Contents

CHAPTER 18

The Fundamental Theorem

The so-called "fundamental theorem of calculus" says that taking the derivative and taking the integral are inverse processes. More precisely, suppose you define a function using a definite integral as follows:

$$F(u) = \int_a^u f(x)\,dx\,,$$

where u is the variable and x is only a "dummy variable" (variable of integration). This $F(u)$ is simply the "area function" we defined before (except that we called it $A(u)$ rather than $F(u)$ in the last section). In the last section we showed that the derivative of this area function is $f(u)$. Thus, taking the derivative of F returns us to the original function f that we integrated to get F. In words, the fundamental theorem says that *if a function is defined by taking the definite integral from a fixed lower limit of integration up to a variable upper limit of integration, then its derivative is obtained by simply dropping the integral sign and taking the integrand with the dummy variable replaced by the variable in the upper limit of integration.* In symbols,

$$\frac{d}{du}F(u) = \frac{d}{du}\int_a^u f(x)dx = f(u).$$

EXAMPLE 18.1. Find d/du of the function $F(u)$ defined as

$$\text{(a)} \int_0^u x^2\,dx;\ \text{(b)} \int_1^u x^2\,dx;\ \text{(c)} \int_0^u \sqrt{1+\tan x}\,dx.$$

In parts (a)–(b), check your answer by finding a formula for $F(u)$ and computing the derivative of this formula.

(a) By the fundamental theorem, $\frac{d}{du}\int_0^u x^2dx = u^2$. Let us check this by finding $F(u) = \int_0^u x^2dx = \frac{1}{3}x^3\,\Big|_0^u = \frac{1}{3}u^3$. Since $\frac{d}{du}\left(\frac{1}{3}u^3\right) = u^2$, the answer checks.

(b) By the fundamental theorem, $\frac{d}{du}\int_1^u x^2dx = u^2$, the same answer as in part (a). But our function $F(u)$ is not quite the same. Namely,

$$F(u) = \int_1^u x^2\,dx = \frac{1}{3}x^3\Big|_1^u = \frac{1}{3}u^2 - \frac{1}{3}\,.$$

Of course, this $F(u)$ still has derivative u^2. This example illustrates the fact that the derivative of a function of the form $F(u) = \int_a^u f(x)dx$ does not depend on what constant value we choose for the lower limit of integration a.

(c) By the fundamental theorem, $\frac{d}{du}\int_0^u \sqrt{1+\tan x}\,dx = \sqrt{1+\tan u}$. Here we cannot find a formula for $F(u)$, since we do not know how to find the anti-derivative of a complicated function like $\sqrt{1+\tan x}$. But thanks to the fundamental theorem, if all we want is the *derivative* of $F(u)$ (and not $F(u)$ itself), we never have to worry about anti-derivatives. The derivative of $F(u)$ can be determined right away without any work.

1. The Fundamental Theorem together with the Chain Rule

Suppose that u is a function of some other variable, like time, and we want to find $\frac{d}{dt}F(u(t)) = \frac{d}{dt}\int_a^{u(t)} f(x)dx$. This can be done using the chain rule:

$$\frac{dF}{dt} = \frac{dF}{du}\cdot\frac{du}{dt} = f(u(t))u'(t).$$

EXAMPLE 18.2. Suppose that in Example 1(c) we have $u = t^2$, and we want to find the time derivative of the integral. Then

$$\frac{d}{dt}\int_0^{t^2}\sqrt{1+\tan x}\,dx = \left(\frac{d}{du}\int_0^u \sqrt{1+\tan x}\,dx\right)\frac{du}{dt} = \sqrt{1+\tan u}\,2t = 2t\sqrt{1+\tan(t^2)}.$$

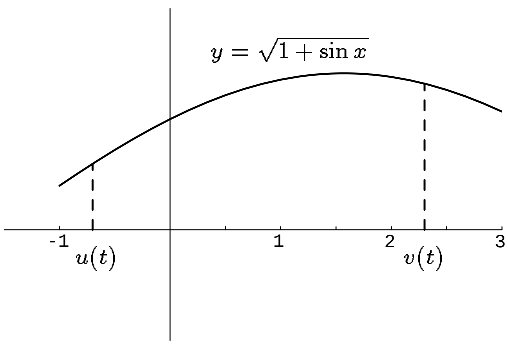
If the function $F(u(t))$ is defined by an integral in which $u(t)$ occurs in the *lower* limit of integration, then simply reverse the upper and lower limits, and put in a minus sign, i.e., $\int_u^a f(x)dx = -\int_a^u f(x)dx$ (see property (3) of definite integrals). What if a function appears in *both* limits of integration, i.e., what if you want to take d/dt of $\int_{u(t)}^{v(t)} f(x)dx$, where u and v are each functions of t? Then choose a constant c and use properties (2) and (3) of definite integrals with a replaced by u and b replaced by v:

$$\int_u^v f(x)\,dx = \int_c^v f(x)\,dx + \int_u^c f(x)\,dx = \int_c^v f(x)\,dx - \int_c^u f(x)\,dx\,.$$

