Contents

CHAPTER 13

Curve Sketching

In this section we discuss how to sketch the graph of a function $y = f(x)$ without plotting many points. Indeed, most parts of a curve are routine and uninteresting and plotting them precisely serves very little purpose. What is really important is to learn the exact points features of the curve change and if we plot these points precisely then we will have a reliable as well as useful graph.

Without having to make up a table of values for the function, we can obtain a good *qualitative* picture of the graph using certain crucial information — local maxima and local minima, inflection points, asymptotes, etc. Our aim is not to draw an exact graph, but rather to get an accurate overall picture of the graph and to pinpoint the points where something special happens.

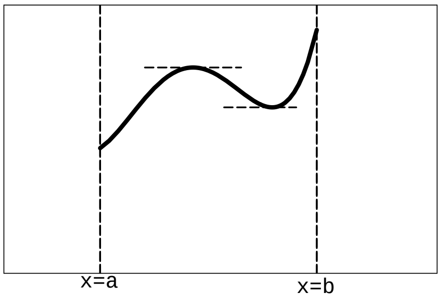
We start by describing the steps to take in curve sketching. For a particular $f(x)$, not all of the steps below will necessarily lead to useful information. The various possibilities will be illustrated later in the examples.

1. Maxima/Minima

If $(p, f(p))$ is a point where $f(p)$ is the highest value as long as the x-values vary in a small interval around p, then $f(p)$ is said to be a **local maximum**value of $f(x)$. Similarly, $f(p)$ is a **local minimum** value, if $f(p)$ is the lowest value as x varies in a small interval near p. Collectively, a local maximum or minimum is called a **local extremum.** The plurals of maximum, minimum and extremum are maxima, minima and extrema respectively.

If the derivative $f'(x)$ exists at $x = p$, or, in other words, if the tangent line to the curve $y = f(x)$ exists at $x = p$, then it can be shown that *the tangent line at that point is horizontal*, i.e., $f'(p) = 0$. The idea of the proof of this claim is easy. If $f'(p) > 0$ then the function is increasing near $x = p$ and there must be values of $f(x)$ bigger than $f(p)$ near (and to the right) of $x = p$. Similarly, if $f'(p) < 0$, then the function is decreasing at $x = p$ and there must be values of $f(x)$ smaller than $f(p)$ near (and to the left) of $x = p$. So, in either case, the function cannot have a local maximum at $x = p$. The case of a local minimum is similar.

Such points can thus be found by setting the derivative equal to zero, and solving for x. (There may be one x for which $f'(x) = 0$, there may be many, or there may be none.) Once you find the x for which $f'(x) = 0$ (and the corresponding y-coordinate of each possible max/min point), you have to determine whether it really is a maximum or minimum. In simple examples it might be obvious. A systematic way to tell is by the second derivative test, which will be described below.



First, we make a few remarks.

1. The reason for using the term **local maximum** point instead of just "maximum " is that it may not actually be a maximum value of the function on the whole domain. As in the above drawing, the local maximum might only be the maximum $f(x)$ for all nearby x — the function might later turn again and go up to a higher value. The reason for the use of the term **local minimum** instead of just "minimum" is similar. Sometimes, local maxima and minima are referred to as "upper turning points" and "lower turning points."

   What is usually understood by the "maximum" will be called the "absolute maximum". Thus $f(p)$ is said to be the absolute maximum value of $f(x)$ on some interval, if it is bigger than or equal to $f(x)$ as x varies over the interval. The term absolute minimum is similar. Some people use the term "global" in place of "absolute".

   If you want to be very precise, you should never use the terms maximum or minimum without some adjective like "local" or "absolute" or "global". If a given word problem does not use these precise terms, then you should use some common sense reasoning to determine what the question must mean!
2. A maximum or minimum might occur at a point where there is no well defined tangent line, i.e., where $f'(x)$ does not exist. For example, $f(x) = |x|$ has its (local as well as absolute) minimum point at $(0, 0)$.

