Contents

CHAPTER 14

Maximum/Minimum Problems

A max/min problem is a story problem where we want to *optimize* something, that is, we want to determine the situation where something is the biggest possible (such as profit, volume of a container, etc.) or where something is the smallest possible (cost, amount of material needed, time of travel, etc.) Solving a max/min problem is the two-step process of (1) reformulating the story problem as a "pure math problem," and (2) solving the "pure math problem". We begin with a brief outline of the procedure, then discuss how to solve the pure math problem, and then present several examples illustrating the procedure.

The **pure math problem** just alluded to was described at the beginning of Chapter 13:

> *Find the maximum (or minimum) value of a function $f(x)$ on the interval from a to b.*

Here is a procedure for solving this type of problem:

1. Find the value of $f(x)$ at each point x between a and b where $f'(x) = 0$.
2. Find the value of $f(x)$ at all points x between a and b where $f'(x)$ does not exist.
3. Find the value of $f(x)$ at the endpoints a and b. (If $a = -\infty$ and/or $b = +\infty$ you have to find the values that $f(x)$ approaches as x approaches $\pm\infty$.)
4. The maximum (minimum) value of $f(x)$ is the largest (smallest) of the values found in steps (1)–(3).

Here are three simple examples which illustrate the procedure.

EXAMPLE 14.1. Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of x on the interval between 0 and 4.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$, and $f(2) = 1$. Next observe that $f'(x)$ is defined for all x, so step (2) gives no x-values. Finally, $f(0) = -3$ and $f(4) = -3$. The largest value of $f(x)$ on the interval between 0 and 4 is $f(2) = 1$.

EXAMPLE 14.2. Again let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval between -1 and $+1$.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$. But $x = 2$ is not in the interval, so we don't use it. Next observe that $f'(x)$ is defined for all x, so step (2) gives no x-values. Finally, $f(-1) = -8$ and $f(1) = 0$. So the largest value of $f(x)$ on the interval between -1 and 1 is $f(1) = 0$.

Note: The maximum value depends not only on the function but on the interval as well. Later in this chapter, we will give examples of story problems where this observation is important.

EXAMPLE 14.3. Find the minimum value of the function $f(x) = 7 + |x - 2|$ for x between 1 and 4.

Note that $|x - 2| = x - 2$ for $x \geq 2$ and $|x - 2| = 2 - x$ for $x < 2$. Thus the function $f(x)$ is given by two separate formulas:

$f(x) = 7 + 2 - x = 9 - x$ for $1 \leq x < 2$ and $f(x) = 7 + x - 2 = 5 + x$ for $2 \leq x \leq 4$.

Thus the derivative of $f(x)$ is given by:

$$f'(x) = -1 \text{ for } 1 < x < 2 \text{ and } f'(x) = 1 \text{ for } 2 < x < 4.$$

Thus, $f'(2)$ is undefined and $f'(x)$ is never zero. So we compute $f(2) = 7$. Finally, we check the end points $f(1) = 8$ and $f(4) = 9$. The smallest of these numbers is $f(2) = 7$, which is, therefore, the minimum value of $f(x)$ on the interval $1 \leq x \leq 4$.

Note that we did not include the endpoints in our investigation of $f'(x)$. They are not needed in our procedure and need special handling even for definition since the derivative at such points can only make sense from one direction

EXAMPLE 14.4. Of all rectangles of area 100, which has the smallest perimeter?

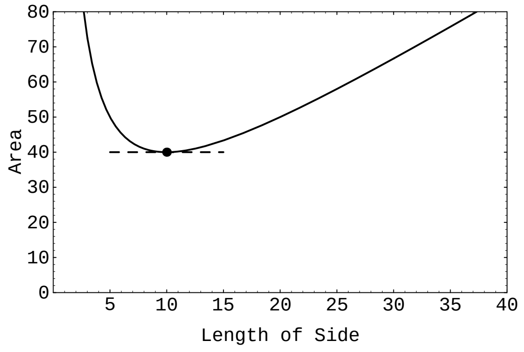
The first step toward solving this problem is to reduce it to a pure math problem. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = \text{perimeter of } x \text{ by } \frac{100}{x} \text{ rectangle} = 2x + 2\frac{100}{x}.$$

The perimeter function is given by the formula $2x + 200/x$, where the *domain* is $x > 0$ (because it makes no sense to talk of a rectangle with negative side). We have reduced the problem to finding the minimum value of $f(x)$ for x between 0 and $+\infty$.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. To solve for x, we clear denominators and then move the constant to the left, obtaining $200 = 2x^2$, which gives us $x = 10$ (actually, $x = \pm 10$, but $x = -10$ does not have practical meaning). Since $f'(x)$ is defined for all $x > 0$, so there are no corners or cusps to check. Our next step is to examine the behavior of $f(x)$ at the endpoints. The sketch this function $f(x) = 2x + 200/x$ in the region of positive x is shown below at the right. We find that as x gets close to 0 the function becomes larger and larger; and the perimeter function also becomes large when x gets very large.

