Contents

CHAPTER 7

Implicit Functions and Implicit Differentiation

1. Implicit Functions

We have seen that not all functions $y = f(x)$ are given by formulas. Sometimes a function is determined by interpolating a table of experimentally determined data (drawing a smooth curve between the points). Another way a function can be determined is *implicitly.* That means that we have a formula which x and y together must satisfy. The equation may take the form $F(x, y)$ =constant, or perhaps $F(x, y) = G(x, y)$.

EXAMPLE 7.1. (a) The circle can be written $x^2 + y^2 = r^2$, which is actually the implicit equation of two functions at the same time: the upper semicircle function $y = \sqrt{r^2 - x^2}$ and the lower semicircle function $y = -\sqrt{r^2 - x^2}$.

(b) We can avoid fractional powers $y = x^{l/m}$ by raising both sides to the m-th power and writing the implicit equation $y^m = x^l$.

(c) We can avoid negative powers by writing $y = x^{-l/m}$ in the form $y^m x^l = 1$.

(d) The points (x, y) on an ellipse with focii at (x_1, y_1) and (x_2, y_2) and semimajor axis a satisfy the geometrical property:

$$(\text{ distance from } (x,y) \text{ to } (x_1,y_1)) + (\text{ distance from } (x,y) \text{ to } (x_2,y_2)) = 2a,$$

which leads to the following implicit equation for the ellipse:

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=2a.$$

(e) In the case when the ellipse's axes of symmetry are parallel to the x- and y-axes, a simpler implicit equation can be given, namely $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$.

With many implicit equations, it is possible to solve for y in terms of x. For example, we can do this for the circle (Example 7.1(a)). In that case we have the choice of whether to use the **explicit form** $y = f(x)$ or the **implicit form** $F(x, y)$ =constant. However, there are other examples where it is impossible to solve algebraically for y in terms of x, and we have no choice but to work with the implicit equation.

2. Implicit Differentiation

Suppose we want to find the derivative y' at a point (x, y) of an implicitly defined curve. What you do is: (1) take d/dx of both sides of the equation; (2) separate the terms involving y' to one side of the equation and the terms without y' to the other side; and (3) solve for y'. In step (1), it is important to remember that you are differentiating with respect to x (or whatever letter is playing the role of the x-variable), and so the chain rule must be used for expressions like y^2. In other words, the letter y stands for a certain function of x, and so taking d/dx of y^2 is just like taking d/dx of $u(x)^2$: $\frac{d}{dx}y^2 = 2y\frac{dy}{dx} = 2y\,y'$.

EXAMPLE 7.2. Find y' at the point (x, y) for each implicitly defined function in Example 7.1.

(a) When we take d/dx of both sides of $x^2 + y^2 = r^2$, remember that (1) the derivative of y^2 is *not* $2y$ but rather $2yy'$, and (2) the derivative of r^2 is *not* $2r$ but rather $2rr' = 0$, since $r' = 0$ as r is a constant.

Thus, taking d/dx of both sides gives: $2x + 2yy' = 0$. We next move the $2x$ to the right, and divide both sides by $2y$ to solve for y': $y' = -2x/(2y) = -x/y$. This answer agrees with our earlier formula for the derivative of $y = \sqrt{r^2 - x^2}$, namely, $y' = -x/\sqrt{r^2 - x^2}$, since y and $\sqrt{r^2 - x^2}$ are the same. Notice that when you differentiate an explicit function $y = f(x)$ your answer is a formula in terms of x, whereas implicit differentiation generally leads to a formula involving *both y and x*.

Note that if we take the lower semicircle function $y = -\sqrt{r^2 - x^2}$, the expression for y' becomes $-\frac{x}{\sqrt{r^2-x^2}}$ which is a different formula. However, the formula $y' = -x/y$ is still the same! Thus the implicit differentiation not only avoids the solution process for y in terms of x, but also gives the right answer for all possible solutions at once.

Parts (b) and (c) of this example will show us how to derive the nx^{n-1} rule when the power of x is a fraction $n = l/m$ or $n = -l/m$, where l and m are positive integers. (In the section on the product rule, we already saw how to get the nx^{n-1} rule for any positive integer n.)

