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CHAPTER 16

Indefinite Integrals

By an **antiderivative** of a function $f(x)$ we mean a function $F(x)$ whose derivative is $f(x)$. The reason for the term is that we're going in the opposite direction from taking the derivative. For example, an antiderivative of the function $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$, since the function $F(x) = \frac{1}{3}x^3$ has derivative $F'(x) = x^2$. But note that $\frac{1}{3}x^3 + 5$ is also an antiderivative of x^2, and in fact so is $\frac{1}{3}x^3 + C$ for any constant C. This is because a constant has derivative zero, and so we can add it to our function $F(x)$ without affecting the derivative $F'(x)$.

Notice that a constant C is the *only* thing that can be added to $F(x)$ without affecting the derivative. If it were possible to add some other function $G(x)$, then we would have to have $F'(x) + G'(x) = F'(x)$, i.e., $G'(x) = 0$. But any function that is not constant must either increase or decrease for certain values of x, and at such x it has a nonzero derivative. To put this another way, the *only* function $G(x)$ whose graph is always pointing horizontally is a constant function. **(There is a subtle point involved here. If you consider any interval contained in the domain of the function $f(x)$, then $G'(x)$ exists and is zero on such an interval, so we can argue that $G(x)$ is a constant. However, if the domain of $f(x)$ is split into disjoint intervals, then the function $G(x)$ can take on distinct constant values on them. Thus it becomes a step function rather than a constant. Usually, we have a single interval in mind and this point is not stressed at all.)**

Thus, if we find one antiderivative $F(x)$ for $f(x)$, then we can write the most general antiderivative as $F(x) + C$, where C is an arbitrary constant. (As explained above, this is valid on some interval in the domain of $f(x)$.)

The notation for the general antiderivative of $f(x)$ is $\int f(x)\,dx$, where the *integral sign* $\int$ says "take the antiderivative," and the "dx" indicates what the variable is. Thus, we write

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

The expression $\int f(x)\,dx$ is said to be an **indefinite integral** of the function $f(x)$. The term "indefinite" is used because of the constant C that always appears in the answer to an antiderivative problem. C is called the "constant of integration." indexconstant of integration The function $f(x)$ that is being integrated is called the "integrand."

Another way to state an antiderivative problem is to say that we want to find a function $y = F(x)$ that satisfies the equation $y' = \frac{dy}{dx} = f(x)$. Such an equation $y' = f(x)$, in which y stands for an *unknown function*, is called a **differential equation.** In general, *a differential equation is an equation for an unknown function in which the derivative (and maybe the second or higher derivatives) occur.* Differential equations will be treated at greater length in higher courses.

The above example $f(x) = x^2$ suggests how we can take the antiderivative of any power function x^n. Just increase the power by 1, and divide by $n+1$ in order to cancel the factor of $n+1$ that arises when taking the derivative of x^{n+1}. In other words,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \qquad \text{for any } n \text{ except } -1$$

(the formula makes no sense if $n = -1$; the antiderivative of the function x^{-1} is a special case, and it will be dealt with in the exercises below. This formula can be regarded as the reverse of the nx^{n-1} rule for taking derivatives. As in that rule, n is allowed to be negative or fractional.

The same formula can be used with other names for the x-variable, for example,

$$\int t^n\,dt = \frac{1}{n+1}t^{n+1} + C; \qquad \int u^n\,du = \frac{1}{n+1}u^{n+1} + C; \qquad \text{etc.}$$

antiderivatives, like derivatives, are **linear**. This means that if we know the antiderivatives of a bunch of functions, then we can compute the antiderivative of any linear combination of those functions. For example, using the above formula for $\int x^n dx$, we can find the antiderivative of any polynomial.

EXAMPLE 16.1.
(a) $\int x^3 - 6x^2 + 3x + 4\,dx = \frac{1}{4}x^4 - 6\cdot\frac{1}{3}x^3 + 3\cdot\frac{1}{2}x^2 + 4x + C = \frac{1}{4}x^4 - 2x^3 + 1.5x^2 + 4x + C$.
(b) $\int\left(\sqrt{x} - \frac{1}{x^2}\right)dx = \int\left(x^{1/2} - x^{-2}\right)dx = \frac{1}{3/2}x^{3/2} - \frac{1}{-1}x^{-1} + C = \frac{2}{3}x^{3/2} + \frac{1}{x} + C$.

In practical problems, often the time t is playing the role of the x-variable, and the function $f(t)$ to be integrated is the rate of change of something. Just as the derivative procedure takes us from the distance function $s(t)$ to the velocity function $v(t)$ and from the velocity to the acceleration $a(t)$, the antiderivative procedure takes us from $a(t)$ to $v(t)$ and then to $s(t)$. Below we will give examples of this.

