Contents

CHAPTER 8

Trig Functions and Sinusoidal Functions

We start by reviewing trig functions of an angle measured in radians. Because this is review material, we will go through it rapidly.

We first draw the unit circle in the xy-plane. The standard way of measuring an angle is to start with the positive x-axis and rotate counterclockwise. As we go through the angle, we sweep around the circumference of the unit circle. The radian measure of the angle is defined to be the distance around the circumference that we travel. That is, a full revolution of $360°$ is equal to 2π radians, half a revolution is $180° = \pi$ radians, and so on. Other frequently used equivalences are: $90° = \pi/2$ rad, $60° = \pi/3$ rad, $45° = \pi/4$ rad, $30° = \pi/6$ rad. *All angles in this course will be assumed to be measured in radians unless otherwise stated. The designation "radian" will often be omitted, e.g., if we write "an angle of $\pi/6$," we mean a $30°$ angle.*

By a *negative angle* we mean a clockwise angle. For example, an angle of $-\pi/2$ (i.e., $-90°$) brings us to the same position as an angle of $3\pi/2$ (which is $270°$).

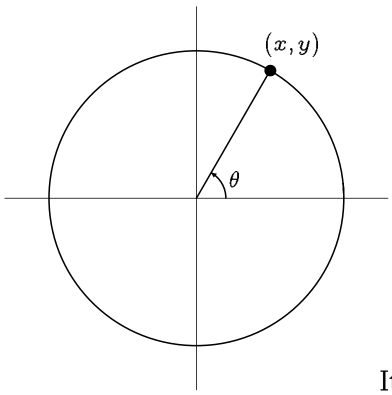
In general, if we have a circle of radius r and sweep out an angle of θ radians, then the distance around the circumference through which we have traveled is simply θr. Of course, one full circumference is $2\pi r$, corresponding to rotation through 2π radians.

Returning to the unit circle (see the drawing below), after rotating through an angle of θ radians let (x, y) be the point where we end up on the circumference. Then the six trig functions are defined as follows:

$$\sin\theta = y \qquad \text{cosecant}\,\theta = \frac{1}{\sin\theta} = \frac{1}{y}$$

$$\cos\theta = x \qquad \text{secant}\,\theta = \frac{1}{\cos\theta} = \frac{1}{x}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$$



It is also customary to use the names csc θ for cosecant θ and sec θ for secant θ.

From the definition of $\sin\theta$ we see that as θ increases from 0 to 2π (i.e., as we go once around the unit circle), $\sin\theta$ goes from 0 to 1 to 0 to -1 and back to 0. Then as θ goes from 2π to 4π (i.e., as we go once again around the circle), the same values of $\sin\theta$ are repeated. We say that the function $\sin\theta$ has **period** 2π, i.e., we have $\sin(\theta + 2\pi) = \sin\theta$. Here are some other relations for the trig functions that follow easily from the above definitions: $\cos(\theta+2\pi) = \cos\theta$, $\sin(\theta+\pi) = -\sin\theta$, $\cos(\theta+\pi) = -\cos\theta$, $\tan(\theta+\pi) = \tan\theta$, $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$, $\sin^2\theta + \cos^2\theta = 1$, and $\cos\theta = \sin(\frac{\pi}{2} - \theta)$. The last of these relations is familiar from high school trigonometry: the cosine of an angle is equal to the sine of the complementary angle.

It can also be seen from our definitions as follows: the x-coordinate of the point on the unit circle you arrive at when you rotate through θ counterclockwise starting at the positive x-axis is the same as the y-coordinate of the point you arrive at when you rotate through θ clockwise starting at the positive y-axis. Sometimes one also needs the following formulas for the trig functions of a sum of two angles: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. As a special case we have the double angle formulas $\sin(2\alpha) = 2\sin\alpha\cos\alpha$ and $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$.

A simple geometric explanation of this will be given in class.

Also notice the sign of the trig functions in the different quadrants: $\sin x$ is positive in the 1st and 2nd quadrants (for angles between 0 and π), $\cos x$ is positive in the 1st and 4th, and $\tan x$ is positive in the 1st and 3rd.

