `v = Pi*r^2*h = 192*Pi`. Therefore, the height in terms of r is 96/r;

```
> h :=196/r^2;
```

$$h := 196 \frac{1}{r^2}$$

We want to minimize the cost, which is `cost = 3*(bottom area) + 1* (side area)`.

```
> c := 3*Pi*r^ 2 + 1*2*Pi*r*h;
```

$$c := 3 \pi r^2 + 392 \frac{\pi}{r}$$

First, find the critical points, and then graph the cost vs. the radius.

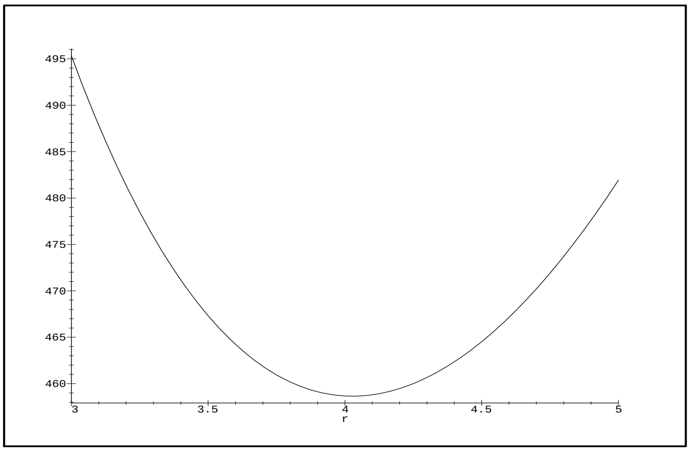
```
> cp := diff(c,r);
```

$$cp := 6 \pi r - 392 \frac{\pi}{r^2}$$

```
> fsolve(cp,r);
```

$$4.027587079$$

```
> plot(c,r=3..5);
```



We see that the best radius is 4 ft.

```
> help(subs);
> subs(r=4,h);
```

$$\frac{49}{4}$$

Therefore, the height is 49/4 ft. Now we'll compute the minimum cost,

```
> cost := subs(r=4,c);
```

$$cost := 146\,\pi$$

```
> evalf(cost);
```

$$458.6725275$$

```
> help(evalf);
```

4. Define a function f by integration as follows:

$$f(\,x\,) := \int_1^x \frac{1}{t}\,dt$$

for all positive x.

(a) What are f '(x) and f "(x)?

Solution : `f ´(x) = 1/x` and `f´´´(x) = - 1/x^2`. Checking to see if Maple knows this –

```
> diff(int(1/t,t=1..x),x);
```

$$\frac{1}{x}$$

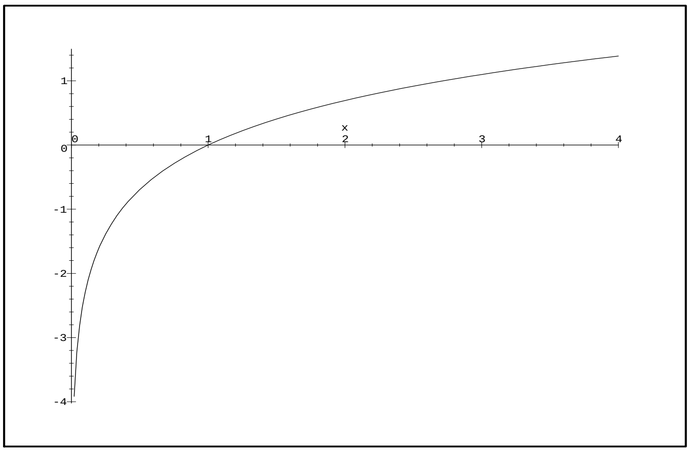
```
> diff(",x);
```

$$-\frac{1}{x^2}$$

(b) Is f an increasing or a decreasing function? Justify your answer.

Solution : Since f ' is positive there then f is increasing on x positive.

```
> plot(int(1/t,t=1..x),x=0..4);
```



(c) Calculate the lower sum and the upper sum for the f on [1,2]. Then show that the lower sum, L2, is 7/12 and the upper sum, U2, is 5/6. So the integral, F(2), is trapped between these two numbers.

5. A diver jumps off a highdive board 24 feet above the water with an upward velocity of 8 feet per second. If we assume the only force thereafter is gravity, when and with what velocity does she hit the water?

Solution : Let a,v, and p be the acceleration, velocity and position, respectively, of the moving lady. The acceleration due to gravity is a constant 32 ft/sec/sec, in the downward direction.

```
> restart;
> a := -32;
```

$$a := -32$$

The velocity is an antiderivative of acceleration.

```
> v := int(a,t) + c;
```

$$v := -32\,t + c$$

Let's evaluate the constant of integration here.

```
> c := solve(subs(t=0,v)=8,c);
```

$$c := 8$$

```
> `velocity = `, v;
```

$$\mathit{velocity}\ =, -32\,t + 8$$

Now we have her velocity at time t. Her position is an antiderivative of v.

```
> p := int(v,t) + c2;
```

$$p := -16\,t^2 + 8\,t + \mathit{c2}$$

Let's find the constant of integration.

```
> c2 := solve(subs(t=0,p)=24,c2);
```

$$\mathit{c2} := 24$$

```
> `position = `, p;
```

$$position\ = , -16\,t^2 + 8\,t + 24$$

Now we have her position at time t. She hits the water when p = 0.

```
> solve(p=0,t);
```

$$-1, \frac{3}{2}$$

She hits the water 3/2 seconds after jumping,

```
> subs(t=3/2,v);
```

$$-40$$

with a velocity of -40 feet per second.

6. Evaluate the following integrals.

(a)

$$\int x\cos(\,x^2\,)\,dx$$

Solution :

```
> int(x*cos(x^2),x);
```

(b)

$$\int_0^1 (\,1 + x\,)\,\sqrt{2\,x + x^2}\,dx$$

Solution :

```
> int((1+x)*sqrt(2*x+x^2),x=0..1);
```

(c)

$$\int 3\sin(\,2\,x\,)\cos(\,2\,x\,)\,dx$$

Solution :

```
> int(3*sin(2*x)*cos(2*x),x);
```

7. Let R be the region bounded by the graphs of `f(x) = 3 - x` and `g(x) = x^2 - 3x`.

(a) Sketch the graphs and shade the region R.

Solution :

```
> line := 3-x;
```

$$line := 3 - x$$

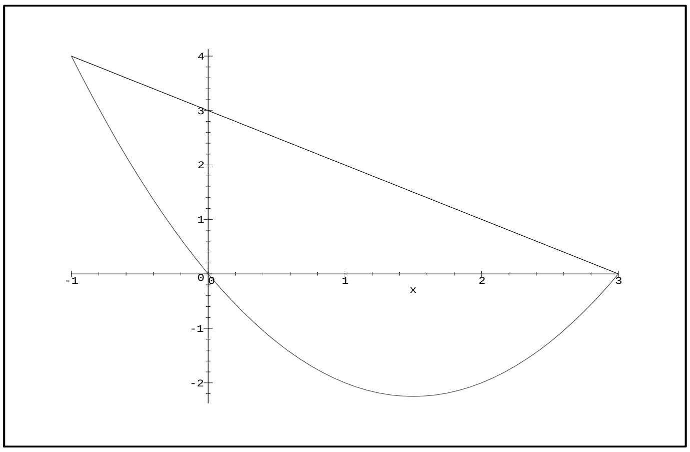
```
> parabola := x^2-3*x;
```

$$parabola := x^2 - 3\,x$$

```
> solve(line=parabola,x);
```

$$-1, 3$$

```
> plot({line,parabola} ,x=-1..3);
```



(b) Write the area of R as an integral and evaluate the integral.

Solution : The line is the top boundary of R, so the area of R is

```
> Int(line-parabola,x=-1..3);
```

$$\int_{-1}^{3} 3 + 2\,x - x^2\,dx$$

```
> value(");
```

$$\frac{32}{3}$$

7. Let R be the region bounded by`f(x) = 1` and `g(x) = (x-10)^2`. (a) Suppose R is revolved around the x-axis. Write an integral which yields the volume of the resulting solid of revolution.

Solution : Here the volume is obtained by integrating the area of the circular cross-section.

```
> volx := Int(Pi*(1-(x-10)^2),x=9..11);
```

$$volx := \int_{9}^{11} \pi\,\left(1 - (\,x - 10\,)^2\right)\,dx$$

```
> volx := value(volx);
```

$$volx := \frac{4}{3}\,\pi$$

(b) Suppose R is revolved around the y-axis. Write an integral which gives the volume of this solid.

Solution : Here we integrate the cylindrical cross-sections.

```
> voly := Int(2*Pi*x*(1-(x-10)^2),x=9..11);
```

$$voly := \int_{9}^{11} 2\,\pi\,x\,\left(1 - (\,x - 10\,)^2\right)\,dx$$

```
> voly := value(voly);
```

$$voly := \frac{80}{3}\,\pi$$

(c) Which of the volumes above is larger? Justify your answer. (Hint: With a good picture, a geometric argument will suffice.)

```
> voly/volx;
```

$$20$$

Notice that the 2nd volume 20 times the 1st volume.

8. A 20 foot chain weighing 100 pounds is suspended by one end from the top of the office tower. Let W be the work done to pull the chain all the way to the top. (a) Set up an approximating sum for the work W. Be sure to specify the meaning and units for any variables used.

