Contents

CHAPTER 5

The Tangent Line Approximation

1. The Power Rule

Before discussing the main theme of this section, we introduce the formula for the derivative of a power function, i.e., a function of the form $y = f(x) = x^n$. Here n can be anything — positive or negative, integer or fraction (or even irrational, like π). The formula is: $f'(x) = n\,x^{n-1}$. That is,

$$\text{(Power Rule)} \qquad \frac{d}{dx}\,x^n = n\,x^{n-1}.$$

We shall use this derivative formula in this section, even though we have not yet explained why it is valid — this will be done later. For some special values of n we did derive this formula in the last section, namely:

$$\frac{d}{dx}x^2 = 2x, \qquad \frac{d}{dx}x^3 = 3x^2,$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = (-1)x^{-2} = -\frac{1}{x^2}, \qquad \frac{d}{dx}\,x^1 = 1\cdot x^0 = 1.$$

Here are some examples of the $n\,x^{n-1}$ formula for other n:

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2\sqrt{x}}, \qquad \frac{d}{dx}\sqrt[3]{x} = \frac{d}{dx}x^{1/3} = \frac{1}{3}x^{-2/3},$$

$$\frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3}.$$

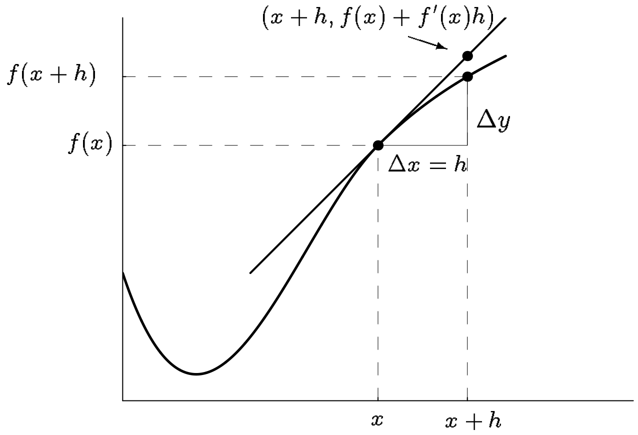
The last of these formulas tells us how to find the slopes of the tangent lines to the graph of $y = x^{2/3}$ in Example 4.5 of the last section. As expected, as x gets closer and closer to 0, the slopes become infinite (since 0 to a negative power is infinite).

2. The Tangent Line Approximation

The tangent line approximation is one of the most fundamental uses of the derivative. Suppose we have a function $f(x)$ which we can easily evaluate at some particular x. For example, the function $f(x) = 1/x$ has an obvious value when $x = 1$ or when $x = 100$; the function $f(x) = \sqrt{x}$ has an obvious value when $x = 4$ or when $x = 0.01$. Now suppose that we want to evaluate our function not at the easy x but rather at some nearby $x + \Delta x = x + h$. For example, we want to find the reciprocal not of 1 but rather of 1.01; or we want to find the square root not of 4 but rather of 3.99. The tangent line approximation is a way of doing this quickly but not with perfect precision — the result will be a little off (the accuracy depends on the particular function and on the size of Δx — the smaller the Δx the better the accuracy).

In a tangent line approximation problem we will know $f(x)$ and want to find $f(x+h)$. The derivative f' will also be relatively easy to compute at the point x. The formula derived below expresses $f(x+h)$ in terms of $f(x)$,$f'(x)$ and of course h.

To derive the formula, we draw a picture of our situation:



We want to find the y-coordinate of the point on the curve above $x+h$. Physically, we can get to that point by starting at our known point $(x, f(x))$ and following the curve until we go a horizontal distance $h = \Delta x$ to the right (we go to the left if h is negative). In doing this, we move upward a distance $\Delta y = f(x+h) - f(x)$. If we can determine Δy, then we know what to add to our known value $f(x)$ to get the value we're interested in: $f(x+h)$. Now $\Delta y/h$ is the slope of the chord joining the two points on the curve. The idea of the tangent line approximation is to use instead the slope of the tangent line to the curve at the point $(x, f(x))$. (This will give us a slightly different value for the change in y — namely, it gives us the change in y if we were to follow the tangent line rather than the actual curve when we move to the right a horizontal distance of h.) That is,

$$\text{(slope of chord)} \approx \text{(slope of tangent line)}$$

or

$$\frac{\Delta y}{h} \approx f'(x)\,.$$

And so

$$\Delta y \approx f'(x)\,h\,.$$

This gives us the **(VERY IMPORTANT!) tangent line approximation** formula

$$\boxed{f(x+h) = f(x) + \Delta y \approx f(x) + (h)f'(x).}$$

In using this formula, the basic steps are: (1) decide what your function f is in the given problem, and what you're taking for x and $x+h$; (2) find the derivative $f'(x)$; and (3) evaluate the formula.

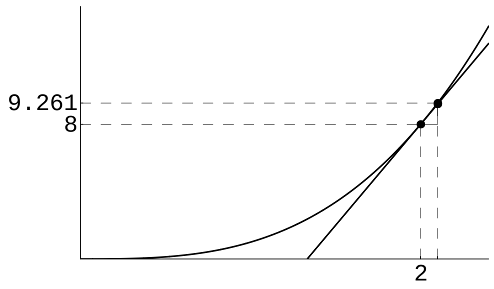
EXAMPLE 5.1. Find 2.1^3.

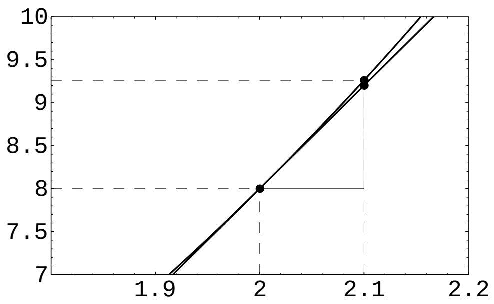
Here our function is $f(x) = x^3$, which we can evaluate in our heads at the point $x = 2$, namely $f(2) = 2^3 = 8$. We want to know its value at the nearby point $x + h = 2.1$. Thus, $h = 0.1$. In the tangent line approximation formula, we need to know $f'(x)$. Now

$\frac{d}{dx}x^3 = 3x^2$ (Use the Power Rule above). Hence $f'(x) = f'(2) = 3 \cdot 2^2 = 12$. We are now ready to use the tangent line approximation formula:

$$f(2 + 0.1) \approx f(2) + (0.1)f'(2) = 8 + 0.1 \cdot 12 = 9.2.$$

Of course, we could easily find $(2.1)^3$ more exactly by calculator (or precisely by multiplying by hand): the result is 9.261.





Since the tangent line approximation does not give the exact answer, and since we could have found 2.1^3 easily enough without it, you might wonder why one bothers with the tangent line approximation. There are several answers: (1) In easy examples, with experience you can perform the tangent line approximation in your head, thereby getting a better "feel" for how the function changes when you make a small change from x to $x + h$. For instance, in Example 5.1 we can say that "the cube of a number near 2 is going to be near 8, and the difference between the cube and 8 is going to be about 12 times the difference between our number and 2." (2) In many practical applications, the tangent line approximation gives simple answers to questions about experimental error, especially *percent error.* (3) Later on, in the section on implicit functions, we will encounter examples where y cannot be written in terms of x, and so there is no way to evaluate the y-coordinate of the point on the curve above $x + h$ except through an approximation method, such as the tangent line approximation.

A more important answer will come much later. A generalization of the tangent line approximation will develop into a systematic method for calculation with any desired precision and this will explain what your calculators or computers are doing. This is the topic concerning Taylor polynomials. In fact, the method can finally be pushed into making precise power series expansions of most functions. This will come much later!

In other words, do not look down upon the imprecise answers, we are just warming up!

EXAMPLE 5.2. Find 1/4.99.

