Contents

CHAPTER 4

The Derivative Function

In the last section we saw how, given a function $y = f(x)$ and some particular value of x, we can speak of the derivative of the function at x, denoted by $f'(x)$. In geometrical terms, $f'(x)$ is the slope of the tangent line which grazes the curve at the point $(x, f(x))$. In this section, instead of taking just one particular value of x (or a few particular values, as in Problem 2 of the last homework), we let x vary. That is, we are interested in the rule which for every x in the domain of our function $f(x)$ determines the value of the derivative $f'(x)$. That gives us a new function f', called "f prime" or "the derivative of f."

EXAMPLE 4.1. Find the derivative function $f'(x)$ for the first function discussed in the last section: $f(x) = \sqrt{625 - x^2}$.

The point on the upper semicircle above x has coordinates $(x, \sqrt{625 - x^2})$. The radius out to that point has slope $\Delta y/\Delta x = (\sqrt{625 - x^2} - 0)/(x - 0) = (\sqrt{625 - x^2})/x$. Since the tangent to the circle at that point is perpendicular to this radius, its slope is the negative reciprocal, i.e., $-x/\sqrt{625 - x^2}$. Thus, $f'(x) = -x/\sqrt{625 - x^2}$. For example, if you plug $x = 20$, 24, -7, or -15 into this formula, you get the answers to Problem 2 of the last homework.

Of course, the above calculation depends on the clever geometrical method for finding the tangent. We need a more reliable method!

Notation. There are various other ways to denote the derivative $f'(x)$. If $y = f(x)$, we can also write y', or else dy/dx, or $\frac{d}{dx}f(x)$. For example, using the different types of notation in Example 1, where $f(x) = \sqrt{625 - x^2}$, we can write:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{625 - x^2}\right) = -\frac{x}{\sqrt{625 - x^2}}.$$

EXAMPLE 4.2. Find the derivative function for the second function in the last section: $y = f(t) = t^2$.

The derivative $f'(t)$ is the **instantaneous velocity** , which we find by taking the limit as $h = \Delta t$ approaches zero of the average velocity during the time interval between t and $t + h$. That average velocity is the difference quotient

$$\begin{aligned}\frac{\text{distance traveled}}{\text{time}} &= \frac{(t+h)^2 - t^2}{h} \\ &= \frac{t^2 + 2t(h) + (h)^2 - t^2}{h} = \frac{2t(h) + (h)^2}{h} = 2t + h.\end{aligned}$$

The limit of this as h approaches zero is $2t$. Thus, $f'(t) = \dfrac{d}{dt}\left(t^2\right) = 2t$. In the case of a distance function, the derivative (the instantaneous velocity) is often denoted with a dot rather than a prime; so we can write: $\dot{y} = \dfrac{d}{dt}\left(t^2\right) = 2t$.

EXAMPLE 4.3. Find the derivative function for the third function in the last section: $y = f(x) = 1/x$.

We go through the procedure at the end of the last section, but with any possible x rather than with 2. The difference quotient — the slope of the chord joining the point $(x, 1/x)$ on the reciprocal curve to the nearby point $(x+h,\ 1/(x+h))$ — is:

$$\frac{f(x+h)-f(x)}{h} = \frac{1/(x+h)-1/x}{h} = \frac{x(x+h)}{x(x+h)} \cdot \frac{1/(x+h)-1/x}{h} =$$
$$= \frac{x-(x+h)}{x(x+h)h} = \frac{-h}{x(x+h)h} = -\frac{1}{x(x+h)}.$$

As h approaches zero, the denominator approaches $x(x+0) = x^2$, and so we get:

$$y' = f'(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

In Example 1 we had a special geometrical situation (the tangent being perpendicular to the radius) which does not apply for any function except circles. But in Examples 2 and 3 we had a more general procedure for finding the derivative function: (1) form the difference quotient; (2) simplify algebraically; and (3) take the limit as h becomes smaller and smaller. In other words, we define the derivative function $f'(x)$ as follows:

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h)-f(x)}{h}.$$

Warning. If you happen to know a bunch of derivative formulas from an earlier course, for the time being you should know that they are true but unproved! In examples like the ones above and the homework below, you are required to know how to find the derivative formula starting from our current definition of the derivative.

1. Corners and Cusps

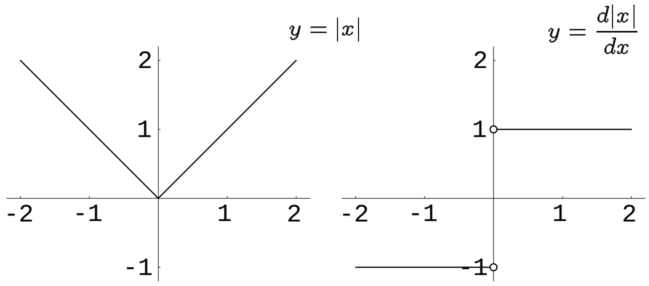
Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no well defined tangent line. At other times, our limiting process can fail if the tangent is vertical. First we consider the lack of a well defined tangent.

