Contents

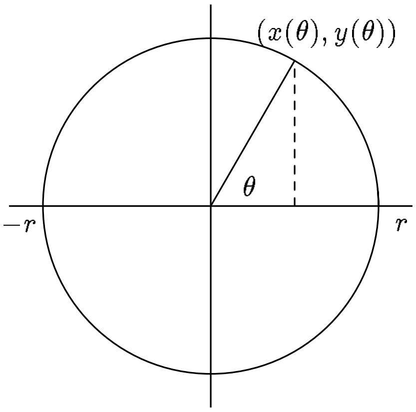
CHAPTER 10

Curves Given Parametrically

So far we have had two different ways of giving a curve. The first was **explicitly**, i.e., in the form $y = f(x)$. For example, the equation $y = \sqrt{r^2 - x^2}$ gives us the upper semicircle of radius r centered at the origin. The second way was **implicitly**, usually in the form $F(x, y)$ =constant. For example, the equation $x^2 + y^2 = r^2$ gives us the circle (both upper and lower semicircles) of radius r centered at the origin.

There is a third fundamental way a curve can be given: **parametrically**. This means that we have a third variable, called the **parameter**, which is used to determine where we are on the curve in the xy-plane. To give a curve parametrically means to give *two* formulas, one for x and one for y, each in terms of the parameter. This parameter might have a geometrical meaning — for example, an angle denoted θ — or it may be the time t. In the case of an angle parameter, our two formulas take the form $x = g(\theta)$, $y = h(\theta)$; in the case of a time parameter, the formulas take the form $x = g(t)$, $y = h(t)$. (Often we don't bother to use separate letters like g and h, instead writing simply $x(t)$ for the formula for x in terms of t and $y(t)$ for the formula for y in terms of t.) The best way to understand parameterized curves is through examples.

EXAMPLE 10.1. (Circles) Suppose we want to describe the points on the circumference of the circle of radius r centered at the origin. A common way to do this is in terms of the angle θ which the line from the point to the origin makes with the positive x-axis (measured counterclockwise). That is, as θ goes from 0 to 2π radians, we sweep once around the circle:



What we want to do is express the x- and y-coordinates of our points on the circumference in terms of θ. To do this, we drop the perpendicular from the point to the x-axis, obtaining a triangle with hypotenuse r and with legs x and y. So we can use the trig functions *sin* and *cos* to express x and y in terms of θ:

$$x = x(\theta) = r\cos\theta, \qquad y = y(\theta) = r\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This says that, as θ goes from 0 to 2π, if we determine x and y by these two equations, then the corresponding point (x, y) goes exactly once around the circumference of the circle.

A special case of this is the **unit circle** — this is simply the case $r = 1$. Thus, the points on the unit circle can be described parametrically by the equations

$$x = x(\theta) = \cos\theta, \qquad y = y(\theta) = \sin\theta, \qquad 0 \le \theta \le 2\pi.$$

The fact that the x-coordinate is simply the cosine and the y-coordinate is simply the sine is not surprising, since we *defined* the cosine and sine of an angle to be the x- and y-coordinates of the point that you get by going around the unit circle through the given angle.

If we want to shift the circle of radius r so that its center is at the point (h, k), we can do this by adding h to the x-formula and k to the y-formula:

$$x = x(\theta) = h + r\cos\theta, \qquad y = y(\theta) = k + r\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This is the circle which is given implicitly by the equation $(x - h)^2 + (y - k)^2 = r^2$.

If instead of $0 \le \theta \le 2\pi$ we had written, say, $0 \le \theta \le \pi$, then we would have described only half of the circle (the upper semicircle). If we had written $0 \le \theta \le 4\pi$, then we would have described going around the circle twice.