(More precisely, the line $x = 0$ is a vertical asymptote, and the line $y = 2x$ is a slanted asymptote that the function approaches as $x \to \infty$.) Consequently, the minimum value of $f(x)$ occurs at $x = 10$, and the desired rectangle is the 10×10-rectangle, whose perimeter is 40. Any other rectangle of area 100 has perimeter greater than 40. In the same way it is not hard to show that for any fixed area A, the rectangle of area A with smallest perimeter is the square of side $\sqrt{A}$.



Another way to see that $x = 10$ gives a minimum is to notice that $f(x)$ decreases for a while as we get away from 0 ; but if we go too far to the right $f(x)$ is increasing. In other words $f'(x) < 0$ if $x < 10$; while $f'(x) > 0$ if $x > 10$. So it is clear that the point $x = 10$ where $f'(x) = 0$ really is a minimum.

EXAMPLE 14.5. You want to sell a certain number x of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

To answer these questions, we first observe that our profit is equal to the number of items sold times the price per item, minus the total cost to us. If P denotes the price, then this profit is $xP - (2000 + 0.50x)$. (The term in parentheses is fixed cost plus number of items times per item cost.)

To find P in terms of x, we first note that the information given allows us to write x in terms of P: $x = 5000 + 1000(1.50 - P)/0.10$ (i.e., 5000 plus 1000 times the number of multiples of 10 cents that P is below \$1.50). Solving for P and simplifying gives: $P = 2 - 0.0001x$. Thus, the profit function that we want to maximize is:

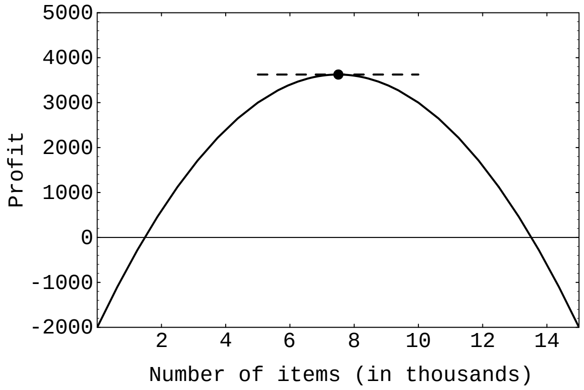
$$\begin{aligned} f(x) &= x\,P - (2000 + 0.50\,x) \\ &= x(2 - 0.0001\,x) - 2000 - 0.50\,x = 1.5x - 0.0001\,x^2 - 2000. \end{aligned}$$

We are now ready to set the derivative equal to zero: $0 = f'(x) = 1.5 - 0.0002x$, from which we obtain $x = 7500$. Thus, we plan to sell 7500 items, which we accomplish by lowering the price to \$1.25. It costs us $\$2000 + 0.50 \cdot 7500 = \$5750$ to produce these items, and we take in $7500 \cdot \$1.25 = \$9375$, leaving us a profit of $f(7500) = \$9375 - \$5750 = \$3625$. This is the largest profit we can get under the assumptions of this problem.

Further analysis. We did not take all the usual steps and we now explain why they do not affect the answer here. For a complete answer, they are necessary!

What is the interval of values for x? Clearly $x > 0$ and x cannot be too large. By looking at the function, clearly x cannot be too large as we cannot be interested in a negative profit function. Thus our interval could be $[0, \infty)$ and neither endpoint will be of interest, since it leads to negative profit. Since we have only one other point (zero of $f'(x)$) to consider, it must be the point with maximum profit.

It is also possible to find a more precise interval for x corresponding to non negative profit, but it is hardly worth the effort!



To be safe, we should check that the value of x which made the profit-function have derivative 0 is really a maximum. It is not hard to compute that $f(x)$ is increasing (i.e., $f'(x) > 0$) when $x < 7500$, and $f(x)$ is decreasing when $x > 7500$, so that $f(x)$ reaches a peak at $x = 7500$. In most of the examples that follow (and when you do the homework problems) it is clear enough from the practical set-up that the value we find really is the maximum point (if we are looking for a maximum) or the minimum point (if we are looking for a minimum). So it is not necessary to belabor the matter. However, we should be cautious, as shown by Example 14.9 below.

