Contents

CHAPTER 12

Related Rates

Suppose we have two variables x and y (in most story problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem where we know one of the rates of change at a given instant — say, dx/dt — and we want to find the other rate dy/dt at that instant.

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

But often x and y will be related in some other way, for example $x = f(y)$, or $F(x, y)$ =const, or perhaps $F(x, y) = G(x, y)$, where $F(x, y)$ and $G(x, y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking $\frac{d}{dt}$ of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem when there is a simple direct relation in the two variables x, y.

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take d/dt of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

Sometimes it may not be convenient or desirable to find the equation between x, y directly. They might in turn be related to each other by a chain of variables and conditions. We will explain such cases in the examples below.

Example 12.1. The radius of a circular puddle is 3 m, and it is increasing at the rate of 1 cm/min. How fast is the puddle's area increasing?

Here the two things that change with time are: the radius r and the area A. We know $dr/dt = 1$ cm/min $= 0.01$ m/min. We want to find dA/dt. Thus, in this problem r is playing the role of x, and A is playing the role of y. The relation between the two variables is: $A = \pi r^2$. Taking d/dt of both sides, we obtain

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}.$$

Notice that, since we are taking derivatives with respect to t, the derivative of r^2 is $2r\frac{dr}{dt}$, by the rule for the derivative of a power of a function (the u^n-rule). In this case we alternately could have used the chain rule for the variables A, r and t:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = 2\pi r \cdot \frac{dr}{dt}.$$

We now plug in the information given. At the instant in question, $r = 3$, and $dr/dt = 0.01$. Thus, $A = 2\pi \cdot 3 \cdot 0.01 = 0.1885$ m^2/min.

Note: A crucial point to notice in this and similar problems is that, even though a numerical value is given for r, r must be regarded as a variable, not a constant. *The value of the variable at the instant in question cannot be plugged into the equation until* **after** *taking d/dt of the equation.*

EXAMPLE 12.2. You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing at the instant when $r = 4$? How fast is the area increasing at the same time?

First we answer the first question.

Here the variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by means of the equation $V = \frac{4}{3}\pi r^3$. Taking d/dt of both sides gives $dV/dt = 4\pi r^2 \cdot dr/dt$. We now substitute the values we know at the instant in question: $7 = 4\pi 4^2 \cdot dr/dt$. Solving for dr/dt, we obtain $dr/dt = 7/(64\pi) = 0.0348$ cm/sec.

Now the second question.

The formula for the area is $A = 4\pi r^2$. Taking d/dt of both sides, we get $dA/dt = 4\pi 2r dr/dt = 8\pi r dr/dt$.

Now that we have worked out the answer to dr/dt, we can easily deduce that

$$dA/dt = 8\pi(4)\frac{7}{64\pi} = 7/2 = 3.5.$$

Note: If only the second question was asked, then we would have first written down the two well known formulas:

$$V = 4\pi/3r^3\ ,\ A = 4\pi r^2.$$

and tried to find a direct relation between V and A. The relation is $36\pi v^2 = a^3$. Then we would try to solve this for A and take derivatives, or take d/dt of both sides and finally arrive at the same result.

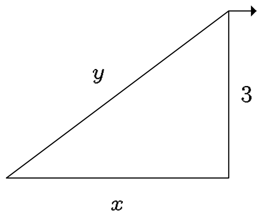
The point is that it is not necessary to eliminate r right away. It is far easier to take d/dt of the two relations first and get equations

$$dV/dt = 4\pi r^2 dr/dt\ ,\ dA/dt = 8\pi r dr/dt \text{ so } 2dV/dt = r dA/dt \text{ or } dA/dt = (2/r)dV/dt.$$

Our final answer is now clear!

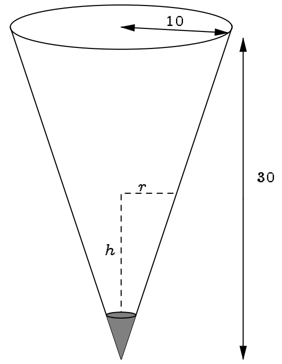
EXAMPLE 12.3. A plane is flying at 500 mph at an altitude of 3 miles in a direction away from where you're standing (i.e., the point on the ground directly beneath the plane is moving away from you). How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