EXAMPLE 10.2. (Ellipses) Let us suppose that we have put our ellipse on the axes in the following convenient way: its center is at the origin, its longer axis of symmetry — of length $2a$ — is along the x-axis, and its shorter axis of symmetry — of length $2b$ — is along the y-axis. Such an ellipse can be obtained by taking the unit circle and stretching it by a factor of a in the x-direction and by a factor of b in the y-direction. (If a or b is less than 1, the "stretching" is actually a contraction.) In terms of the parametric equations, we can accomplish this stretching simply by multiplying the x-equation for the unit circle by a and multiplying the y-equation for the unit circle by b:

$$x = x(\theta) = a\cos\theta, \qquad y = y(\theta) = b\sin\theta, \qquad 0 \le \theta \le 2\pi.$$

This is the parametric form for our ellipse — the ellipse which can be given implicitly in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Note: A word of warning, though, about the parameter θ for an ellipse. It no longer has the meaning of the angle formed by the line from the point (x, y) to the origin. For example, suppose $a = 2$, $b = 1$. Then the point on the ellipse corresponding to the parameter value $\theta = \pi/4 = 45°$ is the point $x = 2\cos(\pi/4) = \sqrt{2}$, $y = \sin(\pi/4) = \sqrt{2}/2$; and it is easy to see that the line joining the point $(1.414, 0.707)$ to the origin does not make a $45°$ angle with the x-axis.

There is a geometric way to interpret the variable θ as a suitable angle anyway and this is best illustrated from the following picture of an ellipse with its auxiliary circle.

1. Trajectories

It is very common to want to describe a curve in terms of the time parameter t. In that case we sometimes call the curve the "path" or "trajectory" of our point (which we think of as a small object or particle). As t goes from an initial time a to a final time b, the equations $x(t)$, $y(t)$ tell us where the particle is at that particular time t.

EXAMPLE 10.3. Suppose a particle is traveling counterclockwise at 3 rev/sec around the circle of radius r centered at the origin. It starts at the point $(r, 0)$ at time $t = 0$. We want to give a parametric description of the motion in terms of t.

To do this, we find an expression in terms of t for the angle θ through which the particle has rotated, and then we substitute this expression in the equations in Example 10.1.

Because we shall be measuring angles in radians, we first translate the angular velocity of 3 rev/sec into units of rad/sec. Since 1 rev = 2π rad, we obtain: $d\theta/dt = 3 \cdot 2\pi = 6\pi$ rad/sec. Thus, θ starts at 0 at time $t = 0$ (because our point starts at $(r, 0)$ on the positive x-axis), and it increases at the constant rate of 6π. Hence, after t seconds it has value $\theta = 6\pi t$. Finally, substituting this expression for θ in the equations in Example 10.1, we obtain:

$$x = x(t) = r\cos(6\pi t), \qquad y = y(t) = r\sin(6\pi t), \qquad 0 \le t \le 1/3,$$

where we chose to take t between 0 and 1/3 so that we would describe exactly one rotation of the particle around the circle.

EXAMPLE 10.4. (Motion at constant velocity between two points)
Suppose that at time $t = 0$ you start at the point $P = (2, 5)$ and travel at constant velocity along the line from P to the point $Q = (6, 3)$, arriving 5 seconds later at Q. Find parametric formulas for this motion.

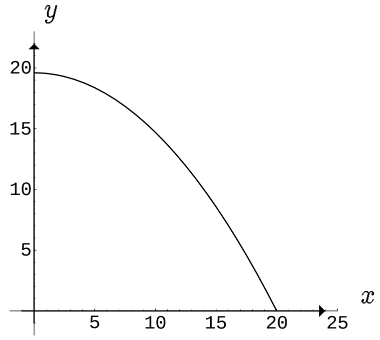
To do this, we use a general principle for setting up parametric equations: **work with the x- and y-directions separately.** In the x-direction in this example we are starting at $x = 2$ and traveling to $x = 6$ in 5 seconds, moving at constant speed. This constant speed — called the "horizontal component of velocity" and denoted $\dot{x} = \frac{dx}{dt}$ — is the distance traveled divided by time: $(6-2)/5 = 0.8$. Thus, the formula for $x(t)$ is $2 + 0.8t$ (i.e., starting x-value plus speed times time). Similarly, in the y-direction we compute the vertical component of velocity $\dot{y} = \frac{dy}{dt}$ to be $(3-5)/5 = -0.4$, so that $y(t)$ is given by the formula $5 - 0.4t$. This is all going on between $t = 0$ and $t = 5$. Thus, our parametric description of the motion is:

$$x = x(t) = 2 + 0.8t, \qquad y = y(t) = 5 - 0.4t, \qquad 0 \le t \le 5.$$

We can check these formulas by noting that when $t = 0$ they give us $x(0) = 2$, $y(0) = 5$ (i.e., we're at the point P), and when $t = 5$ they give us $x(5) = 6$, $y(5) = 3$ (i.e., we're at the point Q).