   Sometimes a curve has a vertical tangent. At such a point, the value of $f'(x)$ will come out undefined as the limiting process for the derivative will lead to $\pm\infty$ as limiting values. For our purposes, we can treat this also as a case of $f'(x)$ undefined, even though it is possible to make a reasonable definition and declare it well defined. A simple example of this is the curve $y =$ [1]$\sqrt[3]{x}$ which has a vertical tangent at $x = 0$.
   1
3. Suppose you want to find the absolute maximum of a function $f(x)$ on the interval from a to b. To do this, you have to try
   (a) all points x where $f'(x) = 0$,
   (b) all points x where $f'(x)$ does not exist (as explained above), and
   (c) the endpoints a and b (for example, in the above drawing the maximum point is at the endpoint b).

   The largest of the function values among these shall come out to be the absolute maximum.
4. Later, we will see examples of points where, even though the slope of the tangent line is 0, we have *neither* a maximum nor a minimum there.

[1] Generally, we forbid taking fractional powers of negative numbers. However, the function that we are talking about here is the "real" cube-root which is well defined even for negative values of x. It can be thought to be the usual cube-root function extended to the negative values by making it an odd function.

2. The Second Derivative Test

Suppose that $x = p$ is a solution of $f'(x) = 0$. The second derivative test is a method that **in most cases** will tell us whether the point $(p, f(p))$ is a local maximum or minimum. The test works as follows.

Compute the *second derivative of the function at the same value* $x = p$, i.e., $f''(p)$. If $f''(p)$ is positive, that means that $f'(x)$ is an increasing function at p, i.e., $f'(x)$ is negative a little to the left of p and is positive a little to the right of p. This is what happens near a *local minimum*. On the other hand, if $f''(p)$ is negative, then $f'(x)$ is positive a little to the left of p and is negative a little to the right of p, which is what happens near a *local maximum*. If it so happens that $f''(p) = 0$ (i.e., if *both* $f'(p)$ and $f''(p)$ are zero at the same x), then the second derivative test doesn't give you any information. To summarize,

$$f''(x) \begin{cases} > 0 & \Longrightarrow \text{min pt;} \\ < 0 & \Longrightarrow \text{max pt;} \\ = 0 & \Longrightarrow \text{can't tell.} \end{cases}$$

To see how and why this test succeeds or fails, you should always consider graphs of the functions x, x^2, x^3, x^4. We have provided a picture below. You should analyze the point $x = 0$ in each case.

3. The First Derivative Test

The Second Derivative Test works like a charm if we can find $f''(x)$ and it is nonzero. If it is difficult to evaluate or zero, then the first derivative case comes in handy. In fact, it can also be used in place of the Second Derivative Test.

Suppose you are testing a point $x = p$ and $f'(p) = 0$. You wish to check if this is a local maximum or local minimum. Evaluate the derivative $f'(x)$ at a point x_1 to the left of p **and near** p. Also evaluate the derivative $f'(x)$ at a point x_2 to the right of p **and near** p.

The test says:

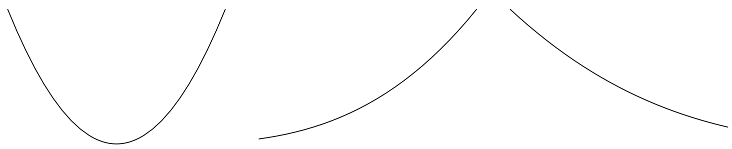
1. If $f'(x_1) > 0$ and $f'(x_2) < 0$, the $x = p$ is a local maximum.
2. If $f'(x_1) < 0$ and $f'(x_2) > 0$, the $x = p$ is a local minimum.
3. If $f'(x_1)$ and $f'(x_2)$ have the same sign, then it is neither a local maximum nor a local minimum. It is an "inflection point" discussed below.

As you can see, this test is definitive. The only problem is to decide how near should the points x_1, x_2 should be. The requirement is that **there should be no critical points of the function** inside $[x_1, p)$ and $(p, x_2]$. If you have only finitely many critical points, then this will be easy to do. If you have infinitely many critical points which cluster around $x = p$, then this test cannot be applied at all!

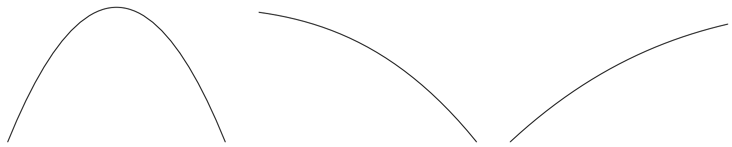
See if you can think of such functions, they don't appear naturally.

