Contents

CHAPTER 3

Instantaneous Rate Of Change: The Derivative

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x about a fixed value. We can measure this change as $\frac{\Delta y}{\Delta x}$, where as usual Δx denotes a change in x and Δy the corresponding change in y. This ratio is called a **Difference Quotient** for the obvious reason that it is a quotient of differences.

If we think of a linear function like $y = mx + b$, then it is easy to see that the ratio is really independent of our choice of the value of x as well as the change Δx; it is simply the slope m of the line. For other functions, life is not so easy!

EXAMPLE 3.1. Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

Let us look at the ratio $\frac{\Delta y}{\Delta x}$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) = f(7.1) - f(7) \\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} = 23.9706 - 24 = -0.0294. \end{aligned}$$

Thus, $\Delta y/\Delta x = -0.0294/0.1 = -0.294$. Geometrically, this means that the **chord** of the circle drawn from the point $(7, 24)$ to the point $(7.1, 23.9706)$ has **slope** equal to -0.294.

In general, if we draw the chord from the point $(7, f(7))$ to a nearby point on the semicircle $(7 + \Delta x,\ f(7 + \Delta x))$, the slope of this chord is the **difference quotient**

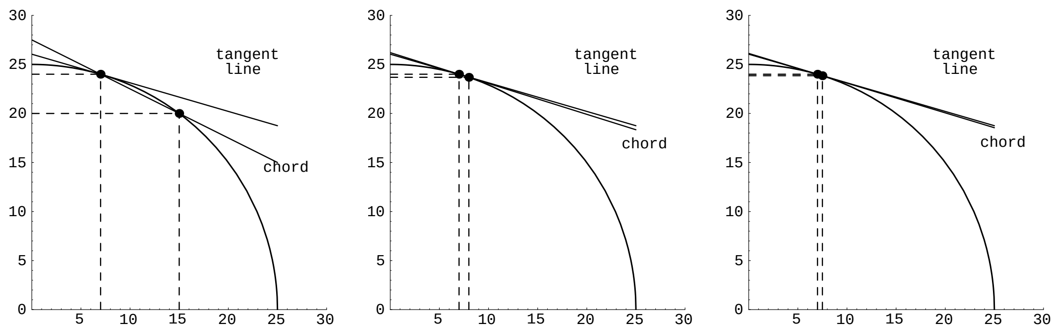
$$\text{slope of chord } = \frac{f(7 + \Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is equal to $(23.997081 - 24)/0.01 = -0.2919$.

Note that unlike the line, the difference quotient depends on x as well as Δx.

As our second x value $7 + \Delta x$ moves in towards 7, the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$ shifts slightly. As can be seen in the picture below, as Δx gets smaller and smaller, the chord joining $(7, 24)$ to $(7 + \Delta x, f(7 + \Delta x))$x gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. Recall that the tangent line is the line that just grazes the circle at that point. In case of the circle, it doesn't meet the circle at any other point, but this is not likely to happen at more complicated functions.

Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line.



$\Delta x = 8$ $\Delta x = 1$ $\Delta x = 0.5$

The slope of the tangent line to the circle at $(7, 24)$ is called the **derivative** of our function $f(x) = \sqrt{625 - x^2}$ at 7. It is denoted as $f'(7)$ (we say "f prime of 7," or equivalently, "the derivative of f at 7"). The slope of the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$, namely, $\Delta y/\Delta x = (f(7 + \Delta x) - f(7))/\Delta x$, gets closer and closer to this value $f'(7)$. We write

$$f'(7) = \lim_{\Delta x \longrightarrow 0} \frac{f(7 + \Delta x) - f(7)}{\Delta x}$$

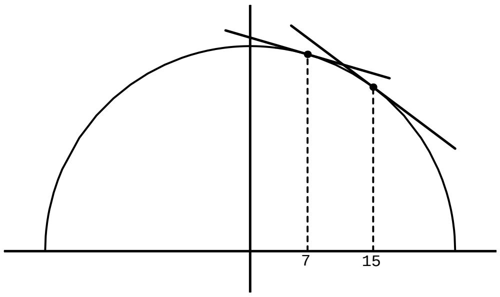
and we say that $f'(7)$ is the **limiting value** or simply the **limit** as Δx approaches zero of the difference quotient $(f(7 + \Delta x) - f(7))/\Delta x$.

In the particular case of a circle, there's a simple way to find the derivative. Namely, the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, and so its slope is the negative reciprocal of the slope of the radius. The radius joining $(0, 0)$ to $(7, 24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24 = -0.29166\cdots$. We write: $f'(7) = -0.29166\cdots$. Notice that when Δx is small, such as 0.01, the slope of the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7 + \Delta x))$ is a good approximation to the value $f'(7)$. For example, when $\Delta x = 0.01$, we saw that this difference quotient is -0.2919, which is close to $f'(7)$.

