Contents

CHAPTER 1

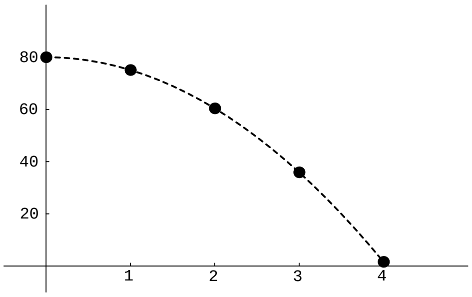
Straight Lines, Polygonal Lines, Slopes, Distance, Circles

Much of the mathematics in this chapter will be a review for you.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. We normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of $45°$ with the x-axis (and also with the y-axis).

But in applications, more descriptive variable names are used, and often different scales are chosen along the two axes. In fact, the quantities represented by the two variables can be so different that the same scale has no meaning! For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter s denote the height in meters. For each t (say, at one-second intervals) you have a corresponding height s. This information can be tabulated, and then plotted on the (t, s) coordinate plane. Here is an example:

t (sec)	s (meters)
0	80.0
1	75.1
2	60.4
3	35.9
4	1.6



We use the word " quadrant " for each of the four regions the plane is divided into: the 1st quadrant is where points have both coordinates positive, and the 2nd, 3rd, and 4th quadrants are counted off counterclockwise as follows:

2nd	1st
3rd	4th

Suppose we have two points $A(2, 1)$ and $B(3, 3)$ in the (x, y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ is "change in" (thus, Δx can be read as "change in x" — it denotes a single number, and should not be read as "delta times x"). In our example, $\Delta x = 3 - 2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3 - 1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) is:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that "the change" is algebraic. For example the values of Δx and Δy in going from B to A will be respectively -1 and -2. You should follow the above formula all the time.

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points A and B in the last paragraph has slope 2.

Note that this slope does not depend on which two points on the line are taken and the order in which they are taken. For instance, if we take the change from B to A, then we will get the ratio of $(-2)/(-1)$ which still evaluates to 2.

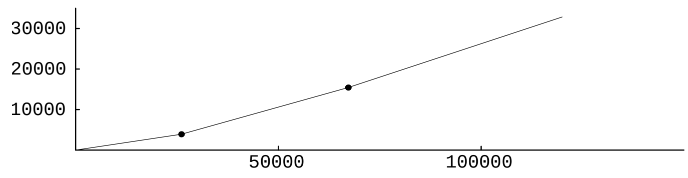
There is a very simple procedure to find all points on the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$. Any point $P(x, y)$ on the line has coordinates

$$x = (1-t)x_1 + tx_2\ ,\ y = (1-t)y_1 + ty_2 \text{ where } t \text{ is any real number.}$$

Convince yourself that $t = 0$ gives the point A, $t = 1$ gives the point B and values of t between $0, 1$ give the line segment from A to B. You should investigate how the different values of t correspond to various subdivisions of the segment AB. Also verify once and for all that the slope is the same if we choose the points A, B or A, P, regardless of the value of t.

Example 1.1. According to the 1990 U.S. federal income tax schedules, a head of household pays 15% on income up to \$26050. If the taxable income is between \$26050 and \$134930, then, in addition, 28% must be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentage, when converted to a decimal (0.15, 0.28, and 0.33), is the *slope* of the straight line which is the graph of tax for the corresponding tax bracket. The tax graph is what's called a *polygonal line* , i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again, and continues with slope 0.33. See the graph below.



The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the *y-intercept* , because it is y-coordinate where the line crosses the y-axis.

If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of the point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the " point-slope " form of the equation of a straight line, which is: $(y - y_1) = m(x - x_1)$. This relation says that "between the point (x_1, y_1) and any other point (x, y) on the line, the change in y divided by the change in x is the slope m of the line."

In general the equation of a line can be expressed as a linear relation of the form $lx + my + n = 0$ where at least one of the coefficients l, m are nonzero. You should convince yourself that every possible line has such an equation and conversely such an equation always gives a line for its graph.