(b) We take d/dx of both sides of $y^m = x^l$, and then solve for y':

$$my^{m-1}y' = lx^{l-1}, \qquad \text{so that} \qquad y' = \frac{l}{m}\frac{x^{l-1}}{y^{m-1}}.$$

The last ratio simplifies algebraically, if we use the fact that $y^{m-1} = y^m/y = x^l/x^{l/m}$. That is,

$$y' = \frac{l}{m}\frac{x^{l-1}}{x^l/x^{l/m}} = \frac{l}{m}x^{l/m-1} = nx^{n-1} \qquad (\text{here } n = \frac{l}{m}).$$

(c) We take d/dx of both sides of $y^m x^l = 1$ (using the product rule), and then solve for y':

$$y^m\left(lx^{l-1}\right) + my^{m-1}y'x^l = 0, \qquad \text{so that} \qquad y' = -\frac{ly^m x^{l-1}}{my^{m-1}x^l}.$$

Canceling $y^m x^l$ from the numerator and denominator in the last expression, we obtain:

$$-\frac{l}{m}\frac{x^{-1}}{y^{-1}} = -\frac{l}{m}\frac{y}{x} = -\frac{l}{m}\frac{x^{-l/m}}{x} = -\frac{l}{m}x^{-\frac{l}{m}-1} = nx^{n-1}, \qquad \text{where} \qquad n = -\frac{l}{m}.$$

(d) When we take d/dx of both sides, we must use the chain rule several times, first with the square root as the outside function and then when we find the derivative of $(x - x_1)^2$, $(y - y_2)^2$, etc. The result is:

$$\frac{(x - x_1) + (y - y_1)y'}{\sqrt{(x - x_1)^2 + (y - y_1)^2}} + \frac{(x - x_2) + (y - y_2)y'}{\sqrt{(x - x_2)^2 + (y - y_2)^2}} = 0.$$

Solving for y' is not difficult, but the expression we end up with is rather messy, so we shall omit it. If we had specific values for the foci (x_1, y_1) and (x_2, y_2), and if we wanted to find the slope y' at a particular point (x, y), then we could substitute all of these numerical values in place of x, y, x_1, y_1, x_2, y_2, ending up with an equation of the form $\alpha + \beta y' = 0$ (where β comes from bringing together all of the terms with y' and α comes from bringing together all the terms without y'). Such an equation has solution simply $y' = -\alpha/\beta$. We will see such situations later in the story problems.

(e) Here we get: $2(x-h)/a^2 + 2(y-k)y'/b^2 = 0$. Solving for y', we obtain

$$y' = -\frac{b^2}{a^2}\frac{x-h}{y-k}.$$

EXAMPLE 7.3. If you know that the curve $(xy)^4 + x^{-2} + y^{-1} = C$ (where C is a constant) passes through the point $(1,2)$, find the y-coordinate of the point $(1.01, ?)$ on the curve near $(1,2)$. Use the tangent line approximation.

(**Note:** The constant C is not given. We could easily compute it from the fact that the point $(1,2)$ satisfies the equation. However, we will never need to know the value of C, so we won't bother to compute it.)

In this problem there is no way you can find an exact expression for the y-coordinate corresponding to $x = 1.01$, because when you substitute $x = 1.01$ you get an equation for y that cannot be solved algebraically. The equation for y could be solved by an approximation method that we'll learn later (Newton's method). In the meantime, the best we can do is to use the tangent line approximation to find the y-coordinate. In this problem we know that the implicitly defined $y = f(x)$ has value 2 when $x = 1$ (this is what it means to say that the curve passes through the point (1,2)). We want to know $f(1.01)$. According to the tangent line approximation, $f(1.01) \approx f(1) + 0.01f'(1) = 2 + 0.01y'$, where y' denotes the derivative at the point $(1,2)$. To find that derivative we first take d/dx of both sides of our implicit equation. We use the chain rule and the product rule to find $\frac{d}{dx}(xy)^4$, and on the right we get 0 (the derivative of any constant is 0):

$$4(xy)^3(xy' + 1\cdot y) + (-2x^{-3}) + (-y^{-2}y') = 0.$$

Once we've taken d/dx we can substitute the particular x and y values of the point where we want the derivative. After that our equation will look a lot simpler — just some numbers together with the unknown y':

$$\begin{aligned} &4(1\cdot 2)^3(y'+2) + (-2\cdot 1) + (-2^{-2}y') = 0, \\ \text{i.e.,}\quad & y'(32-0.25) + 64 - 2 = 0, \\ \text{i.e.,}\quad & y' = -62/31.75 = -1.953. \end{aligned}$$

Finally, substituting this value of $y' = f'(1)$ in the tangent line approximation gives the y-coodinate as $2 + 0.01\cdot(-1.953) = 1.98047$.