Solution : Break the chain into n links. The work required is approximately the sum of the works Wi needed to lift the ith link to the top.

`Wi = (dist from ith link to top) * (weight of ith link) = i *(20/n)* 5 *(20/n).`

```
> approx := sum(i*(20/n)*5*(20/n),i=1..n);
```

$$approx := 1000\,\frac{(n+1)^2}{n^2} - 1000\,\frac{n+1}{n^2}$$

```
> simplify(approx);
```

$$1000\,\frac{n+1}{n}$$

```
> limit(approx,n=infinity);
```

$$1000$$

(b) Use part (a) to write an integral for the work W. Then evaluate the integral.

```
> work := Int(x*5,x=0..20);
```

$$work := \int_0^{20} 5\,x\,dx$$

```
> value (work);
```

$$1000$$

5.1 Linear, Quadratic, and Cubic Functions

A linear function `f(x) = ax + b` is completely determined once a and b are known, and you can get a and b once you know the value of the function at two points or the value at one point and the value of its derivative at one point. Here are a couple of examples worked out in Maple.

Example 1: Find f given that `f(2) = 1`, and `f(-4) = 3`.

Solution :

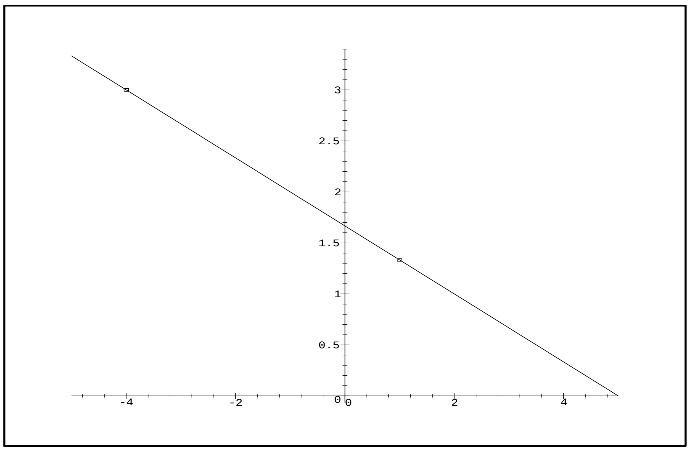
```
> restart;
> f  := x -> a + b*x;  #  set up the unknown function
```

$$f := x \to a + b\,x$$

```
> sol := solve( {f(2)=1,f(-4)=3},{a,b});  #  solve for a and b
```

$$sol := \left\{ b = \frac{-1}{3}, a = \frac{5}{3} \right\}$$

```
> assign(sol);  #  assign the values for a and b.
> pl := plot( f,-5..5):  #  look at the function.
> pts := plot({[1,f(1)],[-4,f(-4)]},style=point,symbol=box):
> plots[display]([pl,pts]);
```



Example 2: Same as above, except f(-4)=3 is replaced with f '(-4) = 3.
Solution :

```
> a := ´a´;  b := ´b´;  #  make a and b variable again
```

$$a := a$$

$$b := b$$

```
> solve( {f(2)=1,D(f)(-4)=3} , {a,b} );  #  solve the system for a and b
```

$$\{a = -5, b = 3\}$$

```
> assign(");  #  assigns the solutions
> plot(f,-5..2);  #  look at the function
```

The same ideas carry over to quadratic and cubic functions. Thus in order to determine f(x) = ax^2 + bx + c, you need to know 3 values of f (that is, x and y coordinates), or 2 values of f and 1 value of f ' , or 1 value each of f, f ', and f " or ... (how many possibilities are there?).

Problems 1. Find the linear function f(x) = ax + b such that f(2) = 213 and f(-12) = 33. Also, given that P = (c,43/154) is a point on the line, find c.

2. Find the parabola which passes through the points (11,13), (1,5), and (21,-70). What is the vertex of this parabola? Find the x-intercepts of the parabola. Plot the parabola. (Hint: To find the parabola, set up three linear equations. Then solve.)

3. The sum s(n) = 1 + 2 + ... + n of the first n positive integers is known to be a quadratic function s(n) = an^2 + bn + c of n. Determine a, b, and c. Then calculate s(2^64)

and `s(52!)`.

```
> sum(i,i=1..n);
```

$$\frac{1}{2}(n+1)^2 - \frac{1}{2}n - \frac{1}{2}$$

```
> sum(i,i=1..2^64);
```

$$170141183460469231740910675752738881536$$

4. A particle is moving along a line with position s(t) so that its acceleration is constant (i.e., `s''(t) = k`). Given `s(5) = 100`, `s'(5) = 371`, and `s(10) = 173`, describe the motion of the particle. Especially, find the position at the time when the velocity of the particle is 0. Make a sketch of the motion.

5. Find the parabola `y = ax^2 + bx + c` which passes through (13,-31) and (15,4), given that its tangent at `x = 3` has slope 5.

6. A cubic function has the values `f(5) = 10`, `f(10) = 25`, `f(15) = 15`, and `f(20) = 25`. Find the values of the coefficients of f. Where do the local maximum value and the local minimum value of occur?

7. Find the cubic function `y = f(x)` at which `f(5)=-2` is a local minimum and `f(15) = 32` is a local maximum.

5.2 A Useful Function – The natural logarithm

The logarithm function was first introduced in the early 1600's by John Napier to reduce the toil of performing multiplication and division of numbers. The idea was quickly accepted worldwide.

Napier's logarithm was defined differently from the way it would be now. The idea of defining the logarithm of x for `x > 1` to be the area under the curve 1/x, that is,

```
> `ln(x) = `, Int(1/t,t=1..x);
```

$$ln(x) \ = , \int_1^x \frac{1}{t}\, dt$$

came along later. It may have been prompted by the observation that this area grows arithmetically as x grows geometrically.

Problems 1. Consider the sequence of numbers 2, 4, 8, 16, ..., `an = 2^n`, ... This is a geometric sequence, that is, the ratio of successive terms is constant (in this case, the common ratio is 2). Geometric sequences grow rapidly when the common ratio is 2 or more. Compute the 64th number in this list and write down its full English name " ... six hundred and sixteen".

Solution :

```
> a := n -> 2^n;
```

$$a := n \to 2^n$$

```
> a(64);
```

$$18446744073709551616$$

Thats 18 quintillion, 446 quadrillion, 744 trillion, 73 billion, 709 million, 551 thousand, and 616.

For n = 1, 2, ..., let bn be the area under the graph of `y = 1/x` between 1 and an. Show that this sequence grows arithmetically, that is, the difference of successive terms is constant. (Hint: compute b(n+1) - bn).

Solution :

```
> b := n -> int(1/t,t=1..a(n));
```

$$b := n \to \int_1^{a(n)} \frac{1}{t}\, dt$$

```
> b(n+1) - b(n);
```

$$\ln(2^{(n+1)}) - \ln(2^n)$$

```
> simplify(");
```

$$\ln(2)$$

2. Estimate ln(2) by Riemann sums to 4 decimal places.

Solution : Here is a procedure `riemann` you can use for ln(2).

```
> restart;
> riemann := proc(n)
>            local Ln,Un,i;
>                Ln := evalf(sum(1/(1+i/n)/n,i = 1 .. n));
>                Un := evalf(sum(1/(1+(i-1)/n)/n,i = 1 .. n));
>                Ln,` <  ln(2)  < `,Un,`  error = `,Un-Ln
>            end;
```

3. Which line `y = m(x-1)` divides the region under the graph of `y = 1/x` between 1 and 2 into two regions with equal area? Sketch the graphs.

Solution :

```
> y := m*(x-1);
```

$$y := m(x-1)$$

```
> b := solve(y=1/x,x);
```

$$b := \frac{1}{2}\frac{m+\sqrt{m^2+4m}}{m}, \frac{1}{2}\frac{m-\sqrt{m^2+4m}}{m}$$

```
> eq := int(1/x-y,x=1..b[1])=1/2*ln(2);
```

$$eq := \ln\left(\frac{1}{2}\frac{m+\sqrt{m^2+4m}}{m}\right) - \frac{1}{2} - \frac{1}{4}m + \frac{1}{4}\sqrt{m^2+4m} = \frac{1}{2}\ln(2)$$

```
> m := fsolve(eq,m);
```

$$m := .7575012634$$

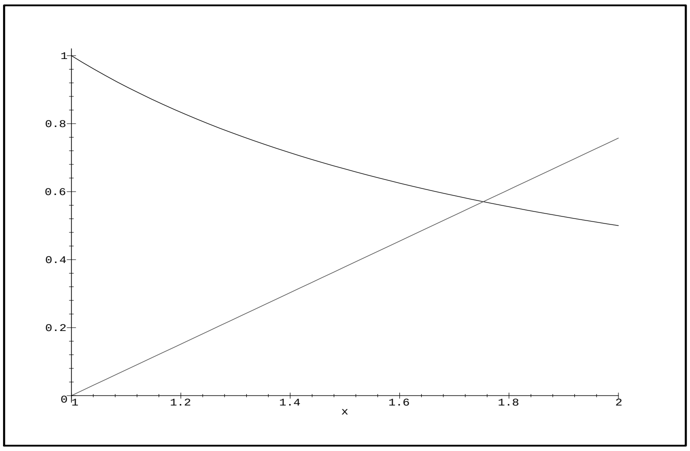
```
> plot({1/x,y},x=1..2);
```

4. Find the parabola `y = x^2 + b` which is tangent to the graph of `y = ln(x)`. Sketch the graph. **Solution** : Let (x1,ln(x1)) be the point of tangency.

```
> b := 'b':eq1 := x^2+b = ln(x);
> eq2 := 2*x = 1/x;
> sol2 := solve(eq2,x);
> x1 := sol2[1]:
```

```
> b := solve(subs(x=x1,eq1),b);
```

```
> plot({ln(x),x^2+b},x=-2..2);
```



5. Find the parabola `y = ax^2` which is tangent to the graph of `y = 1/x`. Sketch the graphs.

6. Find the local extreme values of `y = ln(10 + x(x-1)(x-2))`. Sketch the graph.

7. Sketch the region bounded by `y = ln(1+x^2)` and `y = 10 - x^2`. Find the area of the region.

8. The line `x + y = m` and the graph of `y = 1/x` bound a region of area 10. Find m and plot the graphs.

9. The line `y = 2x + 1` does not intersect the graph of `y = ln(ln(x))`. (Does it?) Find the point on the line which is closest to the graph. Sketch the graphs.