To see what's going on, we first draw a triangle whose hypotenuse is the line from you to the plane (see figure at right). The vertical leg of the triangle is a constant — 3 mi — which does not change in the course of the problem. The horizontal leg x is the variable whose rate we know — its rate $\dot{x}$ is the speed of the plane. The hypotenuse y is the variable whose rate we want. The equation relating x and y is the Pythagorean theorem: $x^2 + 3^2 = y^2$. We next take d/dt of both sides, getting $2x \cdot \dot{x} = 2y \cdot \dot{y}$. At the instant in question we're told that $x = 4$ and $\dot{x} = 500$. What about y? From the Pythagorean theorem, we



have: $y = \sqrt{4^2 + 3^2} = 5$ at the instant in question. Thus, we obtain: $2 \cdot 4 \cdot 500 = 2 \cdot 5 \cdot \dot{y}$, and solving for $\dot{y}$ gives the value 400 mph. That is, the plane is receding from you at 400 mph. (Until you think about this for a while, it might seem counterintuitive that, although the side y is greater than the side x, its rate of increase is less.)

EXAMPLE 12.4. Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down (see side view at right), and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?



In this problem, the water forms a conical shape within the big cone whose height and base radius — and hence also its volume — are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. And we have the relation $V = \frac{1}{3}\pi r^2 h$. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable r whose rate we don't know. But the dimensions of the cone of water must have the same proportions as those of the container. In other words, r and h fit into the above diagram of similar triangles, from which it follows that $\frac{r}{h} = \frac{10}{30}$, i.e., $r = h/3$. So we can eliminate r from the problem entirely: $V = \frac{1}{3}\pi (h/3)^2 h = \frac{\pi}{27} h^3$. We now take d/dt of both sides and then plug in $h = 4$ and $dV/dt = 10$, obtaining $10 = \frac{\pi}{27} 3 \cdot 4^2 \cdot \frac{dh}{dt}$. Thus, $dh/dt = 90/16\pi = 1.79$ cm/sec.

Note: Again we can take the lazy approach and write the equations:

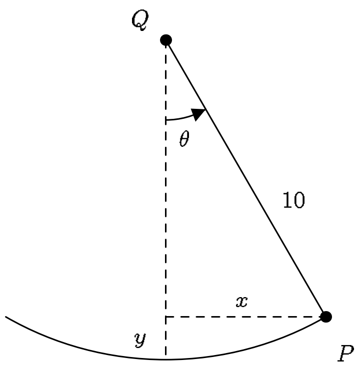
$$V = \frac{1}{3}\pi r^2 h \text{ and } r = h/3$$

that we noticed above. Instead of eliminating the variable r we could have just taken separate derivatives first and then eliminated dr/dt. Of course, here the second equation is so easy to solve (it is already solved for r), we did eliminate r right away in our solution.

In general, the lazy method is better, unless the elimination is very easy.

EXAMPLE 12.5. A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at the speed of 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

In doing this problem, we must start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving **horizontally** at a rate we know. In other words, the **horizontal** coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point Q of attachment. Then the rate we know is dx/dt, and in part (a)



the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram above we see that we have a right triangle whose legs are x and $10-y$, and whose hypotenuse is 10. Hence $x^2+(10-y)^2 = 100$. Taking d/dt of both sides — and recalling that the derivative of the square of a function of t is twice the function being squared times the time derivative of that function — we obtain: $2x\dot{x}+2(10-y)(0-\dot{y}) = 0$. We now look at what we know after 1 sec, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), $y = 2$ (because we get $10 - y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values leaves us with $2 \cdot 6 \cdot 6 - 2 \cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$, getting $\dot{y} = 4.5$ ft/sec.

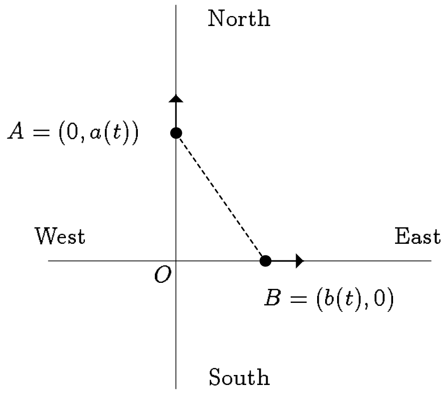
(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time to relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking d/dt of both sides, we obtain $(\cos\theta)\dot{\theta} = 0.1\dot{x}$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6-8-10, the $\cos\theta$ in this equation is 8/10. We also substitute $\dot{x} = 6$ on the right. We get $0.8\dot{\theta} = 0.6$, i.e., $\dot{\theta} = 0.6/0.8 = 0.75$ rad/sec $= 43$ deg/sec.

Sometimes there are several variables that change with time, you know the rates of all but one of them, and you want to know the remaining rate. As in the case when there are just two variables, take d/dt of both sides of the equation relating all of the variables, and then plug in all of the known values and solve for the unknown rate. In other words, our lazy method is the recommended method!