Here we're interested in evaluating the reciprocal function very near to an x-value whose reciprocal we can evaluate in our heads, namely: $f(5) = 1/5 = 0.2$. That is, our function is $f(x) = 1/x$, our x is 5, and $x + h = 4.99$, i.e., $h = -0.01$. Since $f'(x) = -1/x^2 = -1/25 =$

-0.04, we have:

$$\begin{aligned}1/4.99 &= f(4.99) = f(5 + (-0.01)) \approx f(5) + (-0.01)f'(5) \\ &= 0.2 - 0.01 \cdot (-0.04) = 0.2004.\end{aligned}$$

Using a calculator, we can find a better value: $1/4.99 = 0.200400801\cdots$. A more sophisticated calculation can do better: 0.20040080160320641282565130. Thus, in this case the tangent line approximation is accurate to 5 decimal places.

You should consider the question of what decimal expansion, if any, would be a completely precise answer? (**Hint:** In this case, it is a recurring decimal with a pattern of 498 digits!)

EXAMPLE 5.3. On the circle of radius 10 centered at the origin find a point in the first quadrant whose x-coordinate is 6.01. Also, do the same for $x = 5.99$.

First the point $x = 6.01$. Here we know the "easy point" $(6, 8)$ on the circle, and we want to know the y-coordinate of the nearby point $(6.01, ?)$. The function is $y = f(x) = \sqrt{100 - x^2}$, our x is 6, and our $x + h = 6.01$, i.e., $h = 0.01$. Using the technique in the last section (e.g., Problem 1 on the homework), we can find the derivative: $f'(x) = -x/\sqrt{100 - x^2} = -6/\sqrt{100 - 6^2} = -0.75$. Thus, the tangent line approximation tells us that $f(6.01) \approx f(6) + 0.01 \cdot f'(6) = 8 + 0.01(-0.75) = 7.9925$. (A more precise value, computed by calculator, is $7.992490225\cdots$.)

The point $x = 5.99$ can be done similarly by taking $h = -0.01$ instead. So, we get $f(5.99) \approx 8 + (-0.01)(-0.75) = 8.0075$. (A more precise value, computed by calculator, is $8.0074902435\cdots$.)

EXAMPLE 5.4. If 1 quart equals 0.9464 liter, and 1 liter of water occupies exactly 1000 cc (cubic centimeters), i.e., a cubic container 10 cm on a side, then what are the dimensions (in cm) of a cubic container that holds 1 quart of water?

In a story problem like this, we must first analyze the operation that is being performed. The question can be reworded as follows: if a cube holding 1000 cc has side 10 cm, then what is the side of a cube holding 0.9464 as much? The process of going from the volume to the side is the cube-root function. Since "0.9464 as much as 1000 cc" obviously means 946.4 cc, we can rephrase the question once again: if the cube-root of 1000 is 10, then what is the cube-root of 946.4?

Let s be the length of the side of the cube and V its volume, then

$$s = f(V) = \sqrt[3]{V}\,.$$

In the tangent line approximation we take $V = 1000$ and $V + \Delta V = 946.4$, i.e., $\Delta V = h = -53.6$. By the Power Rule, we have $f'(V) = \frac{1}{3}V^{-2/3}$. When $V = 1000$, we have $1000^{-2/3} = 1/100$, and so $f'(1000) = 1/(3 \cdot 100) = 1/300$. So the tangent line approximation says that $\sqrt[3]{946.4} = f(946.4) \approx f(1000) + (-53.6) \cdot f'(1000) = 10 - 53.6/300 = 9.821$ cm. (Note: a more exact value obtained by calculator is $9.81804256\cdots$.)

Alternate Method. Write $V = s^3$. Then by the tangent line approximation $\Delta V \approx 3s^2\Delta s$. Since $s = 10$ and $\Delta V = -53.6$, we have $-53.6 \approx 300\Delta s$. Solving for Δs gives $\Delta s \approx -0.179$, so (new s) $= 10 + \Delta s \approx 10 - 0.179 = 9.821$ cm.

This method illustrates an important concept. It is often easier to use a relation between quantities, rather than one explicit functional evaluation. Solving explicitly for s was not necessary and indeed led to more complicated calculations!

