Contents

CHAPTER 6

Rules For Finding Derivatives

1. Linearity of the Derivative

This means two things:

1. If you know the derivative $f'(x)$ of a function $f(x)$, then the derivative of a constant multiple $cf(x)$ is simply $cf'(x)$. *The derivative of a constant c times a function is equal to c times the derivative of the function*:
$$\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}f(x).$$
This principle is obvious if you think of the derivative as a rate of change. For example, it says that if $f(t)$ is the distance function for a bicycle, and if the distance function for a car is always 5 times the distance for the bicycle, then the car must be going at 5 times the velocity of the bicycle:
$$\frac{d}{dt}\big(5f(t)\big) = 5\frac{d}{dt}f(t)\,.$$
2. If you know the derivatives $f'(x)$ and $g'(x)$ of two functions $f(x)$ and $g(x)$, then the derivative of the sum $f(x) + g(x)$ is simply $f'(x) + g'(x)$. *The derivative of the sum of two functions is equal to the sum of the derivatives of the functions*:
$$\frac{d}{dx}\big(f(x) + g(x)\big) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$
This principle is also obvious when one thinks of rates. It says that the rate at which the sum of two distances is changing is equal to the sum of the rates at which each distance is changing.

If you have already caught on to the idea of the derivative as a limit of the difference quotient $\frac{\Delta y}{\Delta x}$, then these are easy to understand. If y is replaced by cy then clearly the quotient always gets multiplied by the same constant c and so does the derivative which is its limit! Similarly, if $y = w + z$ is a sum of two functions, then their difference quotients clearly add up and give the sum formula.

Principle (2) can be used repeatedly, if we have a sum of more than two functions. It can also be used in combination with the first principle. When combined in this way, the two principles together tell us that it is easy to find the derivative of any **linear combination** of functions whose derivatives we know. By a "linear combination" we mean any function that is constructed using the process of addition and multiplying by constants. For example, a polynomial is a linear combination of power functions: $c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$.

EXAMPLE 6.1. In the section on the derivative function, we found the derivative of t^2 (or x^2), of $mx + b$, and of the reciprocal function. Use that, together with linearity of the derivative, to do Problem 2(a)-(c) of that section much more quickly.

We have:

$$\frac{d}{dt}\left(80-4.9t^2\right)=\frac{d}{dt}80-4.9\frac{d}{dt}t^2=0-4.9(2t)=-9.8t.$$

$$\frac{d}{dx}\left(x^2-\frac{1}{x}\right)=\frac{d}{dx}x^2-\frac{d}{dx}x^{-1}=2x-\left(-x^{-2}\right)=2x+\frac{1}{x^2}.$$

$$\frac{d}{dx}\left(ax^2+bx+c\right)=a\frac{d}{dx}x^2+\frac{d}{dx}(bx+c)=2ax+b.$$

EXAMPLE 6.2. Find the derivative of $y=\left(1-5x^3\right)^2$.

Since $\left(1-5x^3\right)^2=1-10x^3+25x^6$, the derivative is

$$\begin{aligned}\frac{d}{dx}\left(1-10x^3+25x^6\right)=&\frac{d}{dx}1-10\frac{d}{dx}x^3+25\frac{d}{dx}x^6\\ =&0-10\cdot 3x^2+25\cdot 6x^5=-30x^2+150x^5.\end{aligned}$$

2. The Product Rule

Suppose that we want to find the derivative of a function that is written as a product $f(x)=u(x)v(x)$, where each of the two factors $u(x)$ and $v(x)$ is a function whose derivative we know. For example, $f(x)=x^3\sqrt{25-x^2}$ is the product of a function $u(x)=x^3$ whose derivative was already computed (namely, $u'(x)=3x^2$) and a function $v(x)=\sqrt{25-x^2}$ whose derivative we also know (namely, $v'(x)=-x/\sqrt{25-x^2}$).