A convenient way of writing the final equation of a line thru two points (x_1, y_1) and (x_2, y_2) is:

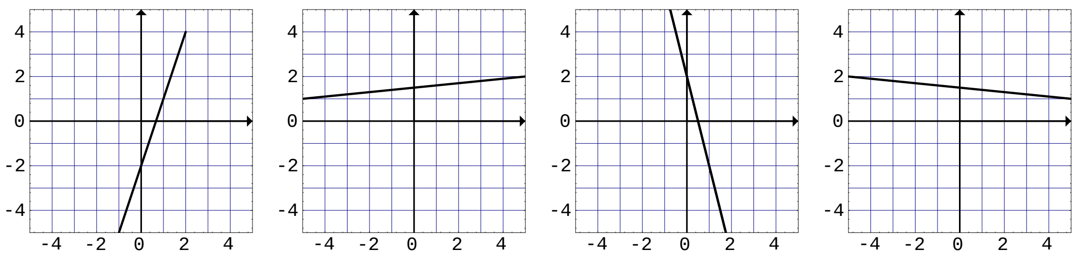
$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$$

You should verify that this is correct by checking that the equation is satisfied by both the points and that it has the correct form for the equation of a line. This is enough justification!

For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 3)$, we can use this formula:

$$\frac{y-1}{3-2} = \frac{x-2}{3-1} \qquad \text{so that} \qquad y - 1 = 2(x - 2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline; while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in the graphs below.



$m = 3$ $\qquad$ $m = 0.1$ $\qquad$ $m = -4$ $\qquad$ $m = -0.1$

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, a vertical line. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A(2, 1)$ and $B(3, 3)$ has x-intercept 1.5. Convince yourself that a vertical line like $x = 5$ has x-intercept 5 and has no y-intercept. The special vertical line $x = 0$ coincides with the y-axis and may be said to have infinitely many y-intercepts.

EXAMPLE 1.2. Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying 110 miles to Seattle; and after driving another half-hour you pass a sign saying 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

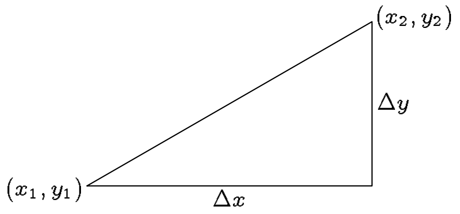
The graph of y versus t is a straight line, because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$. So its slope is $m = (85 - 110)/(1.5 - 1) = -50$. The meaning of the slope is that you are traveling at $|-50| = 50$ mph. The slope m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y - 110}{t - 1} = -50, \qquad \text{so that} \qquad y - 110 = -50(t - 1) \qquad \text{so that } y - 50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t + 160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is: the time when you'll be in Seattle. After traveling 3 hrs 12 min, your distance y from Seattle will be 0.

1. Distance Between Two Points; Circles

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$, and their vertical distance from one another is $\Delta y = y_2 - y_1$. The actual distance from one point to the other is the hypotenuse of a right triangle with legs Δx and Δy:



The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For example, the distance between our two points $A(2, 1)$ and $B(3, 3)$ is equal to $\sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5} = 2.236 \cdots$.

As a special case of the distance formula, suppose we want to know the distance of a point (x, y) to the origin. According to the distance formula, this is equal to $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.

A point (x, y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. This is the **equation of the circle of radius r centered at the origin**. The special case $r = 1$ is called the **unit circle** ; its equation is $x^2 + y^2 = 1$.

Similarly, if $C(h,k)$ is any fixed point, then a point (x,y) is at a distance r from the point C if and only if $\sqrt{(x-h)^2+(y-k)^2}=r$, i.e., if and only if

$$(x-h)^2+(y-k)^2=r^2.$$

This is the **equation of the circle of radius r centered at the point** (h,k). For example, the circle of radius 5 centered at the point $(0,5)$ on the y-axis has equation $(x-0)^2+(y-5)^2=25$. If we expand $(y-5)^2=y^2-10y+25$ and cancel the 25 on both sides, we can rewrite this as: $x^2+y^2-10y=0$.

It is easy to recognize the equation of a circle. Its terms of degree 2 can be always arranged to be x^2+y^2 after dividing the whole equation by a suitable number if needed. For example you should convince yourself that the equation $2x^2+2y^2+4x+8y=5$ describes a circle with center $(-1,-2)$ and radius $\sqrt{15/2}$. Hint: expand the model equation given above and compare coefficients after dividing the given equation by 2.

HOMEWORK 1.

1. For each pair of points $A(x_1, y_1)$ and $B(x_2, y_2)$ find (i) Δx and Δy in going from A to B, (ii) the slope of the line joining A and B, (iii) the equation of the line joining A and B in the form $y = mx + b$, and (iv) the distance from A to B:

(a) $A(2,0)$, $B(4,3)$; (b) $A(1,-1)$, $B(0,2)$;
(c) $A(0,0)$, $B(-2,-2)$; (d) $A(-2,3)$, $B(4,3)$;
(e) $A(-3,-2)$, $B(2,3)$; (f) $A(0.01,-0.01)$, $B(-0.01,0.05)$.