Being able to find antiderivatives is at least as important as being able to find derivatives, for the following reason. Very often in the real world we do not start out knowing a formula for something, but rather we know some information about how it is changing, i.e., about its derivative. For example, in physics we have Newton's law of motion that says that Force=Mass×Acceleration. Usually the basic thing we know about is the forces. This immediately gives us the acceleration function $a(t)$. From that we want to "go backwards" (i.e., find antiderivatives) to get the velocity function $v(t)$, and then finally what we are really interested in, which is the location $s(t)$ of the object. A basic example of this is a falling body problem.

1. From Acceleration to Velocity to Distance; Falling Bodies

Earlier in the course, we encountered the falling body formula $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ for the height of an object above the ground after t seconds. However, we did not give any explanation of how that formula arises — in fact, at first the whole formula (especially the $\frac{1}{2}$ at the beginning) looks "pulled out of a hat." Now we are ready to derive the formula starting from a basic principle of physics: *near the surface of the earth gravity causes a constant force of acceleration.* That "gravitational acceleration" g is 9.806194 m/sec^2 (we'll use simply 9.8 m/sec^2 or 32 ft/sec^2). More precisely, the gravitational acceleration is $-g$ if we take upward to be the positive direction; it is g if downward is taken to be the positive direction. We shall usually take upward to be the positive direction. If we are near the surface of some other planet, the same principle applies, but with a different value of g.

The basic principle can be restated as follows: if no force other than gravity is affecting the object's motion (in particular, we neglect air resistance), then the acceleration function

$a(t)$ is the constant $-g$: $a(t) = -g$. Our object is to derive the formula for the height function $s(t)$. We do this in two steps:

Finding $v(t)$. Since $dv/dt = a(t)$, it follows that $v(t)$ is an antiderivative of $a(t) = -g$. But the antiderivative of the constant $-g$ is $-\int g\,dt = -gt + C$. Thus, $v(t) = \int a(t)\,dt = -\int g\,dt = -gt + C$. How do we find C? Here we have to suppose that we know "initial information," i.e. the value of $v(t)$ when $t = t_0$, which we call v_0. (Often, but not always, the initial time t_0 will be 0.) In order for $v(t_0)$ to equal v_0, we must have $v_0 = -g \cdot t_0 + C$, i.e., our constant of integration is $v_0 + gt_0$. In general, *you solve for the constant of integration using initial information.* Thus, $v(t) = -gt + v_0 + gt_0 = -g(t - t_0) + v_0$. (In this simple situation, we could have obtained the same result by common sense: if something starts at value v_0 and decreases at *constant* rate g, then after $t - t_0$ seconds its value is $v_0 - g(t - t_0)$.) If the initial time is $t = 0$, as is often the case, then $v = -gt + v_0$.

Finding $s(t)$. Let us suppose that $t_0 = 0$, i.e., the starting time for the falling body is taken to be $t = 0$. Since $ds/dt = v(t)$, it follows that $s(t)$ is an antiderivative of $v(t) = -gt + v_0$, i.e.,

$$s(t) = \int v(t)\,dt = \int -gt + v_0\,dt = -g\frac{1}{2}t^2 + v_0 t + C_1 = -\frac{1}{2}gt^2 + v_0 t + C_1.$$

(We used C_1 rather than C for the constant of integration because it has no relation to the C in the last paragraph.) To find the constant of integration C_1 we must again have initial information. Suppose we know the height at time 0: $s(0) = s_0$. Substituting this value for $s(t)$ when $t = 0$ gives $s_0 = s(0) = -\frac{1}{2}g \cdot 0^2 + v_0 \cdot 0 + C_1 = C_1$, i.e., $C_1 = s_0$. We obtain our final formula: $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$. This completes the derivation of the falling body formula.

Having the derivative relations $ds/dt = v(t)$ and $dv/dt = a(t)$, together with a formula for $a(t)$, we can work back to find $s(t)$, provided we also have two pieces of initial information that allow us to find the two constants of integration that arise. We assumed that the two known values are the initial velocity and the initial height.

EXAMPLE 16.2.

Suppose that, starting at $t = 1$, your acceleration is given by the formula $a(t) = 1/t^3$. Your velocity is 2 units/min at that instant, and your location is 0. Find the distance function $s(t)$.

The first step is to find $v(t) = \int a(t)\,dt = \int t^{-3}dt = \frac{1}{-2}t^{-2} + C = -\frac{1}{2t^2} + C$. The initial information is that $v(1) = 2$. Hence, substituting $t = 1$ gives: $2 = v(1) = -1/2 + C$, and so $C = 2.5$. Thus $v(t) = -\frac{1}{2t^2} + 2.5$. The second step is to find

$$s(t) = \int v(t)\,dt = \int (-\frac{1}{2} \cdot t^{-2} + 2.5)dt = -\frac{1}{2} \cdot \frac{1}{-1}t^{-1} + 2.5t + C_1 = \frac{1}{2t} + 2.5t + C_1.$$

Now we use the initial information that $s(1) = 0$: $0 = s(1) = \frac{1}{2} + 2.5 \cdot 1 + C_1$. Solving for C_1, we obtain $C_1 = -3$. Thus, $s(t) = \frac{1}{2t} + 2.5t - 3$.