Now suppose that we choose a different x, say $x = 15$, and we want to know how fast y is changing as x increases a little from 15. As an approximation we could take the chord joining $(15, f(15))$ to $(15 + \Delta x, f(15 + \Delta x))$ when $\Delta x = 0.01$:

$$\frac{\Delta y}{\Delta x} = \frac{f(15.01) - f(15)}{0.01} = \frac{19.992476 - 20}{0.01} = -0.7504.$$

The limiting value of this difference quotient as Δx becomes smaller and smaller is what we mean by $f'(15)$. The value $f'(15)$ is the slope of the tangent to the circle at the point $(15, 20)$. As before, we can find this slope exactly, since the tangent to a circle is perpendicular to the radius. The answer is $f'(15) = -15/20 = -0.75$. Again we see that, if Δx is small (like 0.01), then the slope of the chord joining $(x, f(x))$ to $(x + \Delta x, f(x + \Delta x))$ is very close to this value $f'(x)$ (here $x = 15$).



For different x we generally get different values of the derivative $f'(x)$. In our example of the circle, we had $f'(7) = -7/24$ and $f'(15) = -3/4$.

Warning: A geometrical method used in this example for finding the slope of the tangent line is usually not available for general functions. In fact, historically, clever methods were developed for the conic sections; but the concept of defining the tangent line with an abstract limiting slope took hundreds of years to develop!

The only time $f(x)$ has the same derivative for all different x is when we have a straight line $y = f(x) = mx + b$. Then the difference quotient $\Delta y/\Delta x$ will always be m, and the derivative at any point on the line will also be m. The **tangent line** to the line $y = mx + b$ is simply the line $y = mx + b$ itself, and this has slope m. But except for straight lines, a function $y = f(x)$ has different "slopes" (of the tangent line) for different x.

We wish to calculate the slopes of tangents for several examples. For ease of notation and calculation, let us replace the change Δx by a single letter h. For a function $y = f(x)$, we take two points (x, y) and $(x + h, f(x + h))$ and our difference quotient and the slope of the tangent take the form:

Difference quotient is $\dfrac{f(x+h) - f(x)}{h}$, and the tangent has slope $lim_{h \longrightarrow 0} \dfrac{f(x+h) - f(x)}{h}$.

Of course, the name of the independent variable x can also change by context, as in the next example.

EXAMPLE 3.2. Suppose we are on an airless planet where the gravitational constant is 2 m/sec^2 (the moon has gravitational constant close to this). Then it turns out that an object dropped from rest falls through a distance of exactly t^2 meters after t seconds have elapsed. (Falling body problems will be taken up more systematically later.) That is, if y denotes the distance fallen in meters, we have $y = f(t) = t^2$. Suppose that we are interested in the moment $t = 1$ sec, and we want to know how much the object falls as t increases from 1 to $1 + h$ seconds. The distance fallen in that short time interval is $\Delta y = (1 + h)^2 - 1^2$, which simplifies to $2h + h^2$. For example, in the time from 1 sec to 1.01 sec the object falls a distance $\Delta y = 0.0201$ m. The difference quotient for this 0.01-sec time interval is $\Delta y/\Delta t = 0.0201/0.01 = 2.01$ m/sec. It is customary to calle this difference quotient the *average velocity during the time interval* $[1, 1.01]$. In other words, it is the distance traveled divided by time in this interval. More generally, the **average velocity** between 1 and $1+h$ sec. is

$$\frac{(1+h)^2 - 1^2}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

Now the derivative of $y = f(t) = t^2$ at the point $t = 1$ is the limiting value of this difference quotient as h approaches zero:

$$f'(1) = \lim_{h \longrightarrow 0} 2 + h = 2,$$

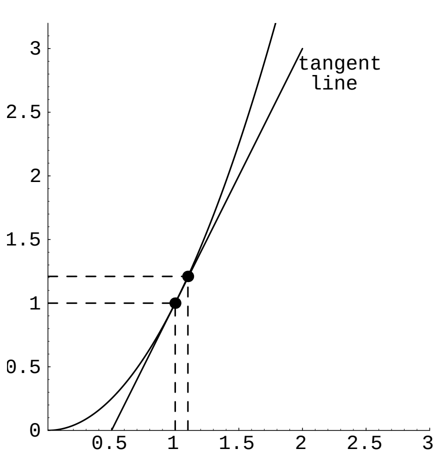
because as h gets closer and closer to 0, the number $2+h$ gets closer and closer to 2. In the case of a distance function like our $f(t)$ (distance fallen), the derivative $f'(1)$ has the practical meaning of the *instantaneous velocity* at time $t=1$. The **instantaneous velocity** at time t is the limiting value of the average velocity during a shorter and shorter time interval from t to $t+h$.