EXAMPLE 10.5. (Falling body in two dimensions)
Again we are interested in an object falling under the influence of gravity (we neglect air resistance). In the past (see Example 6.9) our object was moving only in one dimension — up and down. For example, if we drop a ball from a window 19.6 meters high, its height y at time t is given by the formula $y = -\frac{1}{2}gt^2 + y_0 = -4.9t^2 + 19.6$ (here $g = 9.8$ m/sec^2; the acceleration is negative, i.e., it is $-g$, because we are choosing up to be the positive direction).

Now suppose that, instead of dropping the ball from the window, we throw it horizontally out from the window at 10 m/sec. The ball then moves in the xy-plane, where we have chosen the side of the building as the y-axis and the line along the ground under the ball's path as the x-axis. Its path is a parabola, as we shall soon see. Our task in this example is to write parametric equations for this motion.



To do this, we again consider the x- and y-directions separately. In the x-direction, there is no force affecting the velocity, and so the 10 m/sec horizontal component of velocity of the ball remains unchanged. That is, as in Example 10.4, in the x-direction we are simply traveling at constant speed. Since we start at initial location $x = 0$, the formula for x at time t is: $x(t) = 10t$.

In the vertical direction we have a falling body problem, and so can use the formula $y = -\frac{1}{2}gt^2 + v_0 t + y_0$. Here it is important to understand what v_0 means. v_0 is the initial velocity in the y-direction. In this example v_0 is 0, since the ball's initial velocity is entirely horizontal. In other words, in the y-direction the problem is exactly as if we had simply dropped the ball from a height of 19.6. Thus, we have $y(t) = -4.9t^2 + 19.6$.

Finally, we want to know the time interval where the formulas for x and y apply. The motion starts at time $t = 0$, and it ends when the ball hits the ground. The ball hits the ground when $y = 0$, and so to find when this happens we set $y = -4.9t^2 + 19.6$ equal to 0 and solve for t. We obtain $t^2 = 4$, $t = \pm 2$, and so $t = 2$ (since -2 does not make practical sense). To summarize, we have the following parametric description of the motion:

$$x = x(t) = 10t, \qquad y = y(t) = -4.9t^2 + 19.6, \qquad 0 \le t \le 2.$$

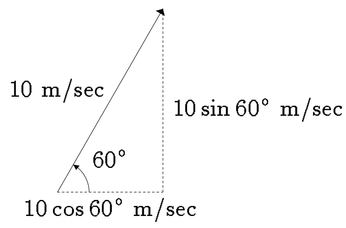
The actual trajectory in Example 10.4 (see the picture above) is a graph in the xy-plane, that is, a function of the form $y = f(x)$. In this example, we can easily find the formula for this function by *eliminating* t in the formulas for $x(t)$ and $y(t)$. That is, since $x = 10t$, we can write $t = 0.1x$, and then substitute this in place of t in the formula for y: $y = -4.9(0.1x)^2 + 19.6 = -0.049x^2 + 19.6$. Thus, the path $f(x)$ is a parabola (more precisely, part of a downward-opening parabola).

2. The Velocity Vector

The velocity of an object moving in the plane is an example of a vector. A **vector** is something which has magnitude and direction. It can be visualized as an arrow of a certain length (or **magnitude**) pointing in a certain direction. The magnitude of the velocity vector is called the **speed** of the object.

EXAMPLE 10.6. The same as Example 10.5, except that at time $t = 0$ the ball is thrown at an angle of 60° above the horizontal.

Here the initial velocity vector makes an angle of 60° above the horizontal and has a magnitude of 10 m/sec. That is, it is partly horizontal and partly vertical. The first step in solving this problem is to **resolve the initial velocity vector into its horizontal and vertical components**. To resolve the velocity vector, we draw the **velocity triangle** having the velocity vector as its hypotenuse (see figure). This triangle has hypotenuse 10 m/sec, making an angle of 60° above the horizontal. Then the horizontal leg of the triangle — the initial x-velocity $v_{0,\text{horiz}} = \dot{x}(0)$ — can be found using the cosine, and the vertical leg of the triangle — the initial y-velocity $v_{0,\text{vert}} = \dot{y}(0)$ — can be found using the sine. We thus have:

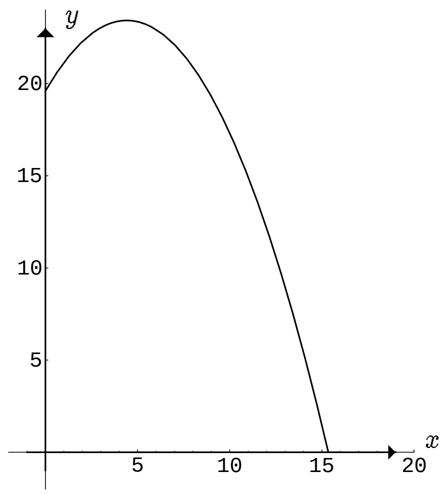


$$v_{0,\text{horiz}} = \dot{x}(0) = 10\cos 60^\circ = 10 \cdot 0.5 = 5\,\text{m/sec},$$
$$v_{0,\text{vert}} = \dot{y}(0) = 10\sin 60^\circ = 10 \cdot 0.866 = 8.66\,\text{m/sec}.$$

We are now ready to write the equations for $x(t)$ and $y(t)$, as we did in Example 10.5. The horizontal component of velocity (which remains constant) is not 10 m/sec, but rather only the horizontal part, namely, 5 m/sec. So $x = 5t$. In the falling body formula $y = -\frac{1}{2}gt^2 + v_0 t + y_0$, we now have a nonzero initial vertical component of velocity $v_0 = v_{0,\text{vert}} = 8.66$ m/sec. Thus, we obtain:

$$x = x(t) = 5t,$$
$$y = y(t) = -4.9t^2 + 8.66t + 19.6.$$

The trajectory of the ball is shown in the figure at the right.



To find the time interval until the ball hits the ground, we again set $y = 0$, and use the quadratic formula to find the time t when we're on the ground. There will be 2 roots, of which we want the positive one:

$$(-8.66 - \sqrt{8.66^2 - 4 \cdot (-4.9) \cdot 19.6})/(2 \cdot (-4.9)) = 3.07 \text{ sec}.$$

So the time interval is $0 \le t \le 3.07$.

Example 10.7. The same as Example 10.6, except that the ball is thrown at an angle of 60° *below* the horizontal.

Here the only change is that the tip of the velocity triangle points downward, i.e., its vertical leg $\dot{y}(0)$ points in the *negative* direction. Thus, in the equations of motion the only thing that changes is the sign of $v_{0,\text{vert}}$ in the falling body formula:

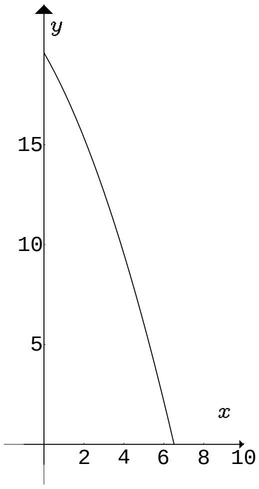
$$x = x(t) = 5t, \qquad y = y(t) = -4.9t^2 - 8.66t + 19.6.$$

Of course, the ball hits the ground much sooner (see figure on the following page).

In all three cases in Examples 10.5–10.7, the trajectory is part of a parabola, as we can see by eliminating t and writing y in terms of x rather than in terms of t.

Suppose that we have a trajectory $x(t)$, $y(t)$, as in Examples 10.3–10.7. There are three types of derivatives we might be interested in. One is the **horizontal component of velocity** $dx/dt = \dot{x}(t)$. The second is the **vertical component of velocity**

$dy/dt = \dot{y}(t)$. The third is not a velocity at all, but rather the slope of the tangent to the curve in the xy-plane: dy/dx. If we find dy/dx at some instant, we can determine the direction in which the object is headed at that instant.



How do we find dy/dx if we do not have y written in terms of x, i.e., if we do not have a formula for the function $y = f(x)$? Well, according to the chain rule, at any instant our three derivatives are connected by the relation

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

In the past we have used the chain rule to find the left hand side. Now we want to find dy/dx. So the form of the chain rule we need is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}.$$

In words, this says that *the slope of the tangent to the path is equal to the vertical component of velocity divided by the horizontal component of velocity.* This makes sense: if the horizontal component of velocity is small and the vertical component of velocity is large, we are traveling at a steep angle; while if the horizontal component of velocity is large and the vertical component of velocity is small, we are traveling at a shallow angle.