We already know how to take the derivative of each of the two terms on the right. Thus,

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(x)dx = f(v(t))\frac{dv}{dt} - f(u(t))\frac{du}{dt}.$$

EXAMPLE 18.3. Suppose that the curve $\sqrt{1+\sin x}$ forms the top of a shape that is bounded below by the x axis and on the left and right by vertical lines whose position changes with time. Suppose that the vertical line on the left is moving steadily to the right at speed 0.1, starting at -1 at time $t = 0$. Meanwhile, the vertical line on the right is oscillating between 1 and 3 according to the formula $2 + \sin(0.25\pi(t-6))$. Find a formula in terms of t for the rate of change of the area of the shape between the two vertical lines.



We have

$$
\begin{aligned}
&\frac{d}{dt}\int_{-1+0.1t}^{2+\sin(0.25\pi(t-6))}\sqrt{1+\sin x}\,dx\\
&\quad=\sqrt{1+\sin(2+\sin(0.25\pi(t-6)))}\left(\frac{d}{dt}(2+\sin(0.25\pi(t-6))\right)\\
&\qquad\quad-\sqrt{1+\sin(-1+0.1t)}\left(\frac{d}{dt}(-1+0.1t)\right)\\
&\quad=0.25\pi\cos(0.25\pi(t-6))\sqrt{1+\sin(2+\sin(0.25\pi(t-6)))}\\
&\qquad\quad-0.1\sqrt{1+\sin(-1+0.1t)}.
\end{aligned}
$$

HOMEWORK 18.

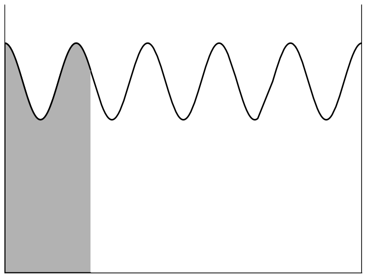
1. Find d/du of the following functions of u:

$$\text{(a)} \int_0^u \sin^2 x\,dx; \quad \text{(b)} \int_{-1}^u \sqrt{1+t+t^2}\,dt; \quad \text{(c)} \int_u^\pi \sqrt{1+\sin^2 y}\,dy;$$
$$\text{(d)} \int_{-u}^u \sin^2 x\,dx; \quad \text{(e)} \int_{-u}^u \sin^3 x\,dx.$$

2. Find d/dt of the following functions of t:

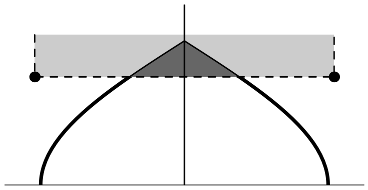
$$\text{(a)} \int_1^{a+bt} \frac{dx}{x}; \qquad \text{(b)} \int_0^{-\frac{1}{2}gt^2+v_0t+s_0} \sin^2 y\,dy;$$
$$\text{(c)} \int_{t^2}^{t^3} \cos(x^2)dx; \quad \text{(d)} \int_{a+bt}^{A\sin(\alpha(t-C))} \sqrt{1+u^2}du.$$

3. The stage of a puppet show has its upper edge defined by $y = 20+10\cos^2(0.1\pi x)$ (the units are cm). Its left side is the y-axis and its lower edge is the x-axis (see the drawing). After the show the curtain closes from left to right at 2 cm/sec. (a) Write a definite integral for the curtain's area as perceived by the audience after t seconds. (b) What is the rate of change of the curtain's area at time t?



4. The same as Problem 3, except that now the upper edge is defined by $y = 40/\left(1+(0.01x-1)^2\right)$.

5. The opening to a stage has its highest point 48 feet above the stage, it is symmetrical about the center line (which we'll take as the y-axis), and it is bounded by the flat stage and the curve $x = \pm 48\cos(\pi y/96)$ (see the drawing). A curtain with flat bottom is lowered by attaching two heavy weights, one at each end, and letting them fall (with acceleration 32 ft/sec^2), bringing down the curtain.



(a) Write a definite integral for the amount of the stage blocked by the curtain after t seconds. (b) At time t, what is the rate of change of the area blocked by the curtain?

6. A flat vertical plate is moving from left to right (back and forth) across a water pipe 1 cm in radius.

(a) Write a definite integral for the area blocked off by the plate when it is u cm to the right of center (the area shaded in the drawing). (b) Suppose that u is a sinusoidal function of time. If u starts at -1 when $t = 0$ and sweeps back and forth once every 2 seconds, find a formula for $u(t)$. (c) Find a formula for the rate of increase of the area blocked (in cm^2/sec) at time t.