We derive the product rule using the tangent line approximation formula for $u(x)$ and $v(x)$:

$$\begin{aligned}u(x+h)&\approx u(x)+(h)u'(x);\\ v(x+h)&\approx v(x)+(h)v'(x).\end{aligned}$$

We now look at the difference quotient for $f(x)=u(x)v(x)$, whose limit as h approaches zero is the derivative $f'(x)$:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{u(x+h)v(x+h)-u(x)v(x)}{h}\\ &\approx\frac{\Big(u(x)+(h)u'(x)\Big)\Big(v(x)+(h)v'(x)\Big)-u(x)v(x)}{h}\\ &=\frac{u(x)v(x)+(h)\Big(u(x)v'(x)+u'(x)v(x)\Big)+(h)^2u'(x)v'(x)-u(x)v(x)}{h}\\ &=\Big(u(x)v'(x)+u'(x)v(x)\Big)+(h)u'(x)v'(x).\end{aligned}$$

As h approaches zero, the last term $(h)u'(x)v'(x)$ drops out, and we are left with $u(x)v'(x)+u'(x)v(x)$. Thus,

$$\text{if } f(x)=u(x)v(x),\qquad\text{then}\qquad f'(x)=u(x)v'(x)+u'(x)v(x).$$

This rule can be written in shorthand as follows:

(Product rule) $$\frac{d}{dx}(uv)=uv'+u'v.$$

In words, *the derivative of a product is equal to the first function times the derivative of the second function plus the derivative of the first function times the second function.*

The above calculations certainly motivate the formula and do give an idea of the proof. They still lack many details for a complete proof. You should think about what is possibly wrong with them. The details will be discussed in class.

WARNING: Never take the derivative of a product by multiplying the derivatives of the terms. With addition you have $(u+v)' = u' + v'$, but with multiplication $(uv)' \neq u'v'$.

EXAMPLE 6.3. Find the derivative of $f(x) = x^3\sqrt{25-x^2}$, and determine the points where the curve has a horizontal tangent line.

By the product rule, $f'(x) = x^3\left(-x/\sqrt{25-x^2}\right) + 3x^2\sqrt{25-x^2}$. To say that a point $(x, f(x))$ has a horizontal tangent line means that the slope of the tangent line is 0, i.e., $f'(x) = 0$. So we must solve the equation $-x^4/\sqrt{25-x^2} + 3x^2\sqrt{25-x^2} = 0$. We do this by first clearing denominators (multiplying everything through by $\sqrt{25-x^2}$). The result is: $-x^4 + 3x^2(25-x^2) = 0$. When you have an equation like this, it is a good idea to try to factor the expression on the left. Right away we can factor out x^2; then we gather terms in what remains: $x^2(-x^2 + 3(25-x^2)) = x^2(-4x^2+75) = 0$. This has roots $x = 0$ or $4x^2 = 75$, i.e., $x = \pm\sqrt{75/4} = \pm 2.5\sqrt{3} = \pm 4.33\cdots$. Thus, there are three points with horizontal tangent lines: $(0,0)$, $(4.33, 202.97)$, and $(-4.33, -202.97)$.

EXAMPLE 6.4. Use the product rule to show how the nx^{n-1} formula can be obtained for $n = 4, 5, 6, \ldots$ (i.e., the successive values of n after the values $n = 0, 1, 2, 3$ for which we already derived the nx^{n-1} formula earlier).

So suppose we don't yet know a formula for the derivative of x^4, but we do know the derivative of x^3 (namely, $3x^2$). Then we can write $x^4 = x^3 \cdot x$, so that, by the product rule,

$$\frac{d}{dx}x^4 = \frac{d}{dx}\left(x^3 \cdot x\right) = x^3\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^3\right)x = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3.$$

This confirms the nx^{n-1} rule when $n = 4$. Now that we have established the rule for the derivative of x^4, we can use the same method to treat x^5:

$$\frac{d}{dx}x^5 = \frac{d}{dx}\left(x^4 \cdot x\right) = x^4\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^4\right)x = x^4 \cdot 1 + 4x^3 \cdot x = 5x^4.$$

And similarly for $n = 6$:

$$\frac{d}{dx}x^6 = \frac{d}{dx}\left(x^5 \cdot x\right) = x^5\left(\frac{d}{dx}x\right) + \left(\frac{d}{dx}x^5\right)x = x^5 \cdot 1 + 5x^4 \cdot x = 6x^5.$$

By now it should be clear that, continuing in this way, we can establish the validity of the nx^{n-1} rule for all positive integers n.