In finding dy/dx it is always important to remember that you take the ratio of *velocities* $\dot{y}$ over $\dot{x}$, **NOT** simply the ratio of y to x. In other words, you must take the derivatives (with respect to t) of the $y(t)$ and $x(t)$ formulas before taking their ratio.

EXAMPLE 10.8. Suppose we want to know the instant when the ball in Example 10.6 reaches the peak of its trajectory. At that instant the path has a horizontal tangent line, i.e., $dy/dx = 0$. Since $dy/dx = \dot{y}/\dot{x}$, this slope is zero if and only if the numerator $\dot{y}$ is 0. Thus, to find the instant when the ball reaches its highest point we set the vertical velocity $\dot{y}$ equal to 0 and solve for t. This is the same principle that we saw in our earlier one-dimensional falling body problems: the ball is at the peak when its vertical velocity is zero. (Warning: Be sure not to confuse this procedure, where we set $\dot{y}$ equal to zero, with setting y itself equal to 0, which we do to find the time when the ball hits the ground.) Since $\dot{y} = \frac{d}{dt}(-4.9t^2 + 8.66t + 19.6) = -9.8t + 8.66$, we have $-9.8t + 8.66 = 0$, i.e., $t = 8.66/9.8 = 0.88$ sec. If we also want to know the maximum height reached by the ball, we take this value of t and substitute it in the formula for the height y, i.e.,

$$y_{\text{max}} = y(0.88) = -4.9 \cdot (0.88)^2 + 8.66 \cdot 0.88 + 19.6 = 23.4 \text{ m}.$$

EXAMPLE 10.9. Suppose that we want to know the direction in which the ball in Example 10.6 is headed after 0.5 sec, and also at the instant when it hits the ground. We first take the time derivatives of y and x:

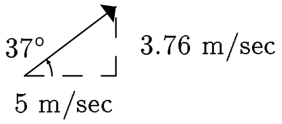
$$\frac{dy}{dt} = \frac{d}{dt}(-4.9t^2 + 8.66t + 19.6) = -9.8t + 8.66 \text{ and} \frac{dx}{dt} = \frac{d}{dt}(5t) = 5\,.$$

Then we have:

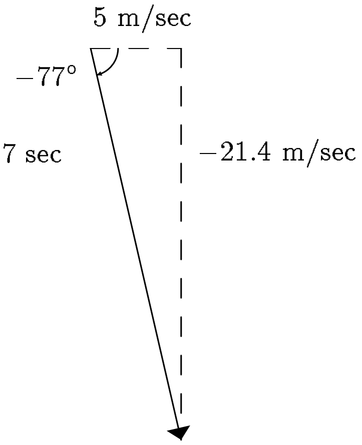
$$\frac{dy}{dx} = \frac{\dot{y}(t)}{\dot{x}(t)} = (-9.8t + 8.66)/5\,.$$

Thus, when $t = 0.5$ we have $dy/dx = (-9.8 \cdot 0.5 + 8.66)/5 = 0.752$. At that instant the ball's location is $(x(0.5), y(0.5)) = (2.5, 22.7)$, so that the tangent line to the path at the point $(2.5, 22.7)$ has slope 0.752. We can also determine the angle at which the ball is traveling at time $t = 0.5$. By this we mean the angle above the horizontal of the tangent line to the path. To do that we observe that the slope of a line $\Delta y/\Delta x$ is opposite over adjacent, i.e., the tan of the angle that the line makes with the horizontal. So to determine the angle we use the inverse-tan on our calculator: $\tan^{-1} 0.752 = 37°$. At the same time $t = 0.5$ we could also construct the velocity triangle, having horizontal leg $\dot{x}(0.5) = 5$ and vertical leg $\dot{y}(0.5) = 3.76$, and compute the hypotenuse (which is the speed):

$$\text{speed at time } 0.5 = \sqrt{5^2 + 3.76^2} = 6.26 \text{ m/sec.}$$



$t = 0.5$ sec



$t = 3.07$ sec

Next, we make the same computations at the instant when the ball hits the ground, i.e., at $t = 3.07$ sec. We have $\dot{x}(3.07) = 5$, $\dot{y}(3.07) = -9.8 \cdot 3.07 + 8.66 = -21.4$, and so speed$=\sqrt{5^2 + (-21.4)^2} = 22.0$. The angle at which the ball hits the ground is $\tan^{-1}(-21.4/5) = -77°$. Thus, the ball hits the ground traveling at 22 m/sec at an angle of 77° below the horizontal.