4. Concavity and Inflection

We know that the sign of the derivative tells whether a function is increasing or decreasing. For example, any time that $f'(x) > 0$, that means that $f(x)$ is increasing. We now explain the geometrical meaning of the sign of the **second** derivative. Suppose that $f''(x) > 0$ for a certain interval of x. This means that as we go from left to right across the interval, the slope of the tangent line increases. We say that the function is "concave upward" over this interval. The pictures below all show concave upward functions.



Conversely, if $f''(x) < 0$ over a certain interval, then the slope of the tangent line decreases, and we say that the function is "concave downward." The pictures at the top of the next page all show concave downward functions.



There are several ways to describe the difference between upward concavity and downward concavity: (1) An upward concave function looks like part of a right-side-up bowl (holds water), while a concave downward function looks like part of an upside-down bowl. (2) If you draw the tangent line to a concave upward function at a point, then the curve sits on the tangent line, whereas in the case of a concave downward function the tangent line rests on the curve. (3) As you go from left to right, the tangent line to a concave upward curve rotates counterclockwise, while the tangent line to a concave downward curve rotates clockwise.

A point that separates a region of upward concavity and a region of downward concavity is called a **point of inflection**. The geometrical meaning of a point of inflection is that, as you roll the tangent line along the curve from left to right, at a point of inflection it reverses its direction of rotation (from clockwise to counterclockwise, or vice-versa). That is, *the sign of second derivative changes as you roll past a point of inflection.* Thus, inflection points can be found by setting $f''(x) = 0$ and solving for x. Each of the drawings for Examples 13.1 and 13.2 below shows a point of inflection at the origin separating a concave downward part of the curve (on the left) from a concave upward part (on the right).

Warning. Notice that $(x, f(x))$ may not be a point of inflection even though $f''(x) = 0$. After finding the points where $f''(x) = 0$, you then must check that $f''(x)$ changes sign. (See Example 13.5 below.)

5. Asymptotes and Other Things to Look For

An asymptote of a curve is a line which will be a tangent to the curve if the curve can be analyzed near infinity. There are precise ways of discussing points at infinity of a curve and making this discussion rigorous. We will, however, take a naive definition and learn how to recognize asymptotes.

Thus, an asymptote for us will be a line such that the graph of the curve approaches the line along some branch of the curve going out to the infinite region of the plane.

The easiest to analyze is a vertical asymptote. A vertical line $x = p$ is an asymptote to $y = f(x)$, if $f(x)$ becomes infinite as x nears p. It is possible that the function gets closer to plus or minus ∞ depending our approach to the point $x = p$. Usually, the formula for the function has a denominator that becomes zero at $x = p$. For example, the reciprocal function has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.).

Note that usually $f(p)$ would not be defined - or at least not naturally defined by the formula. It may be artificially defined in the construction of the function.

Vertical asymptotes are found by asking when the denominator of the formula of a function goes to zero.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Sometimes one also encounters slanted lines for asymptotes. For example, the function $f(x) = 2x + \frac{1}{x}$ has the line $y = 2x$ as a slanted asymptote, because when x is large $2x + \frac{1}{x}$ and $2x$ are very close together.

In general a line $y = mx + c$ shall be an asymptote to $y = f(x)$ if the expression $g(x) = f(x) - (mx + c)$ approaches 0 as x approaches ∞ (or as x approaches $-\infty$.

If the domain of our function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. Thus we get vertical asymptotes. This reconfirms the fact that **in general**, if $x = p$ is a vertical asymptote then $x = p$ is **not in the domain.**

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

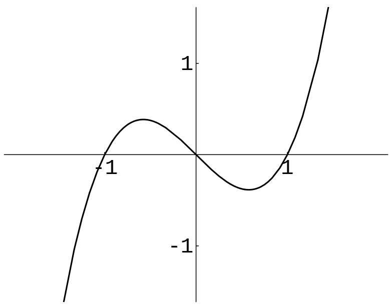
Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." **It is crucial that this property holds throughout the domain of the function. It is not enough to check the equation for a few sample points!** Its graph is symmetric with respect to the y-axis. The most important examples of even functions are: x^n when n is an even number, indeed, this is the reason for the use of the word **even**. Other examples of even functions are: $\cos x$, and $\sin^2 x$. Expressions formed from even functions are also even, for example $x^2 + 1/x^2 - \cos(2x)$.