The next example involves *percent error* . If you measure a value x, then the error (meaning the maximum possible error, or perhaps the maximum error that is likely) is the amount by which the true value could possibly differ from your measured value. We shall denote the error Δx. In most cases the true value could be on either side of the measured value, so the error is actually $\pm\Delta x$. By the **percent error** we mean the percent that this error is of the measured value, namely:

$$\text{percent error} = 100\frac{\Delta x}{x}.$$

For example, if you measure the speed of a car to be 50 mph to within 1 mph on either side, that means you have a percent error of $\pm 100(1)/50 = \pm 2\%$. If you know the percent error, say $\pm p\%$, then to convert this to absolute error h, use the formula: $\Delta x = \frac{p}{100}x$.

The reason for using the percent error is that it is a more realistic estimate of the error size. For instance an error of 10 feet in measuring a mile is not as serious as an error of 10 feet in measuring off a football field of 100 yards! The former percent error is only $100(10)/5280 = 0.189$, but the latter is $100(10)/300 = 3.333$.

EXAMPLE 5.5. If you determine the length of a side of a cubic container by measuring the volume of liquid it holds, then an error of $\pm 1\%$ in measuring the volume will lead to what percent error in your value for the length of a side? An error of $\pm p\%$ will lead to what percent error for the side?

This is a lot like Example 5.4. Again our process is: starting with the volume of a cube, determine its side. That is, our function is again the cube-root function. But we're not given any concrete values — and to answer the question we don't need any.

So let V stand for the measured value of the volume, and let $s = f(V) = \sqrt[3]{V}$ be the length of a side. We want to know the error Δs that is caused if V is replaced by $V \pm \Delta V$, where ΔV is a 1% error, i.e., $\Delta V = 0.01V$. Actually, what we want to know is the *percent* error in the side s, i.e., $100(\Delta s)/s$. According to the tangent line approximation,

$$\Delta s \approx f'(V)\Delta V = \frac{1}{3}V^{-2/3}\cdot 0.01V = \frac{1}{3}\cdot 0.01\sqrt[3]{V},$$

where the last equality comes from the rule of exponents: $V^{-2/3}\cdot V = V^{-2/3}\cdot V^1 = V^{-2/3+1} = V^{1/3} = \sqrt[3]{V}$. But $\sqrt[3]{V}$ is precisely the value $s = f(V)$. So our percent error in s is

$$100\frac{\Delta s}{s} \approx 100\left(\frac{1}{3}\right)\frac{0.01\sqrt[3]{V}}{s} = \frac{1}{3}\%.$$

If we take $-\Delta V$ in place of ΔV, we get $-1/3\,\%$. Thus, the percent error in s is $\pm\frac{1}{3}\%$. This answers our first question.

If the error in V is $\pm p\%$ instead of $\pm 1\%$, everything would go through the same way, except with a factor of p everywhere:

$$100\frac{\Delta s}{s} \approx 100\frac{f'(V)\Delta V}{s} = 100\frac{(1/3)V^{-2/3}\cdot 0.01p\,V}{s} = \frac{1}{3}p,$$

so the error in s is $\pm\frac{1}{3}p\%$. That is, *a certain percent error in measuring the volume of a cube leads to a percent error one-third as great in the value computed for the side.*

Note. Notice where the 1/3 came from in the last example: the exponent in $x^{1/3}$ that comes down in the rule for the derivative of x^n. In the same way, it turns out that whenever your function is of the form $y = f(x) = Cx^n$ (where C is any constant), there is a simple relationship between the percent error in your measurement of x and the percent error in y obtained using the tangent line approximation. More examples of this will show up in the homework.

HOMEWORK 5.

1. Compute the following values using the tangent line approximation with an appropriate $f(x)$, and in each case see how close you are by computing the true value by calculator or computer:
(a) $(2.1)^2$, (b) $1/1.1$, (c) $1/0.99$, (d) $1/101$, (e) $\sqrt{4.01}$, (f) $\sqrt{26}$, (g) $\sqrt{24}$, (h) $\sqrt[3]{8.012}$.

2. What is the y-coordinate of the point on the circle $x^2 + y^2 = 25$ whose x-coordinate is 3.08? Get an approximate answer using the tangent line approximation.