Solution :

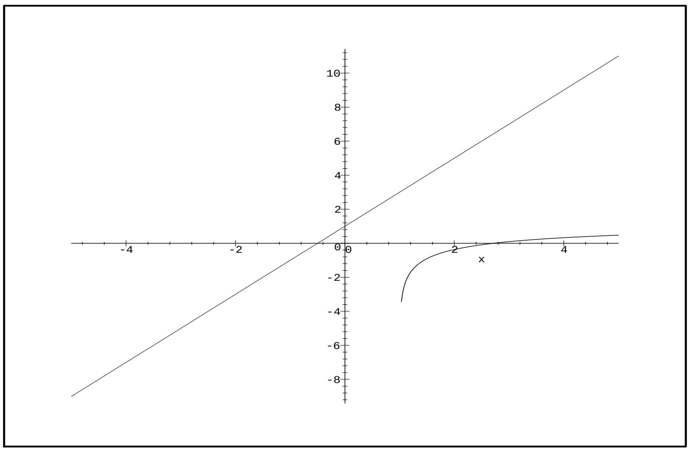
```
> line := 2*x+1;
```

$$line := 2\,x + 1$$

```
> lnln := ln(ln(x));
```

$$lnln := \ln(\ln(x))$$

```
> plot( {line,lnln} ,x=-5..5);
```

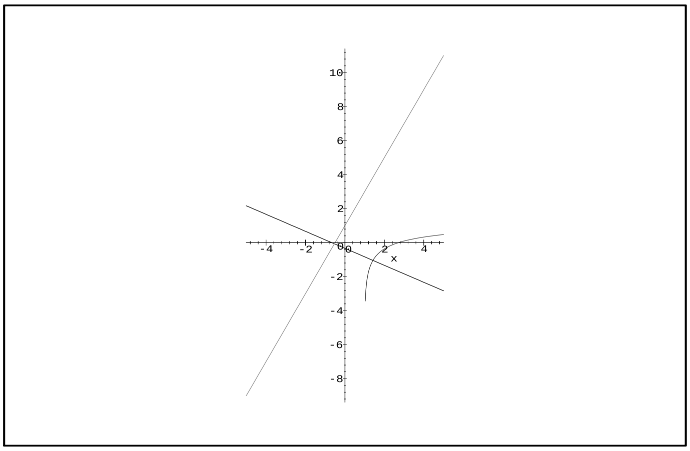


The closest point looks to be near x=1.5

```
> x1:=fsolve(diff(lnln,x)=2,x,1..1.5);
```

$$x1 := \mathbf{1.421529936}$$

```
> plot({line,lnln,-1/2*(x-x1)+subs(x=x1,lnln)},x=-5..5,scaling=constrained);
```



5.3 Inverse Functions – The exponential function

The inverse of the natural logarithm is called the exponential function. It is predefined in Maple as exp. It is called the exponential function because it obeys the 'laws of exponents',

`exp(x+y) = exp(x)*exp(y)` and `exp(-x) = 1/exp(x)`. Remember, we defined e to be the number such that `ln(e) = 1`. This says that `exp(1) = e`. So,

`exp(2) = exp(1+1) = exp(1)*exp(1) = e^2`. Also, `exp(1/2) = e^(1/2)`, since

`exp(1/2)^2 = exp(1/2)*exp(1/2) = exp(1/2 + 1/2) = e`. With some work, you can show that `exp(x) = e^x` for any rational number x, and so it is compatible to define `e^x = exp(x)` for all real numbers x.

Problems 1. The number e can also be shown to be the limit of `(1+1/n)^n` as n gets large. Check this out numerically by writing a maple procedure which takes as input a positive integer n and outputs the decimal value of the above sequence. By experimenting, find the smallest value of n which gives e correct to 5 significant digits. Here is a procedure you can use.

```
> estimate :=proc(n)
>     local i;
>         for i from n by 100 to n+500 do  print(i,evalf((1. +1/i)^i)) od
>     end:
> estimate(100);
```

$$100, 2.704813829$$

$$200, 2.711517123$$

$$300, 2.713764888$$

$$400, 2.714891744$$

$$500, 2.715568521$$

$$600, 2.716020591$$

2. The linear function which "best approximates" the graph of the exponential function `y = exp(x)` at `x = 0` is defined as the tangent line `y = x + 1`. These two functions both have the same value and the same derivative at `x=0`. Now the quadratic function that "best approximates" exp(x) at `x = 0` is defined to be the quadratic function `1 + x + (1/2)x^2`. This function has the same value, derivative, and second derivative as exp(x) at `x = 0`. Plot all three functions `1 + x`, `1 + x + (1/2)x^2`, and exp(x) on the same graph.

```
> plot({ exp(x),1+x,1+x+1/2*x^2 },x=-1..2);
```

3. The functions `y = x^2` and `y = 2^x` cross each other 3 times. Find two of the points of intersection by inspection of the equation `x^2 = 2^x`. Then plot the two functions on the same graph and estimate, from the graph, the coordinates of the third point.

4. The functions `3^x` and `x^3` cross each other 2 times. Find one of the point of intersection by inspection of the equation `3^x = x^3`. Then plot the two functions on the same graph and estimate, from the graph, the coordinates of the second point.

5. Find the value of a between 2 and 3 such that `a^x` and `x^a` cross exactly once. Plot the resulting graphs.

6. Suppose that two cups of soup, the first at 90 C and the second at 100 C are put in a room where the temp is maintained at 20 C. The first cup cooled from 90 C to 60 C after 10 minutes, at which time the second cup is put in a freezer at -5 C. Assume Newton's law of cooling and answer the following questions.

(a) Express the temperature of each cup as a function of time, and draw their graphs.

(b) How much longer will it take the two cups to reach the same temperature?

Solution : Assuming Newton's Law of cooling, we see that the first cup's temperature is given by

```
> restart;
> f := t -> 20 + 70*exp(k*t);
```

$$f := t \to 20 + 70\,e^{(k\,t)}$$

where k is determined by the information given that the first cup cools to 60 C in 10 minutes (i.e. `f(10)=60`).

```
> solve(f(10)=60,k);
```

$$\frac{1}{10}\ln\left(\frac{4}{7}\right)$$

```
> k:=";
```

$$k := \frac{1}{10}\ln\left(\frac{4}{7}\right)$$

Assume that this value of k is also valid for the second cup. (Why shouldn't it be, they are both coffee.) So the second cup's temperature is given by

```
> g := t -> 20 + 80*exp(k*t);
```

$$g := t \to 20 + 80\,e^{(k\,t)}$$

for the first 10 minutes. From that time on, the second cup's temperature is given by

```
> h := t -> -5 + (itemp - (-5))*exp(k*t);
```

$$h := t \to -5 + (\,itemp + 5\,)\,e^{(k\,t)}$$

where itemp is the temperature that the 2nd cup would start at in order for it to cool to g(10) degrees C if it were in the freezer from the start. So itemp satisfies the equation

```
> eq := -5 + (itemp +5)*exp(k*10) = g(10);
```

$$eq := -\frac{15}{7} + \frac{4}{7}\,itemp = \frac{460}{7}$$

Solving for itemp,

```
> itemp := solve(eq,itemp);
```

$$itemp := \frac{475}{4}$$

Now plot the two temperature curves.