On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. **The warning about checking the whole domain should be applied here also.** As before, the main examples of odd functions are: x^n when n is an odd number, and as before this is the reason for the terminology.

Some more examples of odd functions are: $\sin x$, and $\tan x$. Unlike even functions, expressions in odd functions may fail to stay odd. For example the square of an odd function is always even! However, an odd function stays off if multiplied by a constant and sums of odd functions are odd.

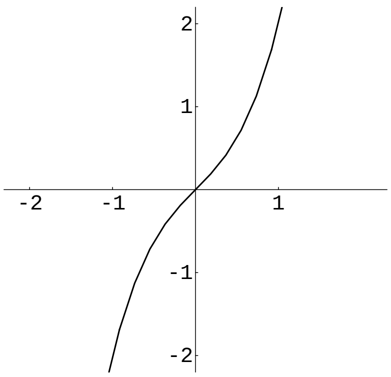
Of course, most functions are neither even nor odd, and do not have any particular symmetry. **A striking fact is that every function $f(x)$ can be written as an odd function $g_1(x)$ and an even function $g_(x)$. You should try to prove this.**

EXAMPLE 13.1. Sketch $y = x^3 - x$.



First, we set $0 = y' = 3x^2 - 1$, which has solutions $x = \pm\sqrt{3}/3 = \pm 0.577$. The corresponding y-coordinates are -0.385 and $+0.385$, i.e., the two "critical points" are $(0.577, -0.385)$ and $(-0.577, 0.385)$. The second derivative test gives $y'' = 6x$, which is positive for the first point and negative for the second. Thus, the first of the two points (in the fourth quadrant) is a local minimum, and the second is a local maximum. Since $y'' > 0$ when $x > 0$ and $y'' < 0$ when $x < 0$, it follows that the curve is concave upward in the right half of the graph and concave downward in the left half. The two halves are separated by the inflection point at the origin. This curve has no asymptotes. It does have symmetry, however, because it is an odd function. Its graph is shown above.

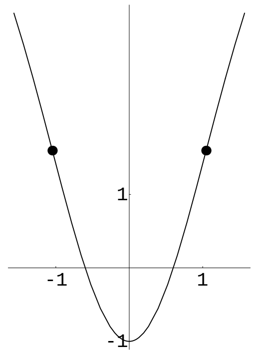
EXAMPLE 13.2. Sketch $y = x^3 + x$.



The only difference with Example 13.1 is the $+$ in front of the x. But this means that the derivative $y' = 3x^2 + 1$ is *never* zero, and hence there are no local maxima or minima. In fact, the function is always increasing, because y' is always positive. The second derivative $y'' = 6x$ is the same as in the last problem, and hence the concavity situation is the same. In particular, this curve also has an inflection point at the origin.

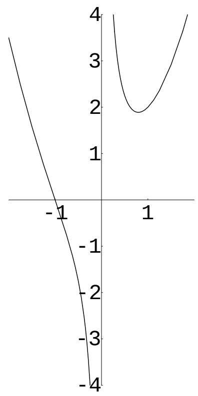
EXAMPLE 13.3. Sketch $y = x^2 - \cos(2x)$ for $-\pi/2 \le x \le \pi/2$.

When we set $0 = y' = 2x + 2\sin(2x)$, we obtain the equality $\sin(2x) = -x$. However, from a quick sketch of the two curves $\sin(2x)$ and $-x$, we immediately see that the only x for which they are equal is $x = 0$. When $x = 0$ the y-coordinate is $0^2 - \cos(2 \cdot 0) = -1$, so our critical point is $(0, -1)$. Since $y'' = 2 + 4\cos(2x)$, which is positive when $x = 0$, the second derivative test tells us that $(0, 1)$ is a local minimum. To find inflection points we set $0 = y'' = 2 + 4\cos(2x)$. This gives $\cos(2x) = -0.5$. Looking at our table in the section on trig functions, we see that in the range from $x = 0$ to $x = \pi/2$ the equality $\cos(2x) = -0.5$ holds when $2x = \frac{2}{3}\pi$, i.e., $x = \pi/3$. Since $\cos(-2x) = \cos(2x)$, the equality also holds when $x = -\pi/3$. Thus, the points $(\pi/3, 1.5966)$ and $(-\pi/3, 1.5966)$ are inflection points. Between these two inflection points the second derivative is positive (concave up), whereas for $x > \pi/3$ and for $x < -\pi/3$ the second derivative is negative (concave downward). Finally, note that $x^2 - \cos(2x)$ is an even function, and so the graph is symmetrical with respect to the y-axis.