HOMEWORK 10.

1. Find parametric equations $x(t)$, $y(t)$ for the point P if:

(a) P travels at constant velocity from (2,3) to (11,−3) between $t = 0$ and $t = 3$;

(b) P travels at constant speed 6.5 units/sec from (1,2) in the direction of (13,7);

(c) P travels at constant speed 10 units/sec from the origin at an angle of 30° above the x-axis.

2. A bicycle has wheels of radius 0.4 meter. Find:

(a) the angular velocity of the wheels in rad/sec if the bicycle is traveling at 10 m/sec;

(b) the angular velocity of the wheels in rev/sec if the bicycle is traveling at 10 m/sec;

(c) the speed of the bicycle in m/sec if the wheels are turning at 5 rev/sec;

(d) the speed of the bicycle in km/hr if the wheels are turning at 15 rad/sec.

3. A truck has wheels of radius 1.6 ft. Find:

(a) the speed of the truck in mph if the wheels are turning at 5 rev/sec;

(b) the angular velocity of the wheels in rad/sec and in rev/sec if the truck is traveling at 60 mph.

4. Find a formula for the distance in feet that a car has traveled during the first t sec if its 10-inch-radius wheels:

(a) are turning at the constant rate of 24 rad/sec;

(b) are turning at the constant rate of 9 rev/sec;

(c) start from rest, and accelerate at the rate of 18 rad/sec per second.

5. A wheel of radius 3 is centered at the origin. Find parameteric equations for the point P on the rim if:

(a) the wheel is spinning counterclockwise at 0.5 rad/sec, and P is at (3,0) at time $t = 0$;

(b) the wheel is spinning clockwise at 2 rev/sec, and P is at (0,3) at time $t = 0$;

(c) the wheel is spinning clockwise at 1 revolution each 5 seconds, and P is at $(0, -3)$ at time $t = 0$;

(d) the wheel is spinning counterclockwise at 10 rev/sec, and P is at $(0, -3)$ at time $t = 0$.

6. A girl is playing on a flatcar which is moving in the positive x-direction at 10 m/sec. At time $t = 0$ she throws the ball straight upward (as viewed from the moving train) with an initial velocity 24.5 m/sec from a height 1 meter above the flatcar. From her point of view, the ball goes straight up and down, and lands on the flatcar directly below the point from which she threw it. However, an observer on the ground sees the ball travel in an arc.

Set up a coordinate system which is fixed to the earth (i.e. one which is not moving along with the train), with the y-axis pointing up and the x-axis pointing in the direction of motion of the train. Choose the origin to be the point in space which is 1 meter below the point from which the ball was released. Neglect air resistance, and take $g = 9.8$ m/sec^2.

(a) Find parametric equations for the motion of the ball in this fixed coordinate system.

(b) Find the instant when the ball reaches the peak of its trajectory.

(c) Find the instant when the ball hits the flatcar.

(d) Describe in words how you would find the angle at which the ball hits the flatcar in these coordinates.

7. Suppose that an object is thrown at 70 ft/sec at an angle of 45° *below* the horizontal from a window 163 ft above the ground.

(a) Express both the horizontal distance x from the window and the height y above the ground as functions of t.

(b) Find dy/dx in terms of t.

(c) Find the **angle** below the horizontal that the object is pointed when $t = 0.5$ sec and when $t = 1$ sec.

(d) Find the **velocity vector** (its magnitude and angle below the horizontal) of the object at the instant when it hits the ground.

8. Suppose that a ball is thrown from ground level at v m/sec at an angle of α above the horizontal. Take the origin to be the point from which the ball is thrown.

(a) Find formulas for $x(t)$ and $y(t)$. The constants v and α should appear in your formulas.

(b) Find the time t when the ball hits the ground, expressing your answer in terms of v and α. There are two ways to do this: (i) Set $y(t) = 0$ and solve for t. (ii) Find the time when the ball reaches its peak (see Example 10.8), and double it (because it takes an equal time to fall back to the ground). Use both methods, and see that your answers agree.

(c) Find the horizontal distance traveled by the time the ball hits the ground, expressing your answer in terms of v and α.