3. The Quotient Rule

Suppose that we want to find the derivative of a function that is written as the quotient of two functions whose derivatives we know: $f(x) = u(x)/v(x)$. The rule is:

(Quotient Rule) $$\frac{d}{dx}\frac{u(x)}{v(x)} = \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2},$$

or, in shorthand notation, $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$. This rule is a little more complicated than the product rule: *the derivative of a quotient is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all over the square of the denominator.* We shall not prove the quotient rule here. We will derive the quotient rule later in this section, when we cover the chain rule.

EXAMPLE 6.5. Use the quotient rule to find the derivative of the derivative of the upper semicircle function $y = \sqrt{25 - x^2}$. (The derivative of the derivative is called the " second derivative.")

Since the derivative is $-x/\sqrt{25 - x^2}$, we want the derivative of an expression of the form $u(x)/v(x)$, where $u(x) = -x$ and $v(x) = \sqrt{25 - x^2}$. Since $u' = -1$ and $v' = -x/\sqrt{25 - x^2}$, the quotient rule gives us:

$$\frac{d}{dx}\frac{-x}{\sqrt{25-x^2}} = \frac{\sqrt{25-x^2}(-1) - (-x)\left(-x/\sqrt{25-x^2}\right)}{(\sqrt{25-x^2})^2} = \frac{-\sqrt{25-x^2} - \frac{x^2}{\sqrt{25-x^2}}}{25-x^2}.$$

The last expression can be simplified by multiplying the top and bottom by $\sqrt{25 - x^2}$. After a little cancelation, we arrive at $-25/(25 - x^2)^{3/2}$.

Alternative method: The above calculations can be simplified by using proper notations and some clever simplification. First set $y = \sqrt{25 - x^2}$ and note that $y' = -x/y$. Applying the quotient rule and linearity we get

$$(y')' = -\frac{yx' - xy'}{y^2} = (-1/y^2)y - x(-x/y) = (-1/y^2)\frac{(y^2+x^2)}{y}.$$

Noticing that $x^2 + y^2 = 25$, we get that

$$y'' = -\frac{25}{y^3} = -\frac{25}{(25-x^2)^{3/2}}.$$

This idea will be developed into implicit differentiation method later.

EXAMPLE 6.6. Find the derivative of $\sqrt{100 - x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

First, using the quotient rule and the fact that $\sqrt{100 - x^2}$ has derivative $\frac{-x}{\sqrt{100-x^2}}$ and $\sqrt{x}$ has derivative $\frac{1}{2\sqrt{x}}$, we obtain:

$$\begin{aligned}\frac{d}{dx}\frac{\sqrt{100-x^2}}{\sqrt{x}} &= \frac{\sqrt{x}(-x/\sqrt{100-x^2}) - (1/2\sqrt{x})\sqrt{100-x^2}}{x} \cdot \frac{2\sqrt{x}\cdot\sqrt{100-x^2}}{2\sqrt{x}\cdot\sqrt{100-x^2}} \\ &= \frac{-x^2-100}{2x^{3/2}\sqrt{100-x^2}}.\end{aligned}$$

Alternately, we could use the product rule if we first write our function in the form $x^{-1/2}\sqrt{100 - x^2}$:

$$\begin{aligned}\frac{d}{dx}\left(x^{-1/2}\sqrt{100-x^2}\right) &= x^{-1/2}\frac{-x}{\sqrt{100-x^2}} - \frac{1}{2}x^{-3/2}\sqrt{100-x^2} \\ &= -\frac{x^{1/2}}{\sqrt{100-x^2}} - \frac{1}{2}x^{-3/2}\sqrt{100-x^2},\end{aligned}$$

which is algebraically equivalent to the answer we got using the quotient rule.

Alternative method: As above, the work can be cleaned up by using a few names. Let $y = \sqrt{100 - x^2}$ and $z = x^{1/2}$. Then $y' = -\frac{x}{\sqrt{100-x^2}} = -\frac{z^2}{y}$. Also, $z' = \frac{1}{2z}$. The derivative of the function becomes:

$$\frac{y}{z} = \frac{zy' - yz'}{z^2} = \frac{1}{z^2}(-\frac{z^3}{y} - \frac{y}{2z}) = -\frac{1}{2yz^3}(2z^4 + y^2).$$

Getting the final answer is now the matter of plugging in the values of y, z and simplifying.