EXAMPLE 13.4. Sketch $y = x^2 + \frac{1}{x}$.

Setting $0 = y' = 2x - \frac{1}{x^2}$, we solve this by bringing $1/x^2$ to the left and clearing denominators: $1 = 2x \cdot x^2 = 2x^3$. So $x = \sqrt[3]{0.5} = 0.7937$. The corresponding y-coordinate is 1.8899. Using the second derivative $y'' = 2 + 2 \cdot x^{-3}$, we see that this is a local minimum. To find inflections, we set $2+2/x^3 = 0$. Clearing denominators and solving for x gives $x^3 = -1$, and so $x = -1$. Thus, the point $(-1, 0)$ is an inflection. In this example we have an asymptote when $x = 0$. To the right of the asymptote (i.e., for positive x), the second derivative is always positive; whereas for negative x the second derivative is positive when $x < -1$ and negative when x is between -1 and 0. Thus, the interval $-1 < x < 0$ is a region of downward concavity; the graph is concave upward outside of this interval. Putting all this together leads to the graph at the right.

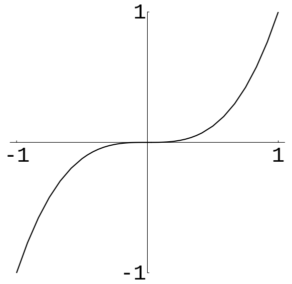


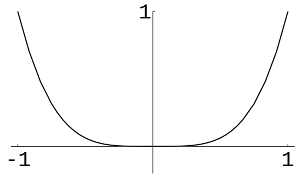
There is another way to think of this example. Our function is the sum of two functions x^2 and $1/x$. The former function is by far the larger of the two when x is large positive or large negative, whereas the reciprocal function is by far the more important when x is near 0. Thus, the graph resembles $1/x$ when x is near 0 and resembles x^2 when x is far from 0. Roughly speaking, the inflection point $(-1, 0)$ and the local minimum $(0.7937, 1.8899)$ mark the transition from behaving like the graph of x^2 to behaving like the graph of $1/x$.

EXAMPLE 13.5. Sketch (a) $y = x^3$ and (b) $y = x^4$.

(a) Setting $0 = y' = 3x^2$, we see that the origin is a possible maximum or minimum. However, the second derivative test tells us nothing, since $y'' = 6x$ also is zero when $x = 0$. In fact, even though $y' = 0$ when $x = 0$, the origin is neither a maximum nor a minimum. Rather, it is a point of inflection, separating the concave downward region in the third quadrant from the concave upward region in the first quadrant.

(b) Again we see that both the first derivative and the second derivative vanish at the origin (and neither derivative is zero anywhere else). This time, however, the origin is a local minimum. Even though the second derivative test doesn't tell us this, we can see directly that, since x^4 is positive for nonzero x, its smallest possible value is when $x = 0$. Note that the origin is *not* a point of inflection, even though $y'' = 0$ there. This is because $y'' = 12x^2 > 0$ both for $x > 0$ and for $x < 0$, so everywhere we have upward concavity. (It is rare for a point where $y'' = 0$ not to be an inflection point; this can occur only when the *third* derivative y''' is also zero at the same point.)

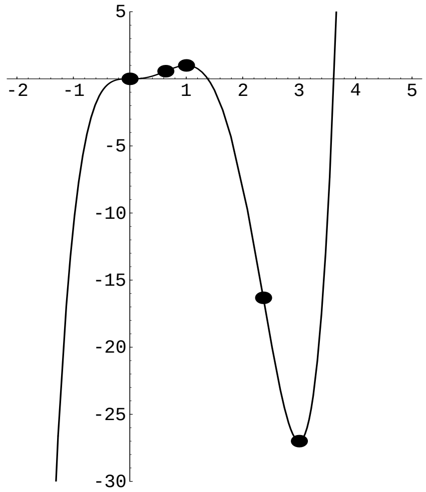




EXAMPLE 13.6. Sketch $y = x^5 - 5x^4 + 5x^3$.