EXAMPLE 12.6. A road going South–North crosses a road going West–East at the point O. Car A is driving North along the first road, and car B is driving East along the second road. At time t car A is $a(t)$ to the North of O, and car B is $b(t)$ to the East of O. Suppose that at some instant you know the values of $a(t)$, $b(t)$, the velocity $v_1(t) = \dot{a}(t)$ of car A, and the velocity $v_2(t) = \dot{b}(t)$ of car B. Find a formula in terms of these known values for the rate at which the distance between the two cars is increasing.

Let c be the distance from car A to car B. By the Pythagorean Theorem, $c^2 = a^2 + b^2$. We now take d/dt of both sides of the Pythagorean Theorem, obtaining $2c\dot{c} = 2a\dot{a} + 2b\dot{b}$. Dividing by $2c$ gives the following formula for the unknown rate:

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{av_1 + bv_2}{\sqrt{a^2 + b^2}}.$$



Notice how this problem differs from Example 12.3. In both cases we took d/dt of the Pythagorean Theorem. However, in Example 12.3 one of the sides was a constant (the altitude of the plane), and so d/dt of the square of that side was simply zero. In Example 12.6, on the other hand, all three sides of the right triangle are variables. As always in a story problem, it's important to read the problem carefully enough to determine at the start what are the variables and what are the constants.

HOMEWORK 12.

Part A. Basic problems:

1. A cylindrical tank has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec?

2. A ladder 13 m long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?

3. A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles downshore from the point A?

4. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?

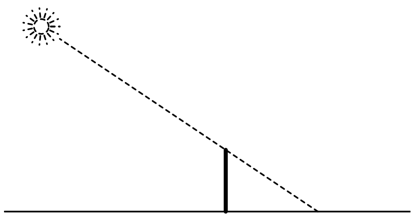
5. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base radius is always half its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

6. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out?

7. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. At that instant a bicycle passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between them increasing 2 seconds later?

8. A cone-like vat has square cross-section (rather than circular cross-section, as in the case of the usual cone). The dimensions at the top are 2 m $\times$ 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any conical shape is $(1/3) \times$ height $\times$ area of base.

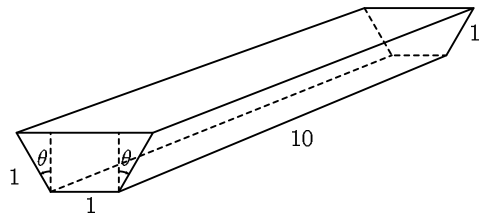
9. The sun is setting at the rate of 1/4 deg/min on a day when the sun passed directly overhead. How fast is the shadow of a 25 meter wall lengthening at the moment when the shadow is 50 meters? (See the picture.)



10. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? Notice that in this problem these rates do not depend on the instant in question — both rates are constant, provided that the woman's speed is constant.

11. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.

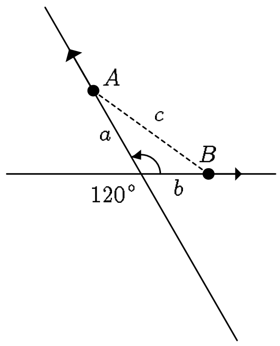
12. The trough shown on the right is constructed by fastening together three slabs of wood of dimensions 10 ft $\times$ 1 ft, and then attaching the construction to a wooden wall at each end. The angle θ was originally $30°$, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in $\mathrm{ft}^3/\mathrm{sec}$) is the water spilling out over the top of the trough if the sides have each fallen to an angle of $45°$, and are collapsing at the rate of $1°$ per second?



13. Suppose that in Example 12.6 the angle between the two roads is $120°$, instead of $90°$ (that is, the South–North road actually goes in a somewhat north-westerly direction from O). In this problem use the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$.

(a) Express $\dot{c}$ in terms of $a, b, \dot{a}, \dot{b}$.

(b) Notice that the angle θ is constant (and so $\cos\theta$ is also constant). State a story problem (perhaps involving a bug on a door, rather than a car on a road) in which not only a, b, c but also θ are all variables. In that case suppose you know the values of a, b, θ and their rates. Find a formula for $\dot{c}$.



Part B. Additional Practice (do these as soon as possible, certainly before the midterm):

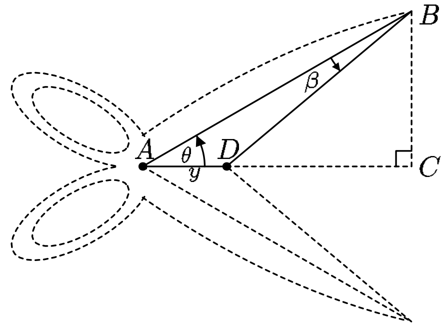
14. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away. How fast is the object's shadow moving on the ground one second later? (Neglect air resistance and take $g = 9.8\ \mathrm{m/sec}^2$ for the falling object.)