In order for the notion of the tangent line at a point to make sense, the curve must be " smooth " at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of — corners and cusps — where there's a sudden change of direction and hence no derivative.

EXAMPLE 4.4. Discuss the derivative of the absolute value function $y = f(x) = |x|$. (This is the function that for negative x drops the minus sign and for nonnegative x leaves it unchanged.)

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve — but there are *two* directions of the curve that come together at the origin. Thus,

$$y' = \frac{d}{dx}|x| = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

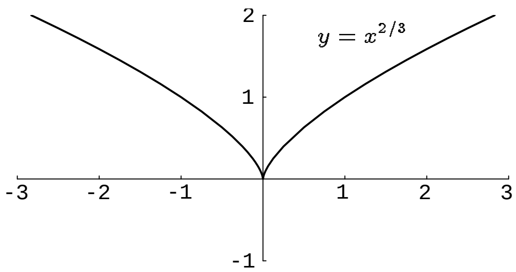


It is worthwhile discussing our difference quotients at $x = 0$. We are interested in $\frac{|0+h|}{h} = \frac{|h|}{h}$. Clearly, this quotient is 1 if $h > 0$ and -1 if $h < 0$. This explains why there is no possible limit as $h \longrightarrow 0$. We can also imagine limits as h goes to 0 thru positive values and deduce that it is $+1$. Similarly, the limit as h goes to 0 thru negative values is -1. In this case, it is possible to justify the statement that the function has two tangents with slopes 1 and -1. More on this later.

Example 4.5. The function

$$y = f(x) = x^{2/3}$$

pictured at the right has a derivative for every nonzero value of x (which we will find a formula for later), but it does not have a derivative at the origin, which is a cusp.

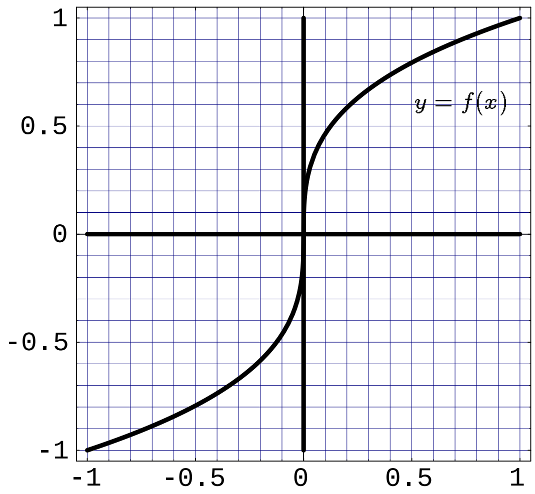


From the picture, it appears that this has a vertical tangent given by the equation $x = 0$. The algebraic procedures for establishing this are too complicated to introduce right now, but we will take it up later.

2. Graphical Differentiation

Often in applications one has a function $y = f(x)$ which is not given by a formula at all, but rather by drawing a smooth curve through points corresponding to a table of experimental data. Whenever the function $f(x)$ is given to you as a curve without any formula, you can find the derivative function $f'(x)$ using a graphical method. Namely, (1) divide up the domain of your function into regular intervals; (2) for each x which is an endpoint of an interval, look at the point over x on the curve; (3) put a small ruler along the curve at that point $(x, f(x))$ so that it just grazes the curve there (i.e., it is tangent to the curve there); (4) find the slope of the ruler, by measuring distance up divided by distance across (for example, if you are doing this against a piece of graph paper, and if you go over to the right 5 little boxes and up 8 little boxes, then the slope is 8/5=1.6); (5) tabulate the slopes obtained for all of the x values; (6) graph these values, drawing a smooth curve between them. The curve you draw is the derivative function $f'(x)$.

EXAMPLE 4.6. Use graphical differentiation to find the graph of $f'(x)$ for the curve $y = f(x)$ pictured below.

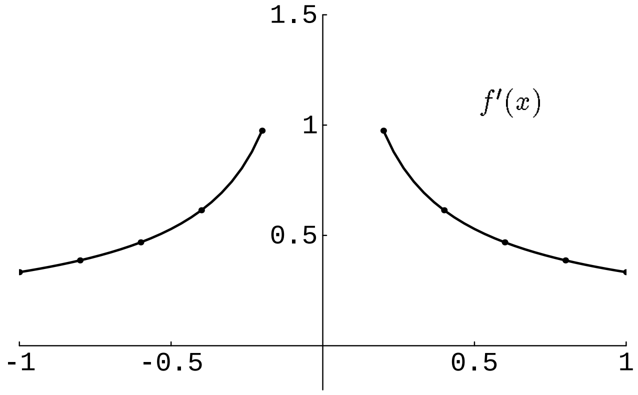


Following the 6-step procedure, we make a table of values of $f'(x)$ for x at intervals of 0.2 from -1 to 1:

x	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$f'(x)$	0.3	0.4	0.5	0.6	1.0	infinite	1.0	0.6	0.5	0.4	0.3

Note that this is a very approximate procedure, so our values of $f'(x)$ can be relied upon only to 1 or 2 significant figures. The estimates of the slopes are also subjective and a function of the accuracy of measurement.