EXAMPLE 14.6. Find the largest rectangle that fits inside the graph of the parabola $y = x^2$ below the line $y = a$. (The top side of the rectangle should be on the horizontal line $y = a$ — see the drawing on the following page.)

Here the problem is to find where along the parabola to put the lower-right corner of the rectangle. That corner will determine all the other corners. Thus, if we let (x, x^2) be the point on the parabola where the lower-right corner is put, then the upper-right corner will be (x, a), the lower-left corner will be $(-x, x^2)$, and the upper-left corner will be $(-x, a)$. We can now write the area of the rectangle in terms of x:

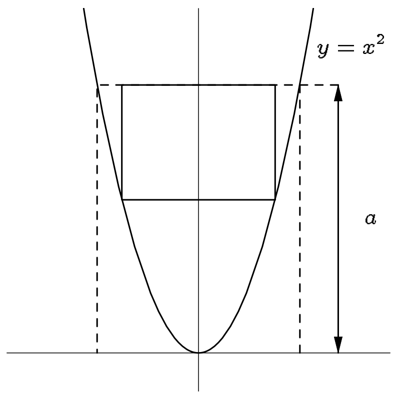
$$f(x) = \text{Area} = (2x)(a - x^2) = 2ax - 2x^3.$$

To maximize this area function, we set

$$0 = f'(x) = 2a - 6x^2,$$

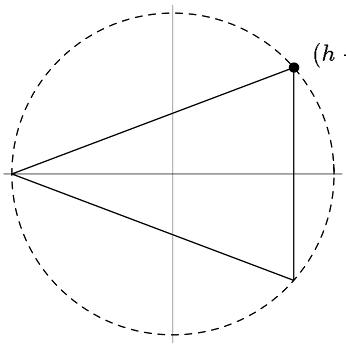
i.e., $x = \sqrt{a/3}$. The resulting rectangle has dimensions $2\sqrt{a/3} \times 2a/3$, and its area is $4(a/3)^{3/2}$.

Additional considerations. It seems reasonable that the biggest possible rectangle should be situated with sides parallel to the x and y axes and corners on the parabola and the line $y = a$ as stipulated. However, this has not been proved. We need to argue this carefully. If the rectangle is situated in a crooked manner, then we should be able to straighten it and then make it slightly bigger! You should think and try to establish these points.



EXAMPLE 14.7. If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\frac{1}{3}\pi r^2 h$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible — equal to R — by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the side view depicted at the right where we have situated the picture in a convenient way relative to the x- and y-axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.



Notice that the function we want to maximize — $\frac{1}{3}\pi r^2 h$ — depends on two variables. This happens in many max/min problems. However, the two variables are related in some way — by a certain condition of the form $C(r,h)$ =constant. So our next step is to find such a condition, and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition can be read off the above drawing: the upper corner of the triangle, whose coordinates are $(h-R,r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Let's do the latter. Then we substitute the result for r in $\frac{1}{3}\pi r^2 h$, obtaining $\frac{1}{3}\pi\big(R^2-(h-R)^2\big)h$.

Notice a simple trick that we used. We do not actually solve for r, but write $r^2 = R^2-(h-R)^2$ and then plug in our result into our function. If you try to solve for r, then you get two roots and you will have to simplify the expression again.

Recall that R is just a constant — the only variable now is h. Let us take $f(h) = \big(R^2-(h-R)^2\big)h = (R^2-h^2+2hR-R^2)h = 2h^2R-h^3$. What happened to the $\frac{1}{3}\pi$? Well, we dropped this constant factor for convenience. The value of h that makes $\frac{1}{3}\pi f(h)$ maximum is the same as the value of h that makes $f(h)$ maximum, so we don't need to keep that constant factor when we take the derivative and set it equal to zero.

We're now ready to set $0 = f'(h) = 4hR - 3h^2 = h(4R-3h)$. This gives us $h = 4R/3$. In other words, we should, in fact, choose h a little larger than R, at the expense of having r somewhat less than R. The exact value of r is determined by the condition above: $(h-R)^2 + r^2 = R^2$. Solving for r here (after setting $h = 4R/3$) gives: $r = (\sqrt{8/9})R$. Finally, the fraction of the sphere occupied by this cone is:

$$\frac{\text{vol of cone}}{\text{vol of sphere}} = \frac{\frac{1}{3}\pi r^2 h}{\frac{4}{3}\pi R^3} = \frac{(8R^2/9)(4R/3)}{4R^3} = \frac{8}{27}.$$

The cone takes up a little less than 30% of the sphere.

Alternative Method: We can use the technique of implicit differentiation to avoid solving for h or r as follows.