We can check this answer by verifying three things: (i) it has the right second derivative $\ddot{s} = 1/t^3$; (ii) its value when $t = 1$ is 0; and (iii) the value of its derivative when $t = 1$ is 2. These computations are easy to do, and they give us the assurance that we did not make any careless errors in the last paragraph.

2. antiderivative of sin$x$ and cos$x$

From the derivative formulas $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$ we obtain the corresponding antiderivative formulas:

$$\int \cos x \, dx = \sin x + C, \qquad \int \sin x \, dx = -\cos x + C.$$

Notice that the derivative of sin is cos, but its antiderivative is *minus* cos.

3. Substitution

In practical applications we rarely encounter the pure sine or cosine function. As we have seen, it is useful to be able to work with the more general sinusoidal function

$$f(t) = A \sin\left(\frac{2\pi}{B}(t - C)\right) + D,$$

where A, B, C, and D stand for various constants, called the amplitude, period, phase shift, and vertical shift, respectively. In order to find antiderivatives of functions of this type and many other functions, we describe a basic method for finding antiderivatives called the **method of substitution.**

In the method of substitution we choose a new variable, which will usually be denoted u, and set u equal to an "inside function" (as in the chain rule), so that the function we are integrating takes on a simpler appearance. Then **EVERYTHING** must be converted to the new variable u. Here "everything" includes the "dx" (or "dt"). We convert dx to du by taking the derivative du/dx. We illustrate by some examples. In each case the reason for having to use du/dx to convert dx to du will become apparent when we check our answer. **(Recall that the answer to an antiderivative problem can always be checked by taking the derivative and verifying that we get our original function back again. It is also strongly recommended that you do this whenever possible.)**

EXAMPLE 16.3. If a and b are constants, find formulas for:

$$\text{(a)} \int (ax+b)^n \, dx \text{ (for any } n \neq -1); \quad \text{(b)} \int \sin(ax+b)dx; \quad \text{(c)} \int \cos(ax+b)dx.$$

(a) We substitute $u = ax + b$. We also find du/dx, which in this case is simply the constant a. We then treat du and dx as if they were algebraic quantities, writing $\frac{du}{dx} = a \Longrightarrow du = a\,dx$, and so $dx = \frac{1}{a}\,du$. Thus, in our integral we change $ax + b$ to u and we change dx to $\frac{1}{a}\,du$, thereby obtaining an integral purely in terms of the new variable u:

$$\int (ax+b)^n dx = \int u^n \frac{1}{a}\,du = \frac{1}{a} \cdot \frac{1}{n+1} u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

(Your final answer should be written in terms of x, not u.)

Explanation: This might be a good place to explain what the substitution means and why it works. To begin with, we have an expression $f(x)\,dx$ inside the integral sign and when we get through with the substitution we convert this to an expression $g(u)\,du$. Where should two such expressions be the same? The answer is that $f(x)$ must be equal to $g(u(x))\frac{du(x)}{dx}$.

First suppose that this is true and we find an antiderivative $G(u)$ for the function $g(u)$. Then we can see that $G(u(x))$ is an antiderivative for $f(x)$, since $\frac{d}{dx}(G(u(x)) = G'(u(x))u'(x) = g(u(x))\frac{du(x)}{dx} = f(x)$, so $G(u(x))$ is an antiderivative of $f(x)$ as needed.

Clearly, the answer $G(u)$ that we want, is given by the symbol $\int G(u)\,du$ and this solves the problem $\int f(x)\,dx$. So we convert the equation

$$f(x) = g(u(x))\frac{du(x)}{dx} \text{ to the equation } f(x)dx = g(u(x))d(u(x)) \text{ or } g(u)du.$$

It is possible to give completely rigorous meaning to the symbols dx and du as abstract (universal) derivatives (called differentials) and make precise sense out of the equation $du = \frac{du}{dx}dx$, but we will not attempt to go into the details here. We are only stating that the symbol dx in $f(x)dx$, is more than just a whimsical notation!

It is instructive to check this answer by taking its derivative. We find ourselves using the chain rule — as always happens when checking an antiderivative that was found by a u-substitution:

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \cdot a = (ax+b)^n.$$

Thus, the $\frac{1}{a}$ term that we obtained when converting dx to du is needed to cancel the factor of $a = du/dx$ that comes out from the chain rule.