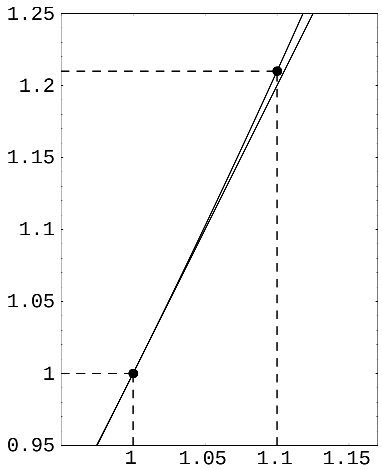
Note that our calculation was independent of the sign of $\Delta t = h$, even though our wording seems to suggest that h is positive. In reality, we need to get the same answer regardless of the sign of h. Leter, we will discuss what happens when it does depend on the sign of h.

To understand the distinction between average and instantaneous velocity, think of a car: the average velocity during a time interval means the distance traveled divided by time, while the instantaneous velocity at a particular instant means the speed indicated on the speedometer at that moment.

The graph below shows distance fallen as a function of time in this example: $y=t^2$. The diagram next to it is an enlargement near the point $t=1$ sec, where $y=1$ meter (i.e., the point (1,1) on the curve). The slope of the line joining $(1,1)$ to $(1.1,1.21)$ on the curve $y=t^2$ is the average velocity of the falling object during the time interval from 1 to 1.1 sec. The slope of the line which just grazes the parabola at the point $(1,1)$ is the instantaneous velocity of the object at time 1 sec. Thus, we have two interpretations of the derivative: as the slope of the tangent line in the graph, or as the instantaneous velocity.

The curve is a parabola and it also has a geometric construction for the tangent. Let O be the vertex and l the tangent at O. At any point P of the parabola drop a perpendicular from P to L and let Q be the foot of the perpendicular. Then the tangent at P is the line PM where M is the midpoint of OQ. (See picture). Of course, this is far more complicated than the circle. You should try to prove it by using our calculations above.





EXAMPLE 3.3. As a final example, we look at the reciprocal function $y=f(x)=1/x$. Suppose we want to find the derivative $f'(x)$ when $x=2$.

One way is to approximate. We can join the point $(2,0.5)$ to the point $(2+h, 1/(2+h))$ for small values of h, say 0.1, 0.01, 0.0001, or 0.0000001, and then compute the slope of the chord. We get the following table:

x	2	2	2	2
$x+h$	2.1	2.01	2.0001	2.0000001
Difference quotient	-0.238095	-0.248756	-0.249988	-0.24999999875

At this point we might guess that the slope of these chords is approaching -0.25, so that the derivative is $-1/4$. But to be more precise, we need an algebraic procedure to determine for sure what the limit is.

That is, as in the example $y = t^2$, we write out the difference quotient, which represents the slope of the chord joining the points $(2, f(2))$ and $(2+h, f(2+h))$, and then we try to simplify:

$$\begin{aligned}\frac{f(2+h)-f(2)}{h} &= \frac{1/(2+h)-1/2}{h} = \frac{2(2+h)}{2(2+h)} \cdot \frac{1/(2+h)-1/2}{h}\\ &= \frac{2-(2+h)}{2(2+h)h} = \frac{-h}{2(2+h)h} = -\frac{1}{2(2+h)}.\end{aligned}$$

As h approaches zero, the denominator $2(2+h)$ approaches $2 \cdot 2 = 4$, and so we get

$$f'(2) = \lim_{h \longrightarrow 0} \frac{f(2+h)-f(2)}{h} = \lim_{h \longrightarrow 0} -\frac{1}{2(2+h)} = -\frac{1}{4}.$$

HOMEWORK 3.

1. Draw the graph of the function $y = f(x) = \sqrt{169 - x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e).

2. Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) -7, (d) -15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.

3. Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average velocity of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average velocity between time 2 and time $2 + \Delta t$. (If you substitute $\Delta t = 1, 0.1, 0.01, 0.001$ in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average velocity (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous velocity. Finally, in your graph of $y = t^2$ draw the straight line through the point $(2, 4)$ whose slope is the instantaneous velocity you just computed.

4. If an object is dropped from an 80-meter high window, its height y above the ground at time t sec is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown at the beginning of the first section.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ sec (it'll be negative, because the object is falling).

5. Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0.

6. Find an algebraic expression for the difference quotient $\big(f(1 + \Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$.

7. Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine the limit as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the limit you just found.