A bit of advice on avoiding the quotient rule. Do **not** use the quotient rule when the numerator is a constant, because it is unnecessarily cumbersome in that case. For example, to find the derivative of $10/x^2$, write this as $10x^{-2}$ and use the power rule: the derivative is $10 \cdot (-2)x^{-3} = -20/x^3$.

On the other hand it is a good idea to try your quotient rule on $1/x$, whose derivative is known to be $-1/x^2$. It helps you to remember that the negative term in the numerator is $-uv'$.

4. The Second Derivative and Acceleration

As mentioned before, the second derivative is simply the derivative of the derivative. If $y = f(x)$ is our function, then there are various ways to denote the second derivative $\frac{d}{dx}\left(\frac{d}{dx} f(x)\right)$:

$$y'' = f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right).$$

EXAMPLE 6.7. Find the second derivative of (a) $y = x^n$, (b) $y = 1/x$, (c) $ax^2 + bx + c$ (where a, b, c are constants).

(a) We have

$$\frac{d}{dx}\left(\frac{d}{dx} x^n\right) = \frac{d}{dx}\left(nx^{n-1}\right) = n(n-1)x^{n-2}.$$

Then (b) is the special case $n = -1$, i.e., the second derivative of x^{-1} is $2x^{-3}$.

In part (c) we have $\frac{d}{dx}(\frac{d}{dx}(ax^2 + bx + c)) = \frac{d}{dx}(2ax + b) = 2a$.

Dot Notation. If the time t is playing the role of the x-variable, i.e., if $y = f(t)$, then we often use dots rather than primes for the derivatives:

$$\dot{y} = \frac{d}{dt} f(t), \quad \ddot{y} = \frac{d^2}{dt^2} f(t).$$

If the y-variable is denoted by a letter other than y, we still use the dot notation. For example, if the distance function is denoted $s = s(t)$, then we write $\ddot{s}(t)$ for the second derivative of the distance function.

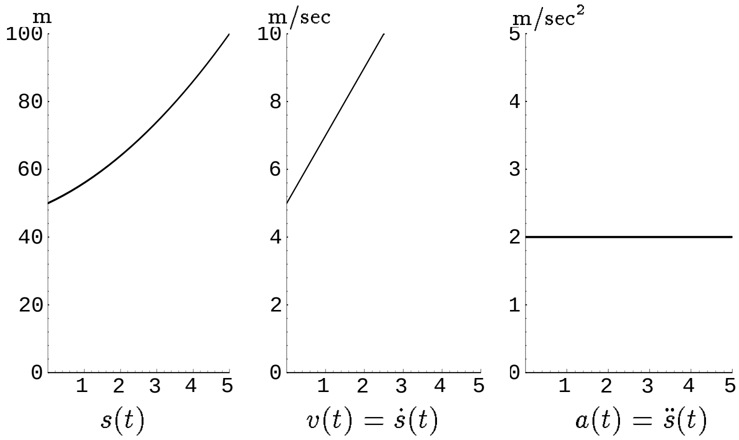
In the case of a distance function $s = s(t)$, both the first derivative function $\dot{s}(t)$ and the second derivative function $\ddot{s}(t)$ have a familiar practical meaning. The first derivative $\dot{s}(t)$ is, as we've seen, the velocity function, which is often denoted $v(t)$: $v(t) = \dot{s}(t) = \frac{d}{dt} s(t)$. The derivative of that — the instantaneous rate at which the velocity is changing — is what is called the **acceleration** , often denoted $a(t)$:

$$a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} s(t) = \ddot{s}(t).$$

Earlier we saw that the velocity at a certain instant t is the slope of the tangent line to the graph of the distance function at the point $(t, s(t))$. Similarly, the acceleration at a certain instant t is the slope of the tangent line to the graph of the velocity function at the point $(t, v(t))$.

EXAMPLE 6.8. A bicycle is traveling along a road (which we take to be the x-axis). Its distance in meters from the reference point (0 on the x-axis) at time t seconds is given by the formula $x(t) = 50 + 5t + t^2$. Find the derivative function $\dot{x}(t)$ and the second derivative function $\ddot{x}(t)$, graph all three functions (distance, velocity, and acceleration), and explain in words what is going on.