3. One liter is 1.0567 quarts. Use the tangent line approximation to answer the question: One quart is how many liters?

4. You want to tile a square floor, which you guess to be 15 feet by 15 feet. Based on your guess, you buy 225 square feet of tile. But after you've used up all of the tile, you still have 5 square feet not covered with tile. What are the actual dimensions of the square floor? Use the tangent line approximation.

5. Roughly speaking, an acre is about 200 feet by 200 feet. More precisely, an acre is 43560 square feet. What are the dimensions of an acre of land in the shape of a square? Use the tangent line approximation.

6. If a right triangle has legs 6 and 8, then the smallest circle in which one can put this right triangle has area 25π (its diameter is the hypotenuse of the triangle, which is 10). Suppose that one leg of the right triangle is known to be exactly 6, but the other leg is known to be 8 with an error of $\pm h$. What is the area of the smallest circle containing the triangle? Give the error estimate for this area in terms of h.

7. You measure a cubic container, and find it to be about 10 cm on a side. From this you conclude that it holds 1000 cc (cubic centimeters). However, your measurement is accurate only to within ± 0.1 cm. In other words, you can only be sure that a side of the cube is between 9.9 and 10.1 cm. Use the tangent line approximation to estimate the error in your value of 1000 for the volume. Then give the *percent error* in your measurement of the length of a side and the *percent error* in your value for the volume.

8. The distance in feet that an object falls in t seconds is given by the formula $y = 16t^2$. (a) If an object is dropped from a height of 400 ft, how long will it take to fall to the ground? (b) If your value of 400 ft for the initial height of the object is accurate only to $\pm 1\%$, how accurate is your answer to part (a)? Determine both the absolute error and the percent error. (c) If your value of 400 ft is accurate to $\pm p\%$, then how accurate (as a percent error, expressed in terms of p) is your answer to part (a).

9. Suppose that you know that a circle has area A with an error of $\pm 1\%$. (a) Find a formula for the radius r in terms of the area A. (b) If your value for A is off by $\pm 1\%$, how far off (as a percent error) is your value for r obtained using the formula in part (a)? (c) What if your value for A is off by $\pm p\%$?

10. If you know that the hypotenuse of a right triangle is exactly 10, and one of the legs is 6 ± 0.20, then what is the other leg? Include an error estimate in your answer.

11. The diameter of a sphere is measured with an error of $\pm p\%$, and this measurement is used to calculate the volume of the sphere. What is the percent error in the value computed for the volume?

12. What is the permissible percent error in measuring the side of a cube if you want at most a $\pm p\%$ error in your value for (a) the surface area of the cube, (b) the volume of the cube.

13. The distance around a sphere's equator is measured with an accuracy of $\pm p\%$. This value is used to compute the sphere's (a) radius, (b) volume, (c) surface area. In each case, determine the percent error.

14. A spherical container is filled with water, and the volume of water the container holds is measured to within $\pm p\%$. This information is then used to compute the surface area of the spherical inside surface of the container. What is the percent error in the value obtained for this surface area?

15. The height of a cylinder is known exactly, but its diameter is measured only to within $\pm p\%$. This information is then used to compute (a) the volume of the cylinder, (b) the circumference of the top of the cylinder, (c) the lateral surface area of the cylinder (top and bottom not included). In each case determine the percent error.

16. Use the tangent line approximation to find an x near 2 such that $x^5 + x^2 = 35$. (Hint: Replace the left hand side by its tangent line approximation at $x = 2$ and solve for h.)

17. A company is making solid cube paperweights. The material used to make the cubes costs 2¢/per cubic cm, while the paint used to paint the 6 square faces costs 0.5¢/per square cm. If the paperweight is $3\,\text{cm} \times 3\,\text{cm} \times 3\,\text{cm}$, then we calculate that the total cost of the material is 54¢ + 27¢ = 81¢. If you want the total cost to be 75¢ rather than 81¢, by how much do you have to decrease the side of the paperweight? Use the tangent line approximation.