1. Graphs of Trig Functions

The most important graphs to be familiar with are the graphs of the functions $y = \sin x$, $y = \cos x$, and $y = \tan x$. (We're now using x for the angle in radians; x no longer denotes a coordinate of a point on the unit circle.) For certain x the trig functions have easily stated exact values, namely, when x is a multiple of $\pi/6$ or of $\pi/4$. Here it is useful to draw the $30^\circ - 60^\circ - 90^\circ$ triangle with hypotenuse 1 and the $45^\circ - 45^\circ - 90^\circ$ triangle with hypotenuse 1. Using these triangles, we arrive at a table of values of the trig functions, from which we can sketch the graphs on the next page.

2. Sinusoidal Functions

A sinusoidal function is a function made up from the function $y = \sin x$ (or $y = \sin t$, if time is playing the role of the x-variable) by inserting constants in various places. The reason for inserting these constants is that the pure sine function $y = \sin x$ almost never occurs in the real world, but curves with a sine-like appearance (this is what the word "sinusoidal" means) arise frequently.

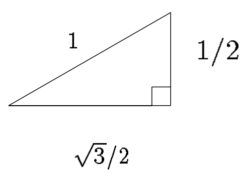
At this point you should review the material at the end of the section on functions: horizontal and vertical shifts, horizontal and vertical expansion, and the last batch of homework problems in that section.

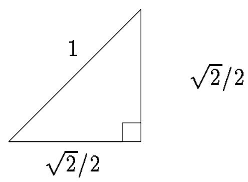
EXAMPLE 8.1. Write the cosine function as a shift of the sine function.

The graphs show that the cosine function is obtained by shifting the sine function $\pi/2$ to the left. Thus, $\cos x = \sin(x - (-\frac{\pi}{2})) = \sin(x + \frac{\pi}{2})$. This relation can also be obtained using the identities listed above: $\cos x = \cos(-x) = \sin(\frac{\pi}{2} - (-x)) = \sin(x + \frac{\pi}{2})$.

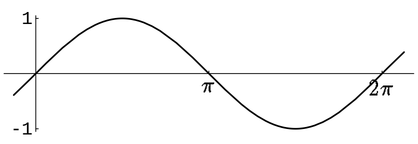
We now give the general form for a sinusoidal function $y = f(x)$. We start with $y = \sin x$.

Step 1. Expansion. Vertically, we expand by a factor of A, which is called the *amplitude* . We do this by replacing $\sin x$ by $A\sin x$. This makes the function go to a maximum of A and a minimum of $-A$ (rather than a maximum of 1 and a minimum of -1 as was the case for $\sin x$). Next, we expand horizontally in such a way that the function repeats its pattern from x to $x + B$ (rather than from x to $x + 2\pi$ as with $\sin x$). B is called the *period*. That is, we want to expand horizontally by a factor of $B/2\pi$. We do this by replacing x by $x/(B/2\pi) = \frac{2\pi}{B}x$ inside the sine. To summarize, the result of the vertical and horizontal expansion is to replace $\sin x$ by the function $A\sin\left(\frac{2\pi}{B}x\right)$. The graph of this

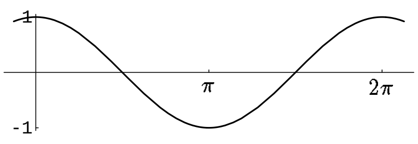




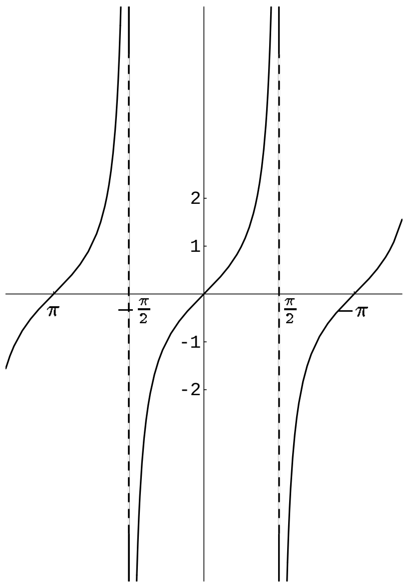
x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
π	0	-1	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	-1	0	∞
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
2π	0	1	0