2. Graph each of the following lines, after changing to the form $y = mx + b$, and also find the y-intercept and x-intercept:

(a) $y - 2x = 2$; (b) $x + y = 6$; (c) $x = 2y - 1$; (d) $3 = 2y$; (e) $2x + 3y + 6 = 0$.

3. Find the equation of the circle of radius 3 centered at: (a) $(0,0)$; (b) $(5,6)$; (c) $(-5,-6)$; (d) $(0,3)$; (e) $(0,-3)$; (f) $(3,0)$.

4. Graph the circles

(a) $x^2 + y^2 + 10y = 0$; (b) $x^2 - 10x + y^2 = 24$; (c) $x^2 - 6x + y^2 - 8y = 0$.

5. Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C (freezing of water) corresponds to 32°F, and a temperature of 100°C (boiling of water) corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x. Graph the line, and find the y- and x-intercepts. What is the practical meaning of the intercepts?

6. A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost y to the number of miles x that you drive the car?

7. An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading will be linear. Let x be the exam score, and let y be the corresponding grade. Find a formula of the form $y = mx + b$ which applies to scores x between 40 and 90.

8. A photocopy store advertises the following prices: 5¢/per copy for the first 20 copies, 4¢/per copy for the 21st through 100th copy, and 3¢/per copy after the 100th copy. Let x be the number of copies, and let y be the total cost of photocopying. (a) Graph the cost as x goes from 0 to 200 copies. (b) Find the equation in the form $y = mx + b$ that tells you the cost of making x copies when x is more than 100.

9. In the Kingdom of Zog the tax system works as follows. Someone who earns less than 100 gold coins per month pays no tax. Someone who earns between 100 and 1000 golds coins pays tax equal to 10% of the amount over 100 gold coins that he or she earns. Someone who earns over 1000 gold coins must hand over to the King all of the money earned over 1000 in addition to the tax on the first 1000. (a) Draw a graph of the tax paid y versus the money earned x, and give formulas for y in terms of x in each of the regions $0 \le x \le 100$, $100 \le x \le 1000$, and $x \ge 1000$. (b) Suppose that the King of Zog decides to use the second of these line segments (for $100 \le x \le 1000$) for $x \le 100$ as well. Explain in practical terms what the King is doing, and what the meaning is of the y-intercept.

10. Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let x be the number of items you can sell, and let P be the price of an item. (a) Express P linearly in terms of x, in other words, express P in the form $P = mx + b$. (b) Express x linearly in terms of P.

11. The tax for a single taxpayer is described in the figure below. Use this information to graph tax versus taxable income (i.e., x is the amount on Form 1040, line 37, and y is the amount on Form 1040, line 38). Find the slope and y-intercept of each line that makes up the polygonal graph.

1990 Tax Rate Schedules

Caution: *use ONLY if your taxable income (form 1040, line 37) is $50,000 or more. If less, use the* **Tax Table**. *(Even though you cannot use the tax rate schedules below if your taxable income is less than $50,000, we show all levels of taxable income so that taxpayers can see the tax rate that applies to each level.)*

Schedule X—Use if your filing status is **Single**

If the amount on Form 1040, line 37, is: Over—	But not over—	Enter on Form 1040, line 38	of the amount over—
$0	$19,450	.... **15%**	**$0**
19,450	47,050	**$2,917.50+28%**	**19,450**
47,050	97,620	**$10,645.50+33%**	**47,050**
97,620	....	Use **Worksheet** below to figure your tax.	

Schedule Z—Use if your filing status is **Head of household**

If the amount on Form 1040, line 37, is: Over—	But not over—	Enter on Form 1040, line 38	of the amount over—
$0	$26,050	.... **15%**	**$0**
26,050	67,200	**$3,907.50+28%**	**26,050**
67,200	134,930	**$15,429.50+33%**	**67,200**
134,930	....	Use **Worksheet** below to figure your tax.	

12. Do the points shown in the following table below lie on a line? If so, find the equation for this line.

x	5.2	5.3	5.4	5.6	5.9	6.5
y	27.8	29.2	30.6	33.4	37.6	46.0

13. If the point $(3, k)$ lies on the line with slope $m = -2$ passing through the point $(2, 5)$, find k.

14. Does the point $(3, -2)$ lie on the line through the points $(8, 0)$ and $(-7, -6)$?

15. Use the slopes to determine whether the points $(7, -1)$, $(10, 1)$, and $(6, 7)$ are the vertices of a right triangle.

16. Is there a value of k so that the graphs of $x^3 - x$ and $kx^2 - 1$ meet in three distinct points? In four points? In 2 points? In one point? Do not meet? If so give specific values of k; if not, explain why.