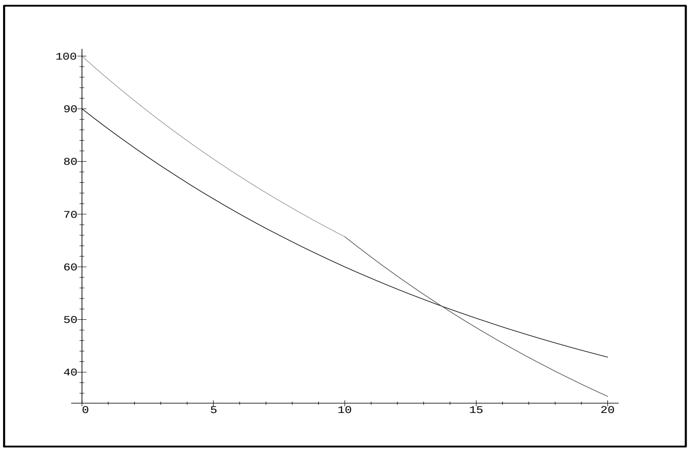
```
> cup1 := [t,f(t),t=0..20];
```

$$cup1 := [\,t, 20 + 70\,e^{(\,1/10\ln(\,4/7\,)\,t\,)}, t = 0..20\,]$$

```
> cup2 := [t,g(t),t=0..10],[t,h(t),t=10..20];
```

$$cup2 := [\,t, 20 + 80\,e^{(\,1/10\ln(\,4/7\,)\,t\,)}, t = 0..10\,],$$
$$\left[t, -5 + \frac{495}{4}\,e^{(\,1/10\ln(\,4/7\,)\,t\,)}, t = 10..20\right]$$

```
> plot( {cup1,cup2} );
```



The cups are at the same temperature before another 5 minutes pass.

```
> etime := solve(f(t)=h(t),t);
```

$$etime := 10\,\frac{\ln\left(\frac{20}{43}\right)}{\ln\left(\frac{4}{7}\right)}$$

At that time the common temperature is

```
> f(etime);
```

$$\frac{2260}{43}$$

5.4 Inverse Functions: The inverse trig functions

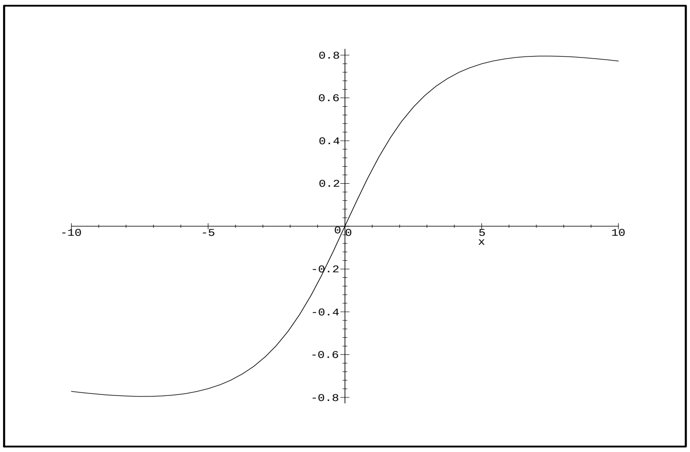
Problems 1. Discover the domain and ranges of the inverse trig functions as defined in Maple. Are they the same as in your text?

2. Find the critical points of `arctan(18/x)-arctan(3/x)`, for positive x. Make a sketch which shows any extreme values.

```
> y := arctan(18/x)-arctan(3/x);
```

$$y := \arctan\left(18\,\frac{1}{x}\right) - \arctan\left(3\,\frac{1}{x}\right)$$

```
> plot(y,x=-10..10);
```




```
> solve(diff(y,x),x);
```

$$-3\sqrt{6}, 3\sqrt{6}$$

There is a maximum at 3*sqrt(6) and a minimum at -3*sqrt(6).

3. Bill, driving a red Mustang at 120 miles per hour, is headed straight toward a railroad crossing in Kansas. When he is a mile from the crossing he notices a train, coming towards the same intersection. He estimates the speed of the train to be 60 mph and the distance of the engine from the intersection to be a mile. The train looks to be about 300 yards long. Keeping his eye on the train during the next minute, he notices that the angle subtended at his eye by the train got larger at first and then began decreasing. How far was Bill from the intersection when the angle was maximum?

Solution :

First, draw a picture. Let's use yards, minutes, and degrees for units. Then Bill's distance from the intersection t minutes from the time Bill first saw the train was `y = 1762 - 2*1762t` yards and the engine's distance was `x = y` yards. The distance of the caboose from the intersection is x + 300 yards. The angle subtended at Bills eye by the train is beta - alpha. Now `tan(beta) = (x+300)/y` and `tan(alpha) = x/y`. So the function to be maximized is `ang := arctan((x+300)/y) - arctan(x/y)`.

```
> restart;
> ang := arctan((x+300)/y)-arctan(x/y);
```

$$ang := \arctan\left(\frac{x+300}{y}\right) - \arctan\left(\frac{x}{y}\right)$$

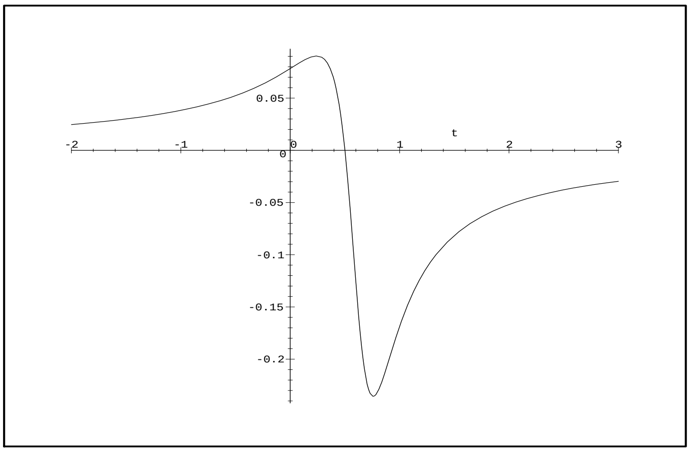
```
> y := 1762*(1-2*t); x:= 1762-1762*t; ang;
```

$$y := 1762 - 3524\,t$$

$$x := 1762 - 1762\,t$$

$$\arctan\left(\frac{2062 - 1762\,t}{1762 - 3524\,t}\right) - \arctan\left(\frac{1762 - 1762\,t}{1762 - 3524\,t}\right)$$

```
> plot(ang,t=-2..3);
```



The plot shows that there is a local maximum viewing angle before Bill gets to the crossing, and an absolute maximum after he crosses. To get the times, distances, and angles do the following:

```
> sol :=solve(diff(ang,t),t) ;
```

$$sol := \frac{1}{2} + \frac{1}{8810}\sqrt{5202305}, \frac{1}{2} - \frac{1}{8810}\sqrt{5202305}$$

```
> evalf([sol]);
```

$$[.7588940067, .2411059933]$$

```
> asol :=evalf([seq(subs(t=sol[i],ang),i=1..2)]);
```

$$asol := [-.2355710897, .0903370978]$$

```
> dsol :=evalf([seq(subs(t=sol[i],[x,y]),i=1..2)]);
```

$$dsol := [[424.8287602, -912.3424796], [1337.171240, 912.3424796]]$$

```
> evalf([seq(convert(asol[i],degrees),i=1..2)]);
```

$$[-13.49722921\, degrees, 5.175934435\, degrees]$$

So Bill's view of the train is locally a maximum of about 5 degrees at about .24 minutes after he sees the train and about 912 yards before he crosses the intersection. His maximum view (about 13.5 degrees) of the train occurs at about .76 minutes and 912 yards beyond the crossing.

4. Sketch the graph of `y = cos(x) + cos(2x)` over two periods. Locate the local extreme values.

5.(a) Find an antiderivative for

$$\int \frac{a}{b + c\,x^2}\,dx$$

(b) Calculate f ” ’(.13) for

$$\mathbf{f}(\ x\ ) = \mathbf{arctan}(\ \mathbf{arcsin}(\ x\ ) + \mathbf{arccos}(\ x\ )\ )$$

6. Two hallways, meeting at right angles, have widths a and b. Find the width of the longest pole which can be slid around the corner.

7. The bottom free corner of a page in a 8 inch by 6 inch book is folded back to the binding and creased so that the crease is as short as possible. What is measure of the angle formed by the crease at the bottom of the page?

5.5 Integration Techniques and Applications

Integration Problems: You can use the student package in maple to practice your integration techniques. First load the student package by typing

```
> with(student):
```

```
Warning: new definition for   D
Warning: new definition for   Int
Warning: new definition for   Sum
```

Then read over the help screens on `changevar` , `intparts` , and `value` , paying particular attention to the examples at the bottom of the screens. Here are some examples.

A Substitution Problem: Find an antiderivative of

$$\frac{1}{\sqrt{1+\sqrt{x}}}$$

```
> F := Int(1/sqrt(1+sqrt(x)),x);
```

$$F := \int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx$$

Let's try the substitution `sqrt(x) = u`.

```
> G := changevar(sqrt(x)=u,F);
```

$$G := \int 2\,\frac{u}{\sqrt{1+u}}\,du$$

This does not seem to help. Lets try `u = 1 + sqrt(x)`

```
> G := changevar(1+sqrt(x)=u,F);
```

$$G := \int 2\,\frac{-1+u}{\sqrt{u}}\,du$$

Now we can do it by inspection, so just finish it off.

```
> G := value(G);
```

$$G := -4\sqrt{u} + \frac{4}{3}\,u^{3/2}$$

Now substitute back and add in the constant.

```
> F := subs(u=sqrt(x),G) + C;
```

$$F := -4\,x^{1/4} + \frac{4}{3}\,x^{3/4} + C$$

An Integration by Parts Problem: Integrate

$$x^2 \arctan(\,x\,)$$

```
> F := Int(x^2*arctan(x),x);
```

$$F := \int x^2 \arctan(\,x\,)\,dx$$

Let's try letting `u = x^2`.

```
> G := intparts(F,x^2);
```

$$G := x^2 \left(x \arctan(x) - \frac{1}{2} \ln(1+x^2) \right) - \int 2x \left(x \arctan(x) - \frac{1}{2} \ln(1+x^2) \right) dx$$

This is a bad choice. Now let `u = arctan(x)`

```
> G := intparts(F,arctan(x));
```

$$G := \frac{1}{3} \arctan(x)\, x^3 - \int \frac{1}{3} \frac{x^3}{1+x^2}\, dx$$

This is much more promising. Split off the integral on the end.

```
> H := op(2,G);
```

$$H := -\int \frac{1}{3} \frac{x^3}{1+x^2}\, dx$$

Now do a partial fractions decomposition of the integrand of H .

```
> H:= Int(convert(integrand(H),parfrac,x),x);
```

$$H := \int \frac{1}{3} x - \frac{x}{3+3x^2}\, dx$$

Now we can do it by hand.

```
> H1 := 1/6*x^2 - 1/3*1/2*ln(1+x^2);
```

$$H1 := \frac{1}{6} x^2 - \frac{1}{6} \ln(1+x^2)$$

Let's check this with the student value.

```
> simplify(value(H-H1));
```

$$-\frac{1}{6} \ln(3)$$

Note the difference of a constant, which is fine for antiderivatives.