EXAMPLE 7.4. Suppose that the ellipse in Example 7.1(d) has foci at the points (0,1) and (3,0), and passes through the point (2,3). Find the y-coordinate of the point $(1.99, ?)$ through which it passes. Use the tangent line approximation.

The tangent line approximation tells us that $f(1.99) \approx f(2) - 0.01f'(2) = 3 - 0.01y'$, where y' is the derivative at the point $(2,3)$. The implicit differentiation of the equation of this ellipse has already been carried out, so it remains just to substitute the particular values we're given: $x_1 = 0$, $y_1 = 1$, $x_2 = 3$, $y_2 = 0$, $x = 2$, $y = 3$:

$$\frac{2+2y'}{\sqrt{8}} + \frac{-1+3y'}{\sqrt{10}} = 0, \qquad \text{so that} \qquad \left(\frac{2}{\sqrt{8}} - \frac{1}{\sqrt{10}}\right) + y'\left(\frac{2}{\sqrt{8}} + \frac{3}{\sqrt{10}}\right) = 0,$$

i.e., $0.39088 + 1.65579y' = 0$. Thus, $y' = -0.39088/1.65579 = -0.23607$, and so $f(1.99) \approx 3 - 0.01y' = 3.00236$.

EXAMPLE 7.5. You are standing at (0,1) and receiving light from two sources: source A located at $(-1,0)$ and source B located at $(1,0)$. You want to start walking in the direction which keeps the total light intensity from the two sources as constant as possible. Assume that light intensity from a source is proportional to the strength of the source and inversely proportional to the square of the distance from the source. (a) If A and B are equally strong, show that you should walk horizontally, i.e., set out along the line $y=1$. (b) If B is twice as strong as A, find the line along which you should move.

We are interested here in the curve of constant total light intensity passing through the point (0,1). That is, we want an equation satisfied by all points (x, y) where the light intensity from A plus the light intensity from B remains constant. Since light intensity is inversely proportional to the square of the distance from the source, this means that

$$\frac{K}{(\text{dist of }(x,y)\text{ from A})^2}+\frac{K}{(\text{dist of }(x,y)\text{ from B})^2}=C.$$

Here the proportionality constant K depends on the strength of the source; since A and B have the same strength in part (a), the constants are the same. In part (b) we change the second K to $2K$, because B is twice as strong as A.

Next, we use the distance formula in the denominators: the square of the distance from (x, y) to $(-1,0)$ is $(x+1)^2+y^2$, and the square of the distance from (x, y) to $(1,0)$ is $(x-1)^2+y^2$. For convenience, we also divide through by K, so that there are no unknown constants on the left. The result is

$$\frac{1}{(x+1)^2+y^2}+\frac{1}{(x-1)^2+y^2}=\frac{C}{K}.$$

The question asks for the direction to go in from the point $(0,1)$ that best keeps to the curve of constant total light intensity. This amounts to asking for the tangent line to the curve at $(0,1)$. To find y', we take d/dx of both sides, and then substitute the point $(x, y)=(0,1)$ where we want to find y'. It is important to note that the fact that you happen to be walking from the point $(0,1)$ is not used until *after* performing the implicit differentiation. When we take d/dx, we use the chain rule, writing the first term on the left as $\left((x+1)^2+y^2\right)^{-1}$, and similarly for the second term on the left (it is more efficient to use the rule for $\frac{d}{dx}u^{-1}$ than to use the quotient rule):

$$-\left((x+1)^2+y^2\right)^{-2}(2(x+1)+2yy')-\left((x-1)^2+y^2\right)^{-2}(2(x-1)+2yy')=0.$$

(Notice that the constants C and K have disappeared, because the derivative of any constant is 0.) Substituting $x=0$, $y=1$ gives:

$$-(1+1)^{-2}(2+2y')-(1+1)^{-2}(-2+2y')=0.$$

Solving this, we find that $y'=0$, i.e., the curve of constant total light intensity has horizontal tangent line at the point $(0,1)$.