Graph of $y = \sin(x)$

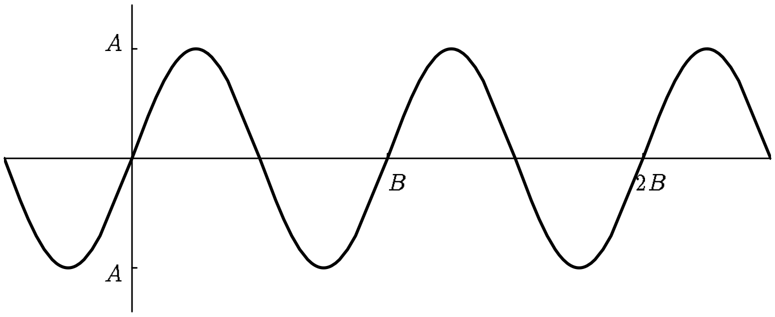


Graph of $y = \cos(x)$



Graph of $y = \tan(x)$

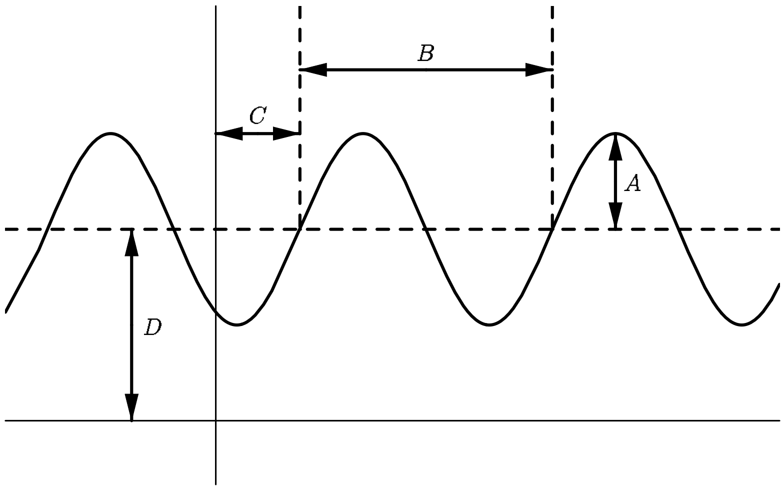
new function is shown below. Notice that as x goes from 0 to B, the value of the function goes from 0 to A to 0 to $-A$ and back to 0.



Step 2. Shifting. We now want to move the whole sine wave upward a certain amount D (called the *vertical shift*) and over to the right a certain amount C (called the *phase shift*). We do this by adding D outside the function, and replacing x by $x - C$ inside the function. The result is:

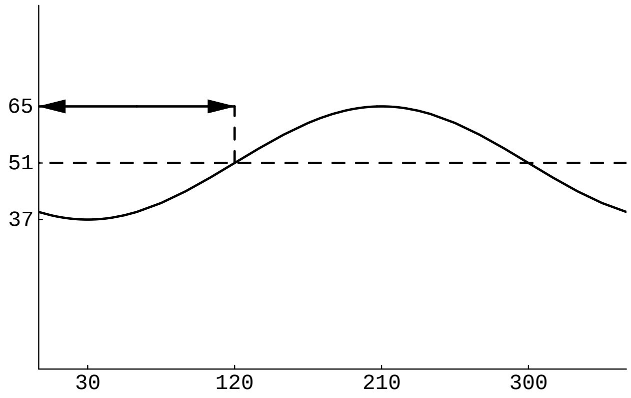
$$y = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

In the graph of this function below, we indicate the meaning of the four constants A, B, C, and D.



EXAMPLE 8.2. Suppose that the mean daily temperature (i.e., the temperature in °F averaged over the 24-hour period) on the t-th day of the year in Seattle is tabulated over a period of many years, and the results for each day are averaged. This is what we call the " normal mean temperature " on the t-th day. For simplicity, suppose that the year consists of 12 months of 30 days each. Take $t = 1$ to be January 1, and so on, until $t = 360$ for December 30. Further assume: (1) the normal mean temperature function $y(t)$ is sinusoidal; (2) the coldest mean temperature is 37° on January 30; and (3) the warmest mean temperature is 65° on July 30. Find a formula for $y(t)$.

We graph $y(t)$ by first plotting the two points $t = 30$, $y = 37$ and $t = 210$ (i.e., July 30), $y = 65$. We then draw a sine wave in such a way that its minimum is at the first point, its maximum is at the second, and its period is a year, i.e., 360 days. We also draw a horizontal dotted line half-way between the coldest and warmest temperature, i.e., at $y = 51$.