We wish to find a maximum value for $F(h,r) = r^2h$ (dropping $\pi/3$ as before), subject to the condition $(h-R)^2 + r^2 = R^2$. Let us think of everything as a function of h and take derivatives with respect to h.

From the derivative of the relation, we get:

$$2(h - R) + 2r dr/dh = 0$$

By calculation of the derivative of $F(h, r)$ and setting it to zero, we get:

$$2rh(dr/dh) + r^2 h = 0$$

EXAMPLE 14.8. You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters for various things: h for the height, r for the base radius, and V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder. Using the area formulas $2\pi r^2$ for the top and bottom together and $2\pi rh$ for the lateral side, we find that the total cost of the material is

$$c(2\pi rh) + Nc(2\pi r^2).$$

Here c and N are constants that are given to us, but r and h are variables. To eliminate one of the two variables, we use the *condition* that the volume must be V, i.e., $\pi r^2 h = V$. We use this relation to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). Thus we solve $h = V/(\pi r^2)$ and substitute in the cost function. As a result, we consider the function:

$$f(r) = \pi r(V/\pi r^2) + N\pi r^2 = \frac{V}{r} + N\pi r^2.$$

(What happened to the c and the 2 in the above formula? For convenience, we can factor out the constant $2c$ and drop it from the expression to be minimized, as in the previous example.) We now set $0 = f'(r) = -V/r^2 + 2N\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Next, we know that $h = V/\pi r^2$, and so the ratio h/r is equal to

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi\big(V/(2N\pi)\big)} = 2N,$$

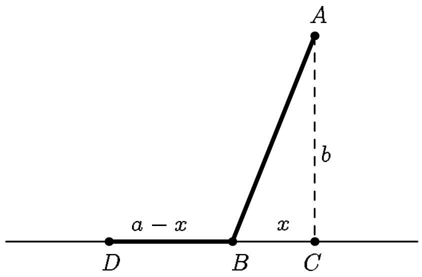
i.e., we choose the height to be $2N$ times the base radius for the most economical container.

EXAMPLE 14.9. Suppose you want to reach a point A that is located across the sand from a nearby road. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. When you're traveling at constant velocity, you have: time=distance/velocity.

In this problem (see the diagram at the right), we travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, we want to minimize the time-of-travel function

$$f(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$



We set $f'(x) = 0$ and next simplify by moving $1/v$ to the left and cross-multiplying:

$$0 = f'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}}, \qquad \text{so that} \qquad w\sqrt{x^2+b^2} = vx.$$

We now square both sides and solve for x, obtaining

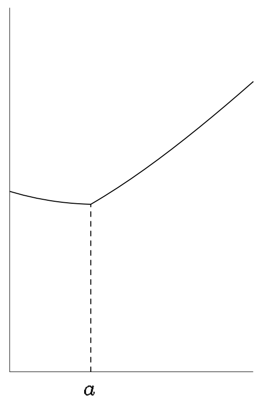
$$w^2b^2 = (v^2 - w^2)x^2, \qquad \text{i.e.,} \qquad x = \frac{wb}{\sqrt{v^2-w^2}}.$$

This is our solution.

Notice, by the way, that our answer turns out not to depend on a; that is, the point where you should turn off the road does not depend on how far back you started. But something's not completely right here. What if a is very small, i.e., suppose $a < wb/\sqrt{v^2-w^2}$. Then our solution is saying that we should go *back* along the road, and then head across the sand. This makes no sense. What have we done wrong in the case of small a? The problem is with the first term $(a-x)/v$ in our function $f(x)$. If $a < x$ that gives us negative time. What we really should have is $|a-x|/v$. In the case when $a < wb/\sqrt{v^2-w^2}$, the graph of our revised function

$$f(x) = \frac{|a-x|}{v} + \frac{\sqrt{x^2+b^2}}{w}$$

is given at the right. The minimum of this function cannot be obtained by setting $f'(x) = 0$ (in fact, the derivative of this function is never 0). Rather, it occurs at a point where there is no derivative — namely, at the "corner" at $x = a$. In practical terms, this says that if $a < wb/\sqrt{v^2-w^2}$, then what you should do is head out immediately across the sand in a straight line to A.



SUMMARY — STEPS IN A MAX/MIN STORY PROBLEM:

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.

2. Write a formula for what you are maximizing or minimizing.

3. Express that formula in terms of only one variable, that is, in the form $f(x)$.

4. Set $f'(x) = 0$ and solve.

5. Be sure that your answer makes sense, and that the max/min value is not really somewhere else.

HOMEWORK 14.