Based on this table of values, we obtain an approximate graph of the derivative function $f'(x)$. Note that the derivative becomes infinite as we get closer and closer to $x = 0$, because the tangent to the curve there is vertical (has infinite slope). We say that $f'(0)$ "does not exist" (or "is infinite").

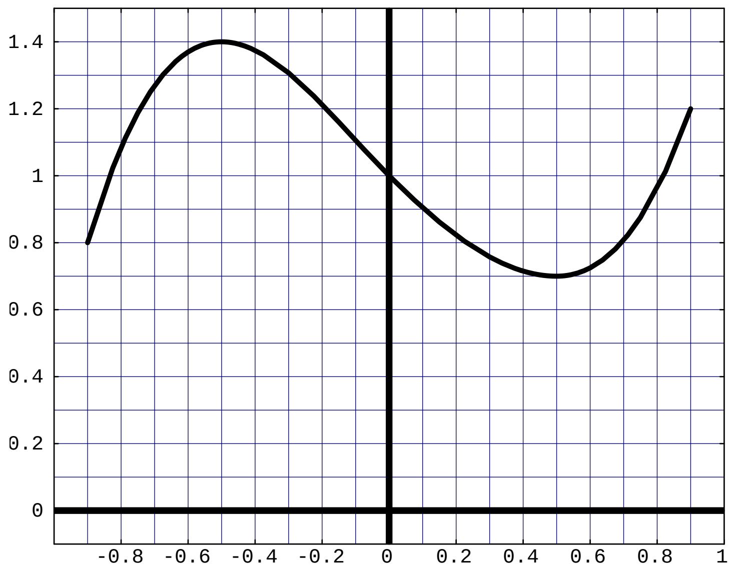


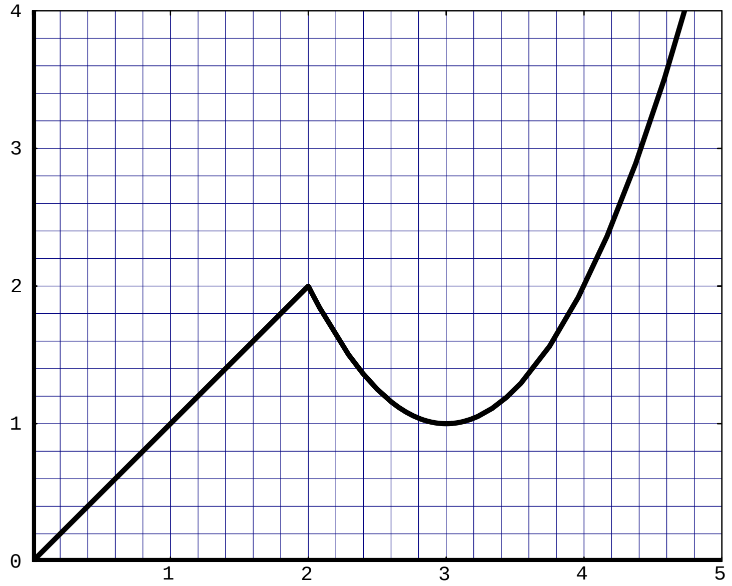
HOMEWORK 4.

1. Using the procedure used above in Example 4.1, find the derivative function for $y = f(x) = \sqrt{169 - x^2}$.

2. Using the procedure that we used in Examples 4.2 and 4.3, find the derivative function for: (a) $y = f(t) = 80 - 4.9t^2$, (b) $y = f(x) = x^2 - (1/x)$, (c) $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants), (d) $y = f(x) = x^3$. Please show every step clearly.

3. For each of the following two graphs $y = f(x)$, use graphical differentiation to sketch the graph of the derivative function. For x at regular intervals from one end of the domain to the other, estimate the derivative $f'(x)$ using a ruler, as explained in the text. Show clearly the points where the derivative is zero and where the derivative does not exist (or is infinite).





4. The 24 graphs below are labeled by letters from (a) to (x). For each of the following graphs of $f(x)$, give the letter of the graph that looks most like it could be the graph of the derivative function $f'(x)$:
(1) b, (2) c, (3) e, (4) g, (5) h, (6) j, (7) l, (8) p, (9) r, (10) s, (11) t, (12) u, (13) w, (14) x.