A Trig Substitution Problem: Find an antiderivative of

$$\frac{x^3}{\sqrt{1+x^2}}$$

```
> F := Int(x^3/sqrt(x^2+1),x);
```

$$F := \int \frac{x^3}{\sqrt{1+x^2}}\, dx$$

The presence of `x^2 + 1` suggests letting `x = tan(t)` .

```
> G := changevar(x=tan(t),F,t);
```

$$G := \int \tan(t)^3 \sqrt{1+\tan(t)^2}\, dt$$

Now use the trig identity `1 + tan(t)^2 = sec(t)^2` .

```
> G := subs(sqrt(1+tan(t)^2)=sec(t),G);
```

$$G := \int \tan(t)^3 \sec(t)\, dt$$

Another substitution.

```
> G := subs(tan(t)^3 = (sec(t)^2-1)*tan(t),G);
```

$$G := \int \left(\sec(t)^2 - 1\right) \tan(t) \sec(t)\, dt$$

Let's make a change of variable,

```
> H := changevar(sec(t)=u,G);
```

$$H := \int u^2 - 1\, du$$

Now we can do it by hand.

```
> H := value(H);
```

$$H := \frac{1}{3} u^3 - u$$

Let's unwind the substitutions.

```
> G := subs(u=sec(t),H);
```

$$G := \frac{1}{3} \sec(t)^3 - \sec(t)$$

```
> F := subs(t = arctan(x),G);
```

$$F := \frac{1}{3} \sec(\arctan(x))^3 - \sec(\arctan(x))$$

```
> F := subs(sec(arctan(x))=sqrt(1+x^2),F) + C;
```

$$F := \frac{1}{3} (1 + x^2)^{3/2} - \sqrt{1 + x^2} + C$$

Checking this calculation:

```
> F1 := int(x^3/sqrt(x^2+1),x);
```

$$F1 := x^2 \sqrt{1 + x^2} - \frac{2}{3} (1 + x^2)^{3/2}$$

It looks different, but is it?

```
> simplify(F-F1);
```

$$C$$

Yes, but only by a constant.

A Partial Fractions Problem: Integrate the rational function

$$\frac{4x^2 + x - 1}{x^2 (x - 1)(1 + x^2)}$$

```
> y :=(4*x^2+x -1 )/(x^2*(x-1)*(x^2+1));
```

$$y := \frac{4x^2 + x - 1}{x^2 (x - 1)(1 + x^2)}$$

First get the partial fractions decomposition of y.

```
> y := convert(y,parfrac,x);
```

$$y := \frac{1}{x^2} + 2\frac{1}{x - 1} - \frac{3 + 2x}{1 + x^2}$$

We can almost do this by hand.

```
> F := Int(y,x);
```

$$F := \int \frac{1}{x^2} + 2\frac{1}{x - 1} - \frac{3 + 2x}{1 + x^2}\, dx$$

```
> F := expand(F);
```

$$F := \int \frac{1}{x^2}\,dx + 2\int \frac{1}{x-1}\,dx - 3\int \frac{1}{1+x^2}\,dx - 2\int \frac{x}{1+x^2}\,dx$$

Now we can do each one by hand. So we'll just use value .

```
> F := value(F) + C;
```

$$F := -\frac{1}{x} + 2\ln(x-1) - 3\arctan(x) - \ln(1+x^2) + C$$

Problems Use the student package to perform the following integrations. (a)

$$\int \frac{x}{x^4+1}\,dx$$

(b)

$$\int \frac{\cos(x)}{\sqrt{1+\sin(x)}}\,dx$$

(c)

$$\int \frac{3x-7}{(x-1)^2(x-2)^3}\,dx$$

(d)

$$\int x^2 \sin(a\,x)\,dx$$

(e)

$$\int \ln\left(x+\sqrt{x}\right)\,dx$$

(f)

$$\int \frac{z^5}{\sqrt{z^2+1}}\,dz$$

(g)

$$\int \frac{1}{e^{(3x)}-1}\,dx$$

(h)

$$\int x \arcsin(2x)\,dx$$

2. Find the area of the region enclosed by the x-axis and the curve `y = x sin(x)` on the interval [0 , Pi]. Sketch the region. Then find the vertical line `x = a` that divides the region in half and plot it.

3. Find the length of the graph of the parabola `y = x^2` from O(0,0) to P(10,100). Find the point `Q(a,a^2)` on the graph which is 10 units from O along the graph. Make a sketch, showing the points O, P, and Q on the graph.

4. Find the volume of the solid of revolution obtained by revolving the region trapped between the the graph of `y = exp(x)*sin(x)` on [0, n*Pi] and the x-axis about the x-axis. Sketch a graph. Does this volume approach a finite limit as n gets large?

5.6 Numerical Integration

Trapezoid and Simpson's Rules If you have a definite integral that you need to evaluate to a certain accuracy, you can avoid finding an antiderivative and using the fundamental theorem by using a numerical integration formula or "rule", such as the trapezoid rule or Simpson's rule. The student package contains words for these two rules. And you may have already discovered on your own that when `evalf(int(f,x=a..b));` is entered, a numerical method is used to calculate an approximation. Here is a maple procedure which estimates the value of the integral using a sequence of trapezoid rules with n, 2n, 3n, 4n, 5n subdivisions.

```
> restart;
> trap := proc(f,a,b,n)
>    local i,k,d;
>        for k to 5 do
>            d := (b-a)/k/n;
>            print(k*n,evalf(1/2*d*sum(f(a+(i-1)*d)+f(a+i*d),i = 1 .. k*n)))
>        od
>    end:
```

Problem . Use trap to estimate

$$\int_0^1 e^{(x^2)}\,dx$$

with 10 through 50 subdivisions, we could enter the following:

```
>    trap(x->exp(x^2),0,1,10);
```

$$10, 1.467174693$$

$$20, 1.463783893$$

$$30, 1.463155037$$

$$40, 1.462934871$$

$$50, 1.462832952$$

By looking at this sequence, we would feel fairly confident, (but not absolutely certain) that the value of the integral is about 1.46.

Problems 1. Use trap to find a value for n so that if

$$\int_0^1 e^{(x^2)}\,dx$$

is estimated with the trapezoid rule with n subdivisions, so that the estimate is correct when rounded to 5 significant digits. (You can use

```
> evalf(int(exp(x^2),x=0..1));
```

$$1.462651746$$

to get the correct value of the integral.)

2. We can modify trap to get a procedure for Simpson's rule. Just change its name to something more appropriate (like simp?) and replace the the print(...); term in trap to the following:

```
> print(k*n,evalf(1/3*d*(f(a) + f(b) + 4*sum(f(a+(2*i-1)*d),i=1..(k*n/2)) + 2*sum(f(a+(2*i)
> *d),i=1..(k*n/2-1)) )  ));
```

$$k\,n, .3333333333 d \left(\mathrm{f}(a) + \mathrm{f}(b) + 4. \left(\sum_{i=1}^{1/2\,k\,n} \mathrm{f}(a + (2\,i - 1)\,d) \right) + 2. \left(\sum_{i=1}^{1/2\,k\,n-1} \mathrm{f}(a + 2\,i\,d) \right) \right)$$

Write a procedure for Simpson's rule and use it to find and to estimate the same integral as in problem 1. Call the `simp` procedure with this command line:

```
> simp(x->exp(x^2),0,1,10);
```

3. Use Simpson's rule with `n = 2` to estimate

$$\int_1^5 x^3\,dx$$

How does it compare with the exact answer? Explain. What does this say about the accuracy of Simpson's rule on any cubic polynomial?

4. Write a version of simp which prints out a sequence of approximations to the integral of f from a to b, stopping when two successive terms of the sequence are equal.

More on Trapezoid and Simpson The error bounds on the trapezoid and Simpson rules are given in most calculus texts. They are

$$\mathit{Trap\ error}: \quad |, \int_a^b \mathrm{f}(x)\,dx - Tn, | \;<\; , \frac{1}{12} M\,(b-a)\,h^2$$

where M is an upper bound on abs(f ´´) over the interval [a,b] and

$$\mathit{Simpson\ error}: \quad |, \int_a^b \mathrm{f}(x)\,dx - Sn, | \;<\; , \frac{1}{180} K\,(b-a)\,h^4$$

where K is an upper bound on absf ´´´ over the interval [a,b]. Let's check to see how good they are for a couple of cases. Here is an example.