Supplement: Differentiation Formulas

In the following formulas, $f(x)$ and $g(x)$ are differentiable functions and c is a constant:

$$
\begin{aligned}
(f+g)'(x) &= f'(x)+g'(x) \\
(f-g)'(x) &= f'(x)-g'(x) \\
(fg)'(x) &= f(x)g'(x)+f'(x)g(x) \\
(cf)'(x) &= cf'(x) \\
(f/g)'(x) &= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2} \\
(f(g(x))' &= f'(g(x))g'(x) \\
g'(x) &= 1/f'(g(x)) \text{ if } g(x) \text{ is the inverse of } f(x)
\end{aligned}
$$

Formulas for common functions:

$\frac{d}{dx}x^a = ax^{a-1}$	$\frac{d}{dx}a^x = (\ln a)a^x$
$\frac{d}{dx}\ln x = 1/x$	$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x} = \frac{\log_a e}{x}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\sec x = \sec(x)\tan(x)$	$\frac{d}{dx}\csc x = -\csc(x)\cot(x)$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\frac{d}{dx}\text{arccot } x = \frac{-1}{1+x^2}$
$\frac{d}{dx}\text{arcsec } x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\text{arccsc } x = \frac{-1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}\sinh x = \cosh x$	$\frac{d}{dx}\cosh x = \sinh x$

Supplement: Derivative Skills Exercises

1. $f(x) = x^{100} + 50x + 1$
2. $f(x) = x^e$
3. $f(x) = (16x)^3$
4. $f(x) = (x^2+1)(2x-7)$
5. $f(x) = x^2 + \frac{1}{x^2}$
6. $f(x) = \frac{1-x^2}{1+x^2}$
7. $f(x) = \cos(x) - 2\ \tan(x)$
8. $f(x) = \frac{\sin(x)}{1+\cos(x)}$
9. $f(x) = x\ \csc(x)$
10. $f(x) = \csc(x)\ \cot(x)$
11. $f(x) = x\ \sin(x)\cos(x)$
12. $f(x) = e^{\sqrt{x}}$
13. $f(x) = x\ e^{2x}$
14. $f(x) = e^{x\ \cos(x)}$
15. $f(x) = \frac{e^{-x^2}}{x}$
16. $f(x) = \frac{e^{3x}}{1+e^x}$
17. $f(x) = \ln(2-x-x^2)$
18. $f(x) = \ln(\ln(x))$
19. $f(x) = \ln(\cos(x))$
20. $f(x) = \cos(\ln(x))$
21. $f(x) = \log_3(x^2-4)$
22. $f(x) = (x+2)^8(x+3)^6$
23. $f(x) = \frac{1}{(x^2-2x-5)^4}$
24. $f(x) = \cos^3(x)$
25. $f(x) = \cos(x^3)$
26. $f(x) = \cos(\tan(x))$
27. $f(x) = x\ \sin(\frac{1}{x})$
28. $f(x) = \frac{\sin^2(x)}{\cos(x)}$
29. $f(x) = -\frac{2}{\sqrt[4]{x^3}}$
30. $f(x) = \sqrt[5]{x\ \tan(x)}$
31. $f(x) = \tan(\sqrt{1-x})$
32. $f(x) = \frac{\sin(mx)}{x}$
33. $f(x) = \cos^2(\tan\ x)$
34. $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$
35. $f(x) = \arcsin(2x-1)$
36. $f(x) = \arctan(x^3)$
37. $f(x) = (\arcsin\ x)^2$
38. $f(x) = \frac{\arccos(x)}{x}$

39. $f(x) = (1 + x^2)\ \arctan(x)$
40. $f(x) = \log_{10}(x^2 - x)$
41. $f(x) = \arcsin(e^x)$
42. $f(x) = \ln(x^2\, e^x)$
43. $f(x) = x^{\sin(x)}$
44. $f(x) = x^{e^x}$
45. $f(x) = \sin(\tan\sqrt{1 + x^3})$
46. $f(x) = \ln(\sqrt[4]{\frac{x^2+x+1}{x^2-x+1}})$

Use implicit differentiation:

47. $x^2 - xy + y^3 = 8$
48. $x^4 + y^4 = 16$
49. $\sqrt{xy} - 2x = \sqrt{y}$
50. $\cos(x - y) = y\ \sin(x)$