(b) When the source B is twice as strong as the source A, the effect is to replace the second K by $2K$, and this means having a 2 rather than 1 always in the numerator of the second term on the left. Our last equation is now

$$-(1+1)^{-2}(2+2y')-2(1+1)^{-2}(-2+2y')=0.$$

We can clear denominators by multiplying through by 2, obtaining: $-1-y'+2-2y'=0$, i.e., $y'=1/3$. Thus, the tangent line has slope $1/3$ and y-intercept 1, so its equation is $y=\frac{1}{3}x+1$.

HOMEWORK 7.

1. In each case use the tangent line approximation to estimate the y-coordinate of the point on the curve whose horizontal distance is 0.01 (to the right) from the given point P:

$$\text{(a) } x^2 + xy + y^2 = 7,\ P = (1, 2);$$

$$\text{(b) } \frac{x}{y} + \frac{y}{x} + \frac{xy^2}{100} = \text{const },\ P = (2, 5);$$

$$\text{(c) } \sqrt{x} + \frac{3}{x} = y - 1 + \frac{1}{\sqrt{y} + 1},\ P = (9, 4).$$

2. Find a formula for the derivative y' at the point (x, y):

$$\text{(a) } y^2 = 1 + x^2; \qquad \text{(b) } x^2 + xy + y^2 = 7; \qquad \text{(c) } x^3 + xy^2 = y^3 + yx^2.$$

3. A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$.

4. In Example 7.5(a), suppose that you are starting from the point $(1, 1)$ rather than $(0, 1)$. Find the equation of the line from this point which is the closest approximation to the curve of constant total light intensity. After you've walked a horizontal distance of 0.1 (to the right), what is your y-coordinate?

5. In the situation of Example 7.5, suppose that you don't know the relative strengths of the sources A and B, but you know that the line $y = 0.6x + 1$ is the line that you should start walking along from the point (0,1) if you want the total light intensity to be as constant as possible. Find how much stronger B is than A.

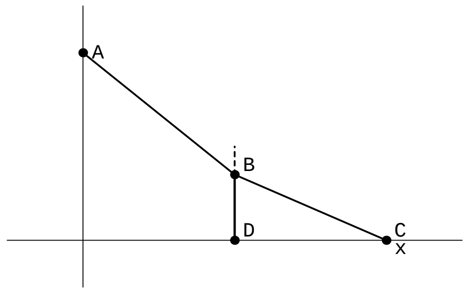
6. Suppose that a light source A is at the origin, and a source B twice as strong is located at $(3, 0)$. Find the slope of the direction you move in from (1,2) in order for the total light intensity to remain as constant as possible.

7. When crude oil flows from a well, water is frequently mixed with it in an emulsion. To remove the water the crude is piped to a device called a heater–treater, which is simply a large tank in which the oil is warmed and the water is allowed to settle out. Operating experience in a particular oil field indicates that the concentration C of water in the treater's output can be modeled by the following equation in the neighborhood of the usual operating point of 135°F and a 2-hour holding time:

$$C = 0.04 - 0.0032\, h^2 - 10^{-6}\, T^2 + 4.1 \cdot 10^{-5} h\, T,$$

where h is the holding time in hours and T is the operating temperature in degrees F. (a) Due to random fluctuations in the well's flow rate, the holding time actually varies slightly around 2 hours. Suppose you are given a simple control device that can change the tank temperature proportionally to the measured change in holding time. What constant of proportionality would best compensate for small holding time fluctuations and keep the water concentration as constant as possible? (b) Now as field equipment ages, its maximum operating settings are generally decreased. Find the equation of the line that best approximates the way in which the holding time would have to be increased as the maximum temperature rating falls slightly below the usual operating temperature.

8. One end of a string (the point A) is attached to the y-axis at a distance 20 meters from the origin; the string is threaded through a hole (the point B) at the end of a pole situated along the line $x = 10$; and the other end of the string (the point C) is attached to a hook that is free to slide along the x-axis and is being pulled by a force to the right that is sufficient to keep the string taut. See the diagram at the right. At time $t = 0$, when the top of the pole is at the point (10,10), the pole starts lowering at the steady rate of 30 cm/sec (=0.30 m/sec). The sliding end of the string reaches the point $x = 20$ at time $t = 10$.



(a) Write an equation relating t and x (the time is an implicit function of the location of the sliding end of the string). (b) Use the tangent line approximation to find the time when the sliding end of the string has moved another 20 cm to the left from the point $x = 20$.