Problem : Use the error bound on Simpson's rule above to estimate how many subdivisions are needed to get the error on ln(6) to within `1*10^(-6)` using Simpsons rule.

Solution :

```
> f := x -> 1/x;      # define the function
```

$$f := x \to \frac{1}{x}$$

```
> f4 := diff(f(x),x,x,x,x);   # and its 4th derivative
```

$$\mathit{f4} := 24\,\frac{1}{x^5}$$

```
> a := 1; b := 6;     # set the limits of integration
```

$$a := 1$$

$$b := 6$$

```
> K := subs(x=a,f4);
```

$$K := 24$$

By putting the smallest value in the interval [1,6] (namely `a = 1`) into f4 we are maximizing the numerator, therefore this process yields an upper bound on K.

```
> h := (b-a)/n;          #  calculate h
```

$$h := 5\,\frac{1}{n}$$

```
> sbound := K*(b-a)/180*h^4;   # calculate the bound
```

$$sbound := \frac{1250}{3}\,\frac{1}{n^4}$$

```
> fsolve(sbound=10^(-6),n,0..infinity);   # solve for n
```

$$142.8720215$$

Thus 143 subdivisions are predicted to suffice for the desired accuracy. Actually, 144 since Simpson requires an even number of subdivisions.

Problems 1. Is the above prediction correct? Can you get a result with fewer subdivisions that still has the desired accuracy? Check the prediction with the simp procedure, or the Simpson rule in the student package.

2. Work out the predicted number of subdivisions needed to assure the desired accuracy for the trapezoid rule. Check your prediction by experimenting (using simp) with fewer subdivisions.

3. Find the number of subdivisions needed to get 3 decimal places of accuracy estimating the following integral

$$\int_0^1 \sqrt{x}\,dx$$

using your simp.

4. If we make the change of variable sqrt(x) = u , then we get

```
> restart;
> with(student):
> Int(sqrt(x),x=0..1)  =  changevar(sqrt(x)=u,Int(sqrt(x),x=0..1));
```

$$\int_0^1 \sqrt{x}\,dx = \int_0^1 2\,u^2\,du$$

How many subdivisions do you need to estimate the new integral? Can you account for this? This example shows that the method of substitution is useful in numerical integration as well as symbolic integration.

4. Use Simpsons rule to construct a procedure F which returns the values of

```
> `F(x)=`,Int(sin(t)/t,t=0..x);
```

$$F(x) =, \int_0^x \frac{\sin(t)}{t}\,dt$$

correct to 4 decimal places in the interval [0, Pi].

Taylor's Theorem

Taylor polynomials œ Suppose the nth derivative of y = f(x) is defined at x = a. Then you can form the nth Taylor polynomial for f at x = a with the following:

```
> p[n] :=  x -> sum((D@@i)(f)(a)/i!*(x-a)^i,i=0..n);
```

$$p_n := x \to \sum_{i=0}^{n} \frac{D^{(i)}(f)(a)\,(x-a)^i}{i!}$$

If the (n+1)st derivative of f is defined on the interval [a,x], we have seen how to get an estimation for the error, namely,

```
> R[n](x) = abs(f(x) - p[n]) , ` < ` , M/(n+1)!* (x-a)^(n+1);
```

$$R_n(x) = \left|\frac{1}{x} - p_n\right|, <, \frac{M\,(x-1)^{(n+1)}}{(n+1)!}$$

where M is an upper bound of the (n+1)st derivative of f on [a,b]. Here are some sample calculations with Taylor polynomials.

Problem

Find the 2nd Taylor polynomial p[2] of f(x) = ln(x)*arctan(exp(x)) at x = 1. Plot both the polynomial and f on the interval [1,1.5]. Determine the maximum error in using p[2] to approximate ln(x) in this interval. **Solution** :

```
> f := x -> ln(x)*arctan(exp(x));
```

$$f := x \to \ln(x)\arctan(e^x)$$

```
> p[2]  := x -> sum((D@@i)(f)(1.)/i!*(x-1.)^i,i=0..2);
```

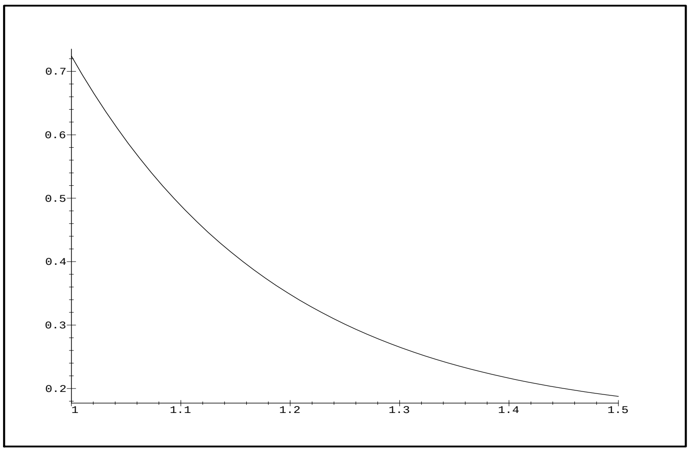
$$p_2 := x \to \sum_{i=0}^{2} \frac{D^{(i)}(f)(1.)\,(x-1.)^i}{i!}$$

```
> p[2](x);
```

$$1.218282905\,x - 1.218282905 - .2851143156\,(x-1.)^2$$

We can estimate the error by plotting the third derivative over the interval.

```
> est := plot((D@@3)(f),1..1.5):  est;
```



So M = (D@@3)(f)(1.) is good.

```
> M := (D@@3)(f)(1.);
```

$$M := .724152878$$

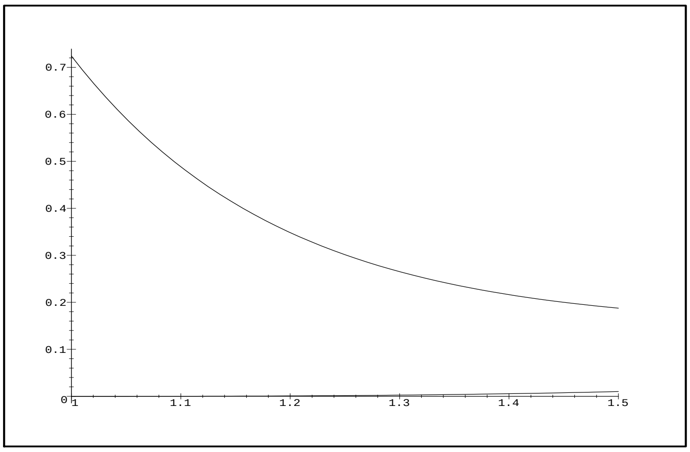
So the remainder R2 = — f(x) - p2(x) — is bounded by

```
> M/3!*(1.5-1)^3;
```

$$.01508651830$$

Let's check this by plotting the actual error and the estimate –

```
> plots[display]([est,plot((f-p[2])(x),x=1..1.5)]);
```



By inspection, the maximum error occurs at x = 1.5. The error at x = 1.5 is

```
> (f-p[2])(1.5);
```

$$.0100268555$$

Our error was well under the predicted error.

Problems 1. How good is the 4th Taylor polynomial p[4] of

$$f(x) =, \ln(x)\arctan(e^x)$$

at x = 1 for approximating f(x) on the interval [1,1.5]? In particular, is it twice as good as the p[2], which we constructed above?

2.(a) Construct the first, third, and fifth Taylor polynomials p[1], p[3], and p[5] of `f(x) = sin(x)` about `x = 0`.

(b) Plot all four functions f, p[1], p[3], p[5] on the same graph over the interval [0, 2*Pi]. Estimate the maximum error `abs(f(x)-p[5](x)` over the interval [0, 2*Pi] using your graph.

(c) Estimate the error in using p[5](x) to approximate sin(x) for x in the interval [0, Pi/4] by getting a bound M on the 6th derivative of the sin(x) and using Taylor's formula with remainder.

(d) Write a procedure to estimate sin(x) for any x by using p[5]. Outline: If x is negative, replace x with -x and use the oddness `sin(x) = -sin(-x)` property. If x is greater than or equal to 2*Pi, then replace x with x-2*Pi and use the periodicity `sin(x) = sin(x-2*Pi)`. Repeat this step until [0, 2*Pi). If `Pi/4 < x < Pi/2`, then use

$$\sin(x) = 2\sqrt{1 - \sin\left(\frac{1}{2}x\right)^2}\,\sin\left(\frac{1}{2}x\right)$$

and approximate sin(x/2) by p[5](x). If `Pi/2 < x < Pi`, then `sin(x) = sin(Pi-x)`. If `Pi < x < 2*Pi`, then `sin(x) = - sin(x - Pi)`.

3. Find the smallest n such that the nth Taylor polynomial p[n](x) for `y = exp(x)` at `x = 0` approximates exp(x) to within `10^(-12)` for x in [0,1]. (You will want to set

```
> Digits := 15;
```

or so to do this problem.)