This also explains the importance of the dx (or dt or du, depending on what the variable of integration is) inside the integral. Keeping track of that part of the integral in the course of a u-substitution will give you the right antiderivative. Do not get into the careless habit of omitting the dx (or dt or du) in an integral.

(b) Again we take $u = ax + b$, so that $dx = \frac{1}{a}du$, and our integral becomes

$$\int \sin(ax+b)\,dx = \int \sin u\,\frac{1}{a}du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

You should check this formula by taking the derivative of $-\frac{1}{a}\cos(ax+b)$.

(c) This works just like part (b). The result is $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$.

EXAMPLE 16.4. Find:

$$\text{(a)} \int x\cos(x^2)\,dx; \qquad \text{(b)} \int \frac{\cos(\pi t)dt}{\sin^2(\pi t)}; \qquad \text{(c)} \int \frac{\sin\frac{\pi t}{10}\,dt}{\sqrt{1+\cos\frac{\pi t}{10}}}.$$

(Note that in parts (b)–(c) we have brought the dt into the numerator, instead of putting it at the far right of the integral.)

(a) Here we take u equal to the expression inside the cosine, namely, $u = x^2$. Then $du/dx = 2x$, and so $du = 2x\,dx$. Fortunately, the integral contains x as well as dx, and so, bringing the x and dx together, we can replace the product $x\,dx$ by $\frac{1}{2}du$:

$$\int x\cos(x^2)\,dx = \int(\cos u)\,\frac{1}{2}du = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

Notice how, when we check by taking $\frac{d}{dx}\left(\frac{1}{2}\sin(x^2)\right)$, we get $x\cos(x^2)$ because of the chain rule.

(b) Here we can greatly simplify the denominator if we take $u = \sin(\pi t)$. Then $\frac{du}{dt} = \pi\cos(\pi t)$, and so $du = \pi\cos(\pi t)dt$, which — fortunately for us — happens to be what is in the numerator (except for the constant π, which does not cause us any problems). That is, we can replace $\cos(\pi t)dt$ by $\frac{1}{\pi}du$:

$$\int \frac{\cos(\pi t)\,dt}{\sin^2(\pi t)} = \int \frac{\frac{1}{\pi}du}{u^2} = \int \frac{1}{\pi}u^{-2}du = \frac{1}{\pi}\frac{1}{-1}u^{-1} + C = -\frac{1}{\pi u} + C = -\frac{1}{\pi\sin(\pi t)} + C.$$

(c) Here we can obtain a much simpler integrand by substituting u in place of the expression in the radical: $u = 1 + \cos\frac{\pi t}{10}$. Then $\frac{du}{dt} = 0 - (\sin\frac{\pi t}{10})(\frac{\pi}{10})$, and so $du = -\frac{\pi}{10}\sin\frac{\pi t}{10}\,dt$. So the numerator in the integral is $-\frac{10}{\pi}du$, and we obtain

$$\int \frac{-10/\pi}{\sqrt{u}}du = \int -\frac{10}{\pi}u^{-1/2}du = -\frac{10}{\pi}\frac{1}{1/2}u^{1/2} + C = -\frac{20}{\pi}\sqrt{1+\cos\frac{\pi t}{10}} + C.$$

HOMEWORK 16.

1. Find the anti-derivative of each of the following:

(a) $\frac{1}{x^4} + x\sqrt{x} + 5$; (b) $(1-t)^9$; (c) $(x^2+1)^2$;
(d) $1/\sqrt[3]{1-5t}$; (e) $\sin^3 x \cos x$; (f) $x\sqrt{100-x^2}$;
(g) $x^2/\sqrt{1-x^3}$; (h) $\sin\left(\frac{2\pi}{B}(t-C)\right) + D$; (i) $\cos(\frac{t}{10})\sqrt{1-\sin(\frac{t}{10})}$;
(j) $\cos(\pi t)\cos\bigl(\sin(\pi t)\bigr)$.

2. Suppose that the $\cos(\pi t)$ term were absent in the numerator of Example 16.4(b). Find the anti-derivative by substituting $u = \pi t$ and then using the formula for the derivative of cot.

3. Find the formula for $s(t)$ for a falling body if you know $v_0 = v(t_0)$ and $s_0 = s(t_0)$ at some time t_0 (not necessarily at time 0).

4. Find the distance function $s(t)$ if $a(t) = 1/t^4$ and both velocity and distance are zero at time $t = 1$.

5. Find the distance function $s(t)$ if the acceleration function is given by the following formula, and if the initial velocity v_0 and the initial distance s_0 at time $t = 0$ are known:

(a) $a(t) = kt$ (where k is a constant); (b) $a(t) = \sqrt{2t+1}$;
(c) $a(t) = \sin(5t)$; (d) $a(t) = \sin(0.1t)/\cos^3(0.1t)$.