15. The two blades of a pair of scissors are fastened at the point A. Let a denote the distance from A to the tip of the blade (the point B). Let β denote the angle at the tip of the blade that is formed by the line $\overline{AB}$ and the bottom edge of the blade (see the diagram). Suppose that a piece of paper is cut in such a way that the center of the scissors at A is fixed, and the paper is also fixed. As the blades are closed (i.e., the angle θ in the diagram is decreased), the distance y between A and D increases, cutting the paper.

(a) Express y in terms of a, θ, and β.

(b) Suppose you know the rate $d\theta/dt$ at which the angle between the blades is decreasing. Express dy/dt in terms of a, θ, β, and $d\theta/dt$.

(c) Suppose that the distance a is 20 cm, and the angle β is $5°$. Further suppose that you are closing the scissors at the rate of 50 deg/sec. At the instant when $\theta = 30°$, find the rate (in cm/sec) at which the paper is being cut.



16. In the previous problem,

(a) express in terms of $d\theta/dt$ the speed of the point on the scissors that moves fastest (this point is B);

(b) if we have $a = 20$ kilometers (i.e., we have scissors of grand proportions) and if $\beta = 0.001$ radians, find a value of $d\theta/dt$ for which dy/dt is a little faster than the speed of light ($c = 300000$ km/sec) when the pair of scissors is almost closed;

(c) would the situation in your answer to part (b) be possible, or would it violate the Theory of Relativity? Explain.

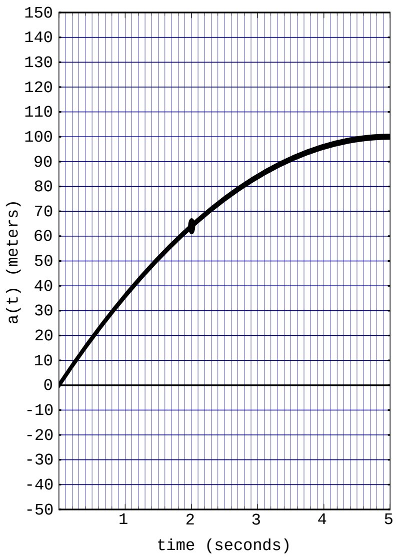
17. In the situation of Example 12.6, find the rate at which the distance between the cars is increasing or decreasing if

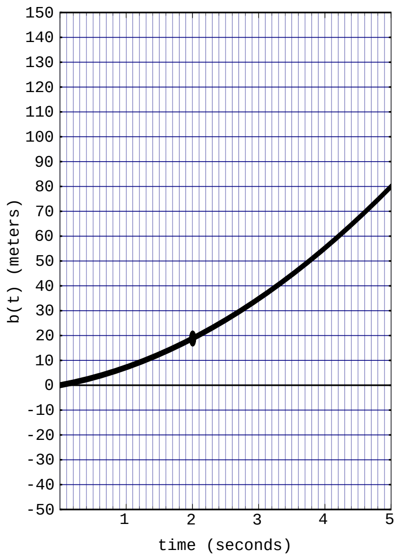
(a) car A is 300 meters north of O, car B is 400 meters east of O, both cars are going at constant speed toward O, and the two cars will collide in 10 seconds;

(b) 8 seconds ago car A started from rest at O and has been picking up speed at the steady rate of 5 m/sec^2, and 6 seconds after car A started car B passed O moving east at constant speed 60 m/sec;

(c) the functions $a(t)$ and $b(t)$ are given by the graphs below and you want to find the rate at which the cars are separating at time $t = 2$ sec.

Hint: Estimate derivatives by drawing tangent lines.





18. In Example 12.6 suppose that instead of car B you have a helicopter flying at speed v_2 to the east of O. Let h be the altitude of the helicopter, and let c be the distance from car A to the helicopter. Then the three-dimensional Pythagorean Theorem states: $c^2 = a^2 + b^2 + h^2$. Find a formula for $\dot{c}$ in terms of a, b, h and the various rates if:

(a) the helicopter is flying at speed $\dot{b}$ at constant altitude;

(b) the helicopter is flying at horizontal speed $\dot{b}$ (i.e., the point on the ground under the helicopter is moving at this speed), and at the same time is gaining altitude at rate $\dot{h}$.