First $v(t) = \dot{x}(t) = \dfrac{d}{dt}(50+5t+t^2) = 5+2t$. Next, $a(t) = \ddot{x}(t) = \dfrac{d}{dt}v(t) = \dfrac{d}{dt}(5+2t) = 2$. The distance function $x(t)$ tells us that the bicycle is at $x(0) = 50$ meters from the reference point at time $t = 0$ and is moving to the right at a certain velocity $v(t)$. The formula for this velocity function $v(t) = 5 + 2t$ says that the speed of the bicycle is 5 m/sec at first ($t = 0$), but it increases steadily. The rate at which the velocity increases is the acceleration function $a(t) = 2$, which happens to be a constant function in this example. The acceleration is measured in "meters per second per second" or "meters per second squared," written 2 m/sec^2. Here are the graphs of the distance, velocity, and acceleration functions:



EXAMPLE 6.9. (Free fall without air resistance)

This is a good place to discuss the vertical motion of an object moving without air resistance under the influence of gravity. Let y denote the height (say in meters) of the object above ground level and let t be the time (in seconds). Suppose that at time $t = 0$ an object is located at height $y = y_0$ and has velocity v_0. (Notice that v_0 is positive means that the object is initially moving upward.) Then y and t are related by the formula

(Falling Body Formula) $$y = y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \,.$$

Taking derivatives gives the formula for the velocity

$$v(t) = \frac{d}{dt}y(t) = -gt + v_0 \,,$$

and taking one more derivative gives the acceleration

$$a(t) = \frac{d}{dt}v(t) = -g \,.$$

In particular, the acceleration of a falling body is constant, it is called the **acceleration due to gravity** . On the surface of the earth $g = 9.8$m/sec^2. (Notice that the acceleration is *negative* because the force of gravity points down.)

Deducing the formulas; We stated the above formulas without proofs. However, if we start with the formula $a(t) = -9$, then the formulas for $v(t)$ and $s(t)$ can be deduced by guesswork and the differentiation formulas that we already know. For instance, the formula for $v(t)$ must be $-gt + C$ for some constant C (think why!). Checking the values at $t = 0$, we can see why $C = v_0$. You should attempt to finish the formula for $y(t)$ yourself.

This is the process of finding antiderivatives and will be developed later.

5. The Chain Rule

The chain rule is the way to find the derivative of a function that is built up from simpler functions by the **composition of two functions** , i.e. by taking a "function of a function." For example, the function $\sqrt{1+x^4}$ is built up from the square root function and the function $1+x^4$. The function $1/\sqrt{5-\sqrt{x}}$ is built up from the reciprocal square root function (i.e., the $(-1/2)$-power function) and the function $5-\sqrt{x}$. We shall write $y = f(x) = g(u(x))$ to indicate that y is obtained by taking a function $g(u)$ (which we sometimes call the "outside function") and applying it to another function $u(x)$ (which we call the "inside function"). In the two examples just mentioned:

$$g(u) = \sqrt{u} \text{ and } u(x) = 1 + x^4 \quad \Longrightarrow \quad y = f(x) = g(u(x)) = \sqrt{1+x^4};$$
$$g(u) = 1/\sqrt{u} \text{ and } u(x) = 5 - \sqrt{x} \quad \Longrightarrow \quad y = f(x) = g(u(x)) = 1/\sqrt{5-\sqrt{x}}.$$

The chain rule enables us to find dy/dx once we know the derivative of the outside function $dy/du = g'(u)$ and the derivative of the inside function $du/dx = u'(x)$.

We shall derive the chain rule using the tangent line approximation. The tangent line approximation formula for the function $y = g(u)$ can be written in the form $\Delta y \approx g'(u)\Delta u$. Similarly, the tangent line approximation formula for the function $u(x)$ can be written $\Delta u \approx u'(x)\Delta x$. Putting these two facts together, we see that a small change Δx produces a change $u'(x)$ times as much in u, which, in turn, produces a change $g'(u)$ times as much in y, i.e., $\Delta y \approx g'(u)\Delta u \approx g'(u)u'(x)h$. Since we also have $\Delta y \approx f'(x)\Delta x$, it follows that the two "proportionality factors" between Δx and Δy must be equal, i.e., $f'(x) = g'(u)u'(x)$. This is the chain rule. It is especially easy to remember when it is written in the dy/dx notation:

$$\text{(Chain Rule)} \qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

It is as if the "dy," "du," and "dx" can be treated individually as algebraic quantities, where we cancel the two "du" on the right.