Part A. Basic problems:

1. Let $f(x) = \begin{cases} 1 + 4x - x^2 & \text{for } x \leq 3 \\ \dfrac{x+5}{2} & \text{for } x > 3 \end{cases}$

(a) Find the maximum value of $f(x)$ for x between 0 and 4.
(b) Find the minimum value of $f(x)$ for x between 0 and 4.
(c) Graph $f(x)$ to check your answers.

2. Find the dimensions of the rectangle of largest area having fixed perimeter P. (Compare with Example 14.4 above.)

3. A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.)

4. You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in ft^2) for the play area?

5. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make?

6. Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle).

7. For a cylinder with given surface area S, find the ratio of height to base radius that maximizes the volume.

8. You want to make cylindrical containers of a given volume V using the least amount of construction material. The lateral side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius.

9. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.)

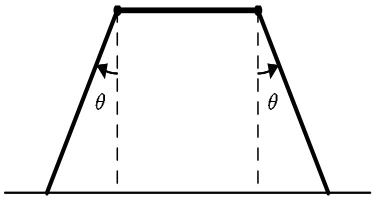
10. In Example 14.9, what happens if $w > v$ (i.e., your speed on sand is greater than your speed on the road)?

11. A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical bottom. (The hemispherical bottom has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the bottom is twice as great as the cost per unit area of the side, and the container is made with no top; (b) the same as in (a), except that the container is made with a circular top, for which the cost per unit area is 1 1/2 times the cost per unit area of the side.

12. (a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b? Note that the side x of the square to be cut from each corner must be less than $a/2$ and $b/2$, and in this way you can eliminate one of the two solutions of $f'(x) = 0$. In addition, use the second derivative test to be sure your value of x really gives the maximum volume.

13. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is some fraction). If the distance from top to bottom (across both the rectangle and the semicircle) is fixed, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light. For what range of values of k does this question make sense?

14. In the same situation as Problem 4, suppose that instead of a rectangular shape you decide to make the play area in the form of a trapezoid, using three sections of fence of length $l/3$ each (see the diagram at the right). Find the angle θ that the sides should be turned outward in order for the enclosed area to be maximal, and find the maximum play area of this form. Is the maximum play area greater or less than in Problem 4?



Part B. Additional Practice (do these as soon as possible, certainly before the final exam):

15. A terrorist sees a plane flying away from him at 500 mph at a height of 2 miles. At time $t = 0$ it is flying over a point on the ground which is 1.5 miles from him. At that instant he shoots a missile at the plane, but because of nervousness and perhaps a subconscious sense of morality, he mistakenly aims directly at the present location of the plane, rather than slightly ahead. For this reason he misses the plane. The purpose of this problem is to determine by how much. Suppose the missile travels at 2000 mph, and **neglect both air resistance and gravity**. First find parametric equations for both the plane and the missile. Then find the time at which the two objects are closest. (Hint: Use the distance formula, and save yourself unnecessary algebra by finding the time at which the *square* of the distance is minimal.) By how much does the missile miss the plane?

16. You are designing a poster to contain a fixed amount A of printing (measured in square inches) and have margins of a inches at the top and bottom and b inches at the sides. Find the ratio of vertical dimension to horizontal dimension of the printing on the poster if you want to minimize the amount of posterboard needed.

17. The strength of a rectangular beam is proportional to the product of its width times the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r.

18. What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?

19. The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back.

20. Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side.

21. A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone.

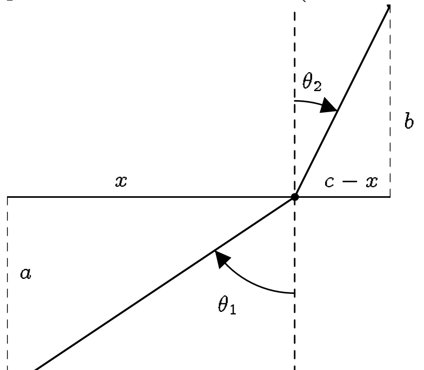
22. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? Compare your answer with the answer to Example 14.7.

23. Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source.

24. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.

25. How are your answers to Problem 5 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced.

26. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed v_1 on land and at a slower speed v_2 in the water. Your perpendicular distance from the side of the pool is a, the child's perpendicular distance is b, and the distance along the side of the pool between the closest point to you and the closest point to the child is c (see the drawing). Without even stopping to do any calculus, you instinctively choose the quickest route (shown in the diagram at the right) and save the child. Our purpose is to derive a relation between the angle θ_1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ_2 your path makes with the perpendicular when you're in the water. To do this, let x be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in



terms of x (and also the constants a, b, c, v_1, v_2). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water.