4. Suppose `f '(x) = 1/(x^2 + 1)`, and `f(0) = 0`. Find p[6](1). What is f(1)? How do they compare?

5.7 Sequences and Series

Periodic Points of functions. Let f be a function, and let a be a point in the domain of f. If n is a positive integer such that

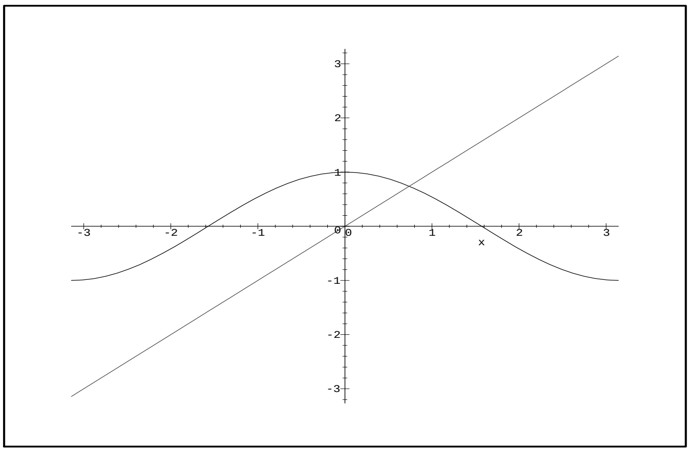
$$(f^{-n})(a) = a$$

but

$$(f^{-i})(a) \neq a$$

for all integers `0 < i < n`, then a is called a periodic point of period n for f. Period one points are called fixed points. You can locate the fixed points of a function by looking to see where the graphs of `y = f(x)` and `y = x` cross. For example, the cosine function has one fixed point, as we can see.

```
> plot({cos(x),x},x=-Pi..Pi);
```



To find the fixed point more precisely, use `fsolve` .

```
> fix := fsolve(cos(x)=x,x,0..Pi);
```

$$\mathit{fix} := .7390851332$$

An attracting fixed point is a fixed point a with the property that for points b close to a,

```
> limit(b[n],n=infinity) = a;
```

$$\lim_{n \to \infty} b_n = a$$

where `b1 = b`, and `bn = f(bn-1)` for `n = 2, 3, ... .`

A repelling fixed point is a fixed point a with the property that for points b close (but not equal) to a,

```
> limit(b[n],n=infinity) <> a;
```

$$\lim_{n \to \infty} b_n \neq a$$

where `b1 = b`, and `bn = f(bn-1)` for `n = 2, 3, ....`

Here is a simple procedure you can use to investigate periodic points and fixed points of a function.

```
> iterate := proc(f,n,x)
>     local a,i,s;
>         a := evalf(x); s := a; for i to n do  a := f(a); s := s,a od
>     end:
```

For example, to investigate whether the fixed point of the cosine function is attracting or not, we can iterate the function at a point near the fixed point.

```
> iterate(cos,10,fix-1),fix;
```

$$-.2609148668, .9661543793, .5684675409, .8427269503, .6654297419, .7866515363, .7062199575, .7608204115, .7242705678, .7489829351, .7323817612, .7390851332$$

The fixed point seems to be an attracting one.

Problems 1. Find the fixed points of `y = cx(1-x)`, for `c= 2,3,4`. Which of them are attracting fixed points? Which of them are repelling?

2. Find the periodic points of period 2, 3, and 4 of `y = cx(1-x)`, for `c= 2,3,4`. (Hint: the period points of order two of f would be the fixed points of `f@@2` which are not fixed points of f.

3. Let a1, a2, a3, ..., an, ... be a sequence of positive numbers converging to 0. Imagine yourself starting at the origin and traveling east a1 miles, then turning north and going a2 miles, then west a3 miles, and so forth, cycling through the directions as you go through the sequence an. (1) How far do you travel? (2) Where do you end up?

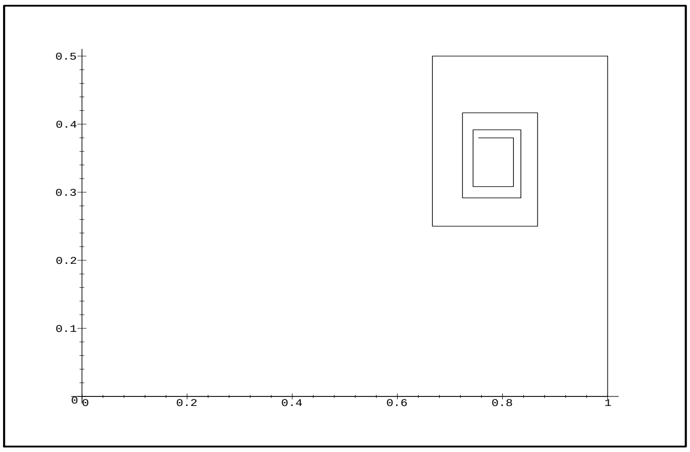
Work the answers out for the sequences 1/n and `1/2^n`. Approximations are acceptable.

Suppose we wanted to draw the paths described in the problem. Here is a procedure which will do that. Use it to draw the paths for the sequences `an = 1/n`, and `an = 1/2^n`.

```
> cycle := proc(a,m)    # name the procedure
> local path,i, dir;      # declare local variables.
> path := 0,0;             # start the path.
> dir := 1,0;                # first go east, then north 0,1, etc
> for i from 1 to m do  # construct the path
>   path := path,path[2*i-1]+dir[1]*a(i), path[2*i]+dir[2]*a(i);
>   dir := -dir[2],dir[1];  #rotate the direction 90 degrees.
> od;
> plot([path],style=LINE);   # draw the path.
> end;
```

```
cycle :=
    proc(a,m)
    local path,i,dir;
        path := 0,0;
        dir := 1,0;
        for i to m do
            path := path,path[2*i-1]+dir[1]*a(i),path[2*i]+dir[2]*a(i);
            dir := -dir[2],dir[1]
        od;
        plot([path],style = LINE)
    end
```

```
> cycle(n->1/n,15);
```



5.8 Drawing procedures

Most of the words in the Maple vocabulary are computational words, that is, the output is a number or an expression which has been computed by the execution of the word. Some words, such as leftbox in the student package, are what we will call drawing words, that is, the output is a picture. Drawing words are a natural thing to define during the analysis of a problem in order to 'get an idea' of what is going on. After the problem is solved, 'draw a picture' of the solution. Sometimes this can be as challenging as solving the original problem.

The Snowflake Curve The Snowflake Curve was initially described by Koch as an affirmative solution to the problem of whether there is a continuous curve that has no tangent line at any point on the curve.

It is defined as the limit of the sequence of curves generated by the procedure snowflake given below. Type in the words defined below and then enter snow(4) to get a feel for what the snowflake curve looks like. The first word performs a basic operation on any segment, which we think of as a list of its endpoints. It replaces the segment of length d with a sequence of four segments of length d/3 obtained in the following way: Start at the left endpoint of the segment and go d/3 of the way along the segment, turn left 60 degrees and go the same distance, turn right 60 degrees and go the same distance, then turn left 60 degrees and proceed d/3 units along the original segment to the right endpoint. Note that the resulting path has three points where there is no tangent.

```
> basic := proc(p1,p2)
> local dx,dy, p3,p4,p5;
> dx := (p2[1]-p1[1])/3.;
> dy := (p2[2]-p1[2])/3.;
> p3 := p1[1]+dx,p1[2]+dy;
> p4 := p1[1]+2*dx,p1[2]+2*dy;
> p5 := p1[1]+1.5*dx-sqrt(3.)/2*dy,p1[2]+1.5*dy+sqrt(3.)/2.*dx;
> p3,p5,p4,op(p2);
> end;
```

```
basic :=
proc(p1,p2)
local dx,dy,p3,p4,p5;
    dx := .3333333333*p2[1]-.3333333333*p1[1];
    dy := .3333333333*p2[2]-.3333333333*p1[2];
    p3 := p1[1]+dx,p1[2]+dy;
    p4 := p1[1]+2*dx,p1[2]+2*dy;
    p5 :=
       p1[1]+1.5*dx-1/2*sqrt(3. )*dy,p1[2]+1.5*dy+.5000000000*sqrt(3. )*dx
       ;
    p3,p5,p4,op(p2)
end
```

So for example, if we feed the points [0,0], [1,0] into basic, we get out the following sequence of numbers:

```
> basic([0,0],[1,0]);
```

$$.3333333333, 0, .5000000000, .2886751347, .6666666666, 0, 1, 0$$

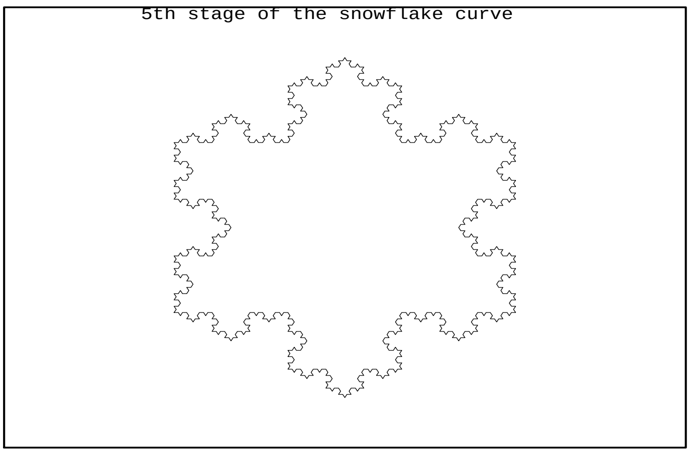
Note the left endpoint of the original segment is missing from this list. That is for programming reasons that become clear in the definition of the word flake which uses basic.