Indeed, another way of formulating the above calculations is to write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

which is precise. Now we take the limits as Δx goes to 0. We have to see why Δu also goes to zero and then the formula will be proved. We leave the precise treatment for the appendix.

Schematically, we can regard a function of a function as a 2-step procedure, first going from x to u and then from u to y. The chain rule says that the derivative from x to u must be multiplied by the derivative from u to y to get the derivative from x to y.

Before giving some computational examples, we shall give a story problem to show that the chain rule makes sense from a practical point of view.

EXAMPLE 6.10. Suppose your income is in the 28% tax bracket, and it is increasing at the rate of \$1000 per year. At what rate is your tax increasing?

As before, let us use x to denote income and y to denote the income tax. Recall that "28% tax bracket" means that y as a function of x has slope 0.28 over the range of values of x (your income) in question. In this problem y depends on x, which in turn depends on the time t. We could write that $y = g(x)$ and $x = x(t)$ together give $y = f(t) = g(x(t))$. (Thus, x is playing the rule of u in the chain rule, and t is playing the role of x.) The chain rule says that

$$\begin{aligned}\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} &= (28\%) \cdot (1000 \text{ dollars/ year}) \\ &= 0.28 \cdot 1000 \text{ dollars/year} = 280 \text{ dollars/year}.\end{aligned}$$

That is, tax is increasing at the rate of \$0.28 for every dollar increase in income, and income is increasing at the rate of \$1000 for every year increase in time; hence, tax is increasing at the rate $0.28 \cdot 1000 = 280$ dollars every year.

EXAMPLE 6.11. (a) Use the chain rule to find the derivative of the upper semicircle function $y = \sqrt{r^2 - x^2}$ (where r is a constant), and see that the answer agrees with our earlier formula for the derivative obtained using geometry. Also find the derivative of the two examples mentioned above: (b) $y = \sqrt{1 + x^4}$, (c) $y = 1/\sqrt{5 - \sqrt{x}}$.

(a) We take $y = g(u) = \sqrt{u}$ for the outside function and $u = r^2 - x^2$ for the inside function. Then, since $du/dx = -2x$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot (-2x) = -\frac{x}{\sqrt{u}} = -\frac{x}{\sqrt{r^2 - x^2}},$$

as expected.

(b) Here the outside function is $y = g(u) = \sqrt{u}$, and the inside function is $u = 1 + x^4$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 4x^3 = \frac{4x^3}{2\sqrt{1 + x^4}}$$

(the answer should be written in terms of x, i.e., in $g'(u)$ you should substitute $u = u(x)$, which in this case is $1 + x^4$).

(c) Here the outside function is $y = g(u) = u^{-1/2}$, and the inside function is $u = 5 - \sqrt{x}$. So

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{2} u^{-3/2} \left(-\frac{1}{2\sqrt{x}} \right) = \frac{1}{4(5 - \sqrt{x})^{3/2} \sqrt{x}}.$$

When the outside function $g(u)$ is the power function u^n, we can write the chain rule in the form

$$\frac{d}{dx} (u(x)^n) = n\, u(x)^{n-1} \frac{du}{dx}.$$

For example, the derivative of the square of a function is $\frac{d}{dx} u(x)^2 = 2u(x)u'(x)$.

EXAMPLE 6.12. Find the derivative of $(1 + 3x)^2$ in two ways: (a) by the chain rule; (b) by expanding the square.

(a) In the above formula for the derivative of $u(x)^2$, we take $u(x) = 1 + 3x$, so that $u'(x) = 3$. Then $\frac{d}{dx}(1 + 3x)^2 = 2(1 + 3x) \cdot 3 = 6(1 + 3x)$.

(b) We write $(1+3x)^2 = 1+6x+9x^2$, and then $\frac{d}{dx}(1+6x+9x^2) = 6+9 \cdot 2x = 6+18x = 6(1 + 3x)$, the same answer as in part (a).