First, we set $0 = y' = 5x^4 - 20x^3 + 15x^2$. To solve this, we factor out what we can, namely $5x^2$. This leaves a quadratic that can be factored either by inspection or by the quadratic formula. The result is $0 = y' = 5x^2(x-1)(x-3)$. Thus, the critical points are $(0,0)$, $(1,1)$, and $(3,-27)$. Using $y'' = 20x^3 - 60x^2 + 30x = 10x(2x^2 - 6x + 3)$, we see from the second derivative test that $(1,1)$ is a local maximum and $(3,-27)$ is a local minimum, but we get no information about $(0,0)$. Setting $0 = y''$ and using the quadratic formula to find the roots of $2x^2 - 6x + 3$, we find the following three points of inflection: $(0,0)$, $(0.634, 0.569)$, $(2.366, -16.32)$.

In a complicated case like this, it is also worthwhile to see what the function is doing when x is large positive or large negative. If x is large, the x^5 term in our function dominates (is greater in absolute value than all the other terms). Thus, the function heads upward steeply into the first quadrant as $x \longrightarrow +\infty$, and it heads steeply down into the third quadrant as $x \longrightarrow -\infty$. Putting this information together, we obtain the graph shown above. The curve is concave downward in the third quadrant, and also between the two points of inflection $(0.634, 0.569)$ and $(2.366, -16.32)$. For $0 < x < 0.634$ and for $x > 2.366$ the curve is concave upward.



HOMEWORK 13.

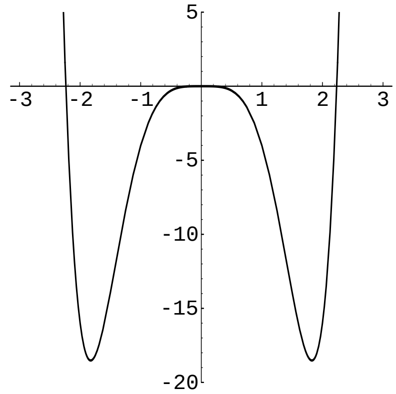
1. Suppose that n is an integer greater than 2. On the curve $y = f(x) = x^n$, what sort of point is the origin? Sketch the curve, and indicate the concavity.

2. In parts (a)-(j) below, find each max/min point (using the second derivative test to be sure what type of point it is), point of inflection, and asymptote (vertical, horizontal, or slanted), if there are any. Also indicate where the curve is concave up and concave down. Sketch the graph.
(a) $y = x^2 - x$, (b) $y = 2 + 3x - x^3$, (c) $y = x^3 - 9x^2 + 24x$, (d) $y = x^4 - 2x^2 + 3$, (e) $y = 3x^4 - 4x^3$, (f) $y = (x^2 - 1)/x$, (g) $y = 3x^2 - (1/x^2)$, (h) $y = \cos(2x) - x$, (i) $y = \sin x + \cos x$, (j) $y = \tan(x/2) - x$.

3. Suppose $\begin{cases} f'(x) > 0 & \text{for } |x| > 2; \\ f'(x) < 0 & \text{for } |x| < 2; \\ f''(x) < 0 & \text{for } x < 0; \text{ and} \\ f''(x) > 0 & \text{for } x > 0 \end{cases}$.
From the given information, sketch a possible graph of $f(x)$. How could your answer vary and still be correct?

4. Suppose $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$. Suppose also that $f(1) = 1$. Sketch a possible graph of $f(x)$, assuming that
i) $f'(1) = 0$, ii) $f'(1) > 0$ and iii) $f'(1) < 0$.

5. At the right is a sketch of $y = x^6 - 5x^4$.



(a) Find the exact coordinates (x, y) of the local maxima and local minima.

(b) For what values of x is the function concave upward?

(c) Find the exact x-coordinates of all points of inflection. (**Hint:** Color the part of the curve that is concave up blue and color the part that is concave down red. Points of inflection occur *only* where the curve *changes* color!)

(d) Explain how you *know from your calculations* (not from the sketch you were given) which of your answers to part (a) are minima and which are maxima.