The word flake takes a list of points, which represents a sequence of line segments laid end to end, and returns a list representing 4 times as many line segments, where each segment in the original list has been replaced by the 4 segments returned by basic.

```
> flake := proc(fl)
>     local i,curve;
>         curve := [fl[1],fl[2]];
>         for i by 2 to nops(fl)-2 do
>             curve := [op(curve),basic([fl[i],fl[i+1]],[fl[i+2],fl[i+3]])]
>         od
>     end:
```

Now we can define the nth stage of the snowflake curve.

```
> snowflake := proc(n)
>  local i,curve,ti;
>      curve := [0,0,1/2,1/2*sqrt(3),1,0,0,0];
>      for i from 2 to n do  curve := flake(curve) od;
>      ti := cat(n,`th stage of the snowflake curve`);
>      plot(curve,style = LINE,axes = NONE,scaling = CONSTRAINED,title = ti)
>  end:
> snowflake(5);
```



Problem : What is the length of the snowflake curve?

Solution: To calculate the length of the snowflake curve we see that the first stage has length s1 = 3 units, the 2nd stage has length

`s2 = s1 + 1/3 s1 = 4/3 s1`, the 3rd stage has length

`s3 = s2 + 1/3 s2 = 4/3 s2 = s1 (4/3)^2`, in general the nth stage has length

`sn = s(n-1) + 1/3 s(n-1) = 4/3 s(n-1) = s1 (4/3)^(n-1)`. So the length of the nth stage goes to infinity as n gets large. Since the nth stage of the curve is a polygonal approximation of it, this says the snowflake curve is infinitely long.

Finding the sum of a series Use the help screens for the Maple words sum and limit to see how to sum series.

Suppose you have established somehow, either directly or by some test, that a series

$$\sum_{n=0}^{\infty} a_n$$

converges, that is, the sequence of partial sums of the series is known to converge to a number. How do we calculate this number to any specified accuracy? If the series converges fast enough you can look at the sequence of partial sums and get the desired accuracy. Here is the Maple word psums which you can use to get an approximate value for a series you know converges. For example, after you have this word entered into the vocabulary, a call to `psums(1/n^3,n,100,10)` will generate a sequence of ten partial sums of the 3-series starting with the 100th one.

```
> psums := proc(term,var,start,inc)
> local n,s,i,m;
> s := 0.;
> for i from 1 to start do
>      s := evalf(subs(var=i,term)) + s;
> od;
> print(start,s);
> for n from start by inc to start+10*inc do
>     for m from 1 to inc do
>          s := s + evalf(subs(var=n+m,term));
>     od;
>     print(n,s);
> od;
> end;
> psums := proc(term,var,start,inc)
>          local n,s,i,m;
>              s := 0;
>              for i to start do  s := evalf(subs(var = i,term))+s od;
>              print(start,s);
>              for n from start by inc to start+10*inc do
>                  for m to inc do  s := s+evalf(subs(var = n+m,term)) od;
>                  print(n,s)
>              od
>          end:
> psums(1/n,n,100,10);
```

$$100, 5.187377520$$

$$100, 5.282234601$$

$$110, 5.368868290$$

$$120, 5.448591341$$

$$130, 5.522425268$$

$$140, 5.591180593$$

$$150, 5.655511229$$

$$160, 5.715952399$$

$$170, 5.772947726$$

$$180, 5.826869012$$

$$190, 5.878030952$$

$$200, 5.926702262$$

Hmmm. You can't really tell by looking at these that the harmonic series doesn't converge.

Problems In each of the problems below, determine whether the series converges or diverges. Give a reason in each case. For the convergent series, get an estimate correct to 2 decimal places of the sum of the series using psums or some other word of your own devising. You can check with sum to see if Maple can sum it.

1.

```
> Sum( 1/(3+2*n)^2,n=1..infinity);
```

$$\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$$

2.

```
> Sum(1/(n*(ln (n))^2 ),n=2..infinity);
```

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^2}$$

3.

```
> Sum(1/(2*n+1)^(1/3),n=0..infinity);
```

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{1/3}}$$

4.

```
> Sum((1+2^n)/(1+3^n),n=0..infinity);
```

$$\sum_{n=0}^{\infty} \frac{1+2^n}{1+3^n}$$

5.

```
> Sum(( n^5+4*n^3+1)/(2*n^9+n^4+2 ),n=0..infinity);
```

$$\sum_{n=0}^{\infty} \frac{n^5 + 4\,n^3 + 1}{2\,n^9 + n^4 + 2}$$

6.

```
> Sum(sin (1/n^4 ),n=1..infinity);
```

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^4}\right)$$

7.

```
> Sum( n*e^(-n^2),n=1..infinity);
```

$$\sum_{n=1}^{\infty} n\, e^{(-n^2)}$$

A Spacefiller David Hilbert in 1891 described a continuous curve whose range is the unit square! This was contrary to the intuition of the time, which was that the range of a path had to be 1-dimensional. Here is a Maple word that draws an approximation to Hilbert's space filling curve . The approach to defining the drawing procedure peano (below) is similar to that used to define snowflake (above).

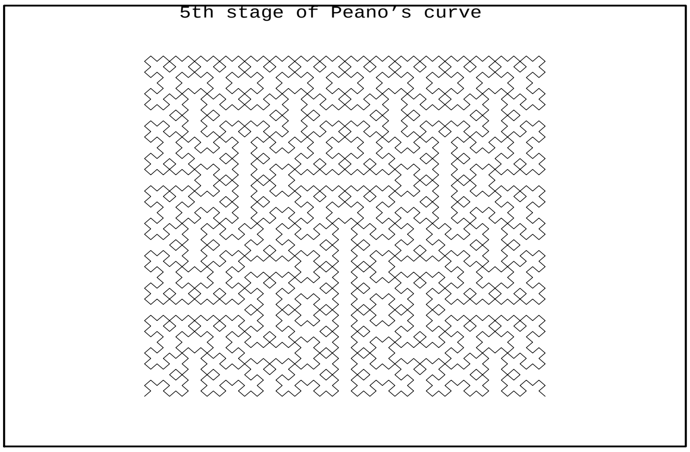
```
> restart;
> basic := proc(p1,p2,p3)
> local dx,dy,p4,p5,p6,p7,p8,p9;
>      p4 := (p1[1]+p2[1])/2.,(p1[2]+p2[2])/2.;
>      p9 := (p2[1]+p3[1])/2.,(p2[2]+p3[2])/2.;
>      p5 := p4[1]+(p2[1]-p9[1]),p4[2]+(p2[2]-p9[2]);
>      p6 := p2[1]+(p2[1]-p9[1]),p2[2]+(p2[2]-p9[2]);
>      p7 := p2[1]+(p4[1]-p1[1]),p2[2]+(p4[2]-p1[2]);
>      p8 := p9[1]+(p4[1]-p1[1]),p9[2]+(p4[2]-p1[2]);
>      p4,p5,p6,op(p2),p7,p8,p9,op(p3);
> end:
> peano := proc(fl)
> local i,cur;
> cur := [fl[1],fl[2]];
> for i from 1 by 4 to nops(fl)-5 do
>     cur := [op(cur),basic([fl[i],fl[i+1]],[fl[i+2],fl[i+3]],[fl[i+4],fl[i+5]])];
> od;
> end:
> fl := [0,0,1,1,2,0];
```

$$fl := [\,0, 0, 1, 1, 2, 0\,]$$

```
> for i from 1 to 6 do fl := peano(fl) od:
> ti := cat(6,`th stage of Peano´s curve`);
```

$$ti := \textit{6th stage of Peano's curve}$$

```
> plot(fl,style=LINE,axes=NONE,title=ti,scaling=CONSTRAINED);
```



A Power Series Problem For the power series

```
> Sum(n*x^n,n=0..infinity);
```

$$\sum_{n=0}^{\infty} n\, x^n$$

what is the radius of convergence? Hint: Use the ratio test. Check your hand calculations by plotting some of the partial sums of the series.

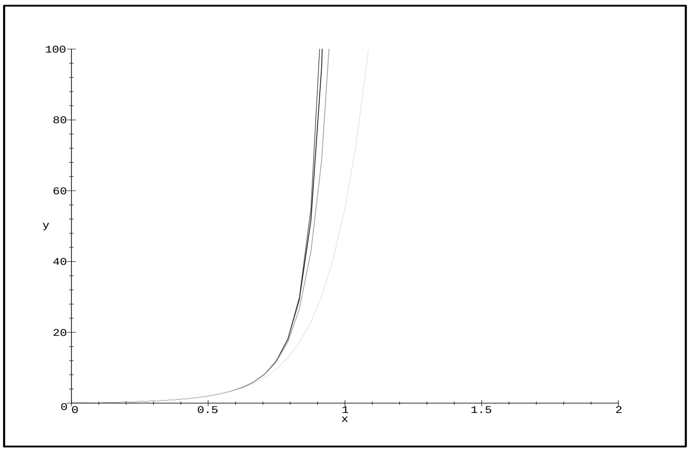
```
> cat(`radius of convergence is `,limit((n+1)/n,n=infinity));
```

radius of convergence is 1

```
> i:=´i´: s := n -> sum(i*x^i,i=0..n);
```

$$s := n \to \sum_{i=0}^{n} i\, x^i$$

```
> plot({s(10),s(20),s(30),s(40)},x=0..2,y=0..100);
```



Perform the same task with the power series expansion of

```
> Sum(n/2^n*x^n,n=0..infinity);
```

$$\sum_{n=0}^{\infty} \frac{n\,x^n}{2^n}$$

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