We next use the chain rule to show how we get the quotient rule. We write the quotient $u(x)/v(x)$ in the form of a product $u(x)(v(x)^{-1})$. Then we apply the chain rule to find the derivative of $v(x)^{-1}$, where the outside function is the reciprocal function and the inside function is $v(x)$. According to the chain rule,

$$\frac{d}{dx}\left(v(x)^{-1}\right) = -\frac{1}{v(x)^2}v'(x).$$

We are now ready to apply the product rule:

$$\begin{aligned}\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) &= \frac{d}{dx}\left(u(x)(v(x)^{-1})\right) = u(x)\frac{d}{dx}\left(v(x)^{-1}\right) + u'(x)(v(x)^{-1}) \\ &= u(x)\left(-\frac{1}{v(x)^2}v'(x)\right) + \frac{u'(x)}{v(x)} = \frac{-u(x)v'(x)}{v(x)^2} + \frac{v(x)}{v(x)}\cdot\frac{u'(x)}{v(x)} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{v(x)^2},\end{aligned}$$

which is the quotient rule.

Another comment to make about the chain rule is that it can be used to find the derivative of a function of a function of a function, or even a function of a function of a function of a function, and so on, where in each case we build up a function by applying a chain of simple functions. Here is an example to illustrate this.

EXAMPLE 6.13. Find the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$.

The outermost function is $g(u) = \sqrt{u}$, where $u(x) = 1 + \sqrt{1+\sqrt{x}}$. In the chain rule we need $u'(x)$, which in this case we find by using the chain rule again:

$$u'(x) = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{d}{dx}(1+\sqrt{x}) = \frac{1}{4\sqrt{1+\sqrt{x}}\sqrt{x}}.$$

Finally,

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}u'(x) = \frac{1}{2\sqrt{1+\sqrt{1+\sqrt{x}}}}\cdot\frac{1}{4\sqrt{1+\sqrt{x}}\sqrt{x}} \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}.\end{aligned}$$

HOMEWORK 6.

1. Find d/dx of

(a) $x^4 - 3x^3 + 0.5x^2 + 7x - \pi$, (b) $x^3 - 2x^2 + 4\sqrt{x}$,
(c) $(x^2+1)^3$ (expand out the cube first),
(d) $(x^2+1)^3$ (use the chain rule), (e) $x\sqrt{169 - x^2}$,
(f) $(x^2 - 4x + 5)\sqrt{25 - x^2}$, (g) $(x^2 + x + 1)/(1 - x)$,
(h) $\sqrt{25 - x^2}/x$, (i) $\sqrt{(169/x) - x}$ (use the chain rule),
(j) $\sqrt{(169/x) - x}$ (factor out the $1/x$ and use the product or quotient rule),
(k) $\sqrt{x^3 - x^2 - (1/x)}$, (l) $100/(100 - x^2)^{3/2}$,
(m) $\sqrt[3]{x + x^3}$, (n) $\sqrt{(x^2+1)^2 + \sqrt{1 + (x^2+1)^2}}$.

2. Find the first derivative d/dt and also the second derivative d^2/dt^2 of the following functions of t (here a, b, and c are constants): (a) $-4.9t^2 + 20t + 50$, (b) $at + b$, (c) $\sqrt{at + b}$, (d) $\sqrt{at^2 + bt + c}$.

3. Suppose that the bicycle in Example 6.8 has distance function $x(t) = 50 - 5t + t^2$. Find $v(t) = \dot{x}(t)$ and $a(t) = \ddot{x}(t)$. Graph $x(t)$ and $v(t)$, and explain in words what the bicycle is doing.

4. If $f(x) = \sqrt{3 + x^n}$, then $f(1) = 2$. Use the tangent line approximation to find $f(1 + h)$. Express your answer in terms of n and Δx.

5. You want a ladder to reach a window which is about 24 feet above the ground. You place the bottom of the ladder exactly 7 feet away from the side of the building. If the window were exactly 24 feet high, you compute that you would need 25 feet of ladder to reach it. Suppose that the error in your value of 24 for the height of the window is $\pm h$. Find the error in the figure of 25 feet for the length of ladder.

6. In the situation of Problem 6 of the last homework, suppose that you want to fit the triangle in the smallest possible *sphere* (rather than circle). In that case the triangle's hypotenuse becomes a diameter of the sphere. Find the volume of that smallest sphere, and include an error estimate obtained using the tangent line approximation.

7. Suppose that an object falling from a height of 80 meters (see Problem 4 of Chapter 3) is viewed from a height of 2 meters by a person standing 10 meters from the side of the building where the object is falling. (a) Find a formula in terms of t for the distance from the person's eye to the object. (b) Find a formula in terms of t for the rate at which that distance is changing.