Now we read the four constants A, B, C, D off the graph. The amplitude A is 14, the period B is 360, and the vertical shift (the distance up to the dotted line) is 51. To find C we want to know when the curve crosses the dotted line on its way from a minimum to a maximum, i.e., when the curve looks like $\sin x$ looks when it crosses the origin. This can be found by taking the t-value half-way between the coldest day (which is $t = 30$) and the warmest day (which is $t = 210$). This t-value is $(30 + 210)/2 = 120$ (i.e., April 30). Thus, the phase shift is $C = 120$, and our equation is

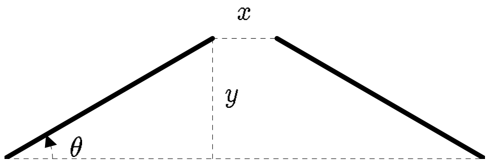
$$y(t) = 14 \sin\left(\frac{2\pi}{360}(t - 120)\right) + 51.$$

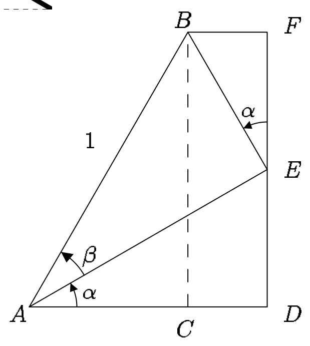
HOMEWORK 8.

1. Convert to radians: (a) 15°, (b) 120°, (c) a quarter revolution, (d) 10 revolutions.

2. A bicycle wheel has radius 40 cm. Find the rate in rad/sec at which the wheel is turning if the bicycle is traveling at (a) 10 m/sec, (b) 10 km/hr.

3. A drawbridge whose two spans are each 30 meters long is opened as shown below. Express in terms of θ: (a) the height y of the end of a span, (b) the slope of the left span, (c) the distance x between the ends of the spans.





4. Using the diagram at the right, show that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.$$

5. Sketch the graph of each of the following functions:

(a) $y = 1 + \sin(\pi x)$, (b) $y = 2\sin\left(\frac{\pi}{3}t\right)$,
(c) $y = 2 + 2\sin\left(\frac{\pi}{2}(t-1)\right)$,
(d) $y = 3 + 2\sin(2t+1)$.

6. The mean daily temperature in Fairbanks, Alaska was tabulated over a 30-year period, the average was taken over the 30 values for each day, and the result was found to be very close to the sinusoidal function $37\sin\left(\frac{2\pi}{365}(t-101)\right)+25$, where t is the day of the year.[1] (Here we are NOT assuming 30 days per month.) Sketch the graph of this temperature function, find the maximum and minimum mean daily temperatures, and find the warmest and coolest days of the year.

7. Studies are made of the variation in population of foxes and rabbits in a certain large forest. The number changes from year to year. Roughly speaking, when there are a lot of rabbits, the foxes will start to increase in number, because they can eat rabbits plentifully; but then this decreases the number of rabbits, causing the foxes to be short of food, and this causes a decrease in the fox population; and, finally, when there are fewer foxes the rabbits will be able to increase in number without being eaten, so we are back to where we started. Thus, we might expect a sinusoidal oscillation in the number of foxes and in the number of rabbits. (This is the so-called " predator-prey" problem in mathematical biology.) Suppose that t stands for the number of years since the study began, F stands for the number of foxes, and R stands for the number of rabbits. Suppose that F and R are each a sinusoidal function of t. Use the following table of data (which gives the maximum and minimum population of each) to find a formula for F and for R in terms of t:

t	0	1	2	3	4
F	150	200	150	100	150
R	10000	7500	5000	7500	10000

[1] This example is from an article by B. M. Lando and C. A. Lando in *The Mathematics Teacher*, Sept. 1977.

8. Write a formula for each of the following functions of t, assuming that the function is sinusoidal (also suppose that a year consists of 12 months of 30 days each):

(a) the mean low daily temperature, if the coldest is 20°F on January 19 and the highest low temperature reading is 60°F on July 19;

(b) the time of sunset each day, if the sun sets at 9 p.m. on June 21 and at 5 p.m. on December 21;

(c) the temperature in a well-insulated building, which varies between 25°C at 5 p.m. (which is 17:00 on the 24-hour clock) and 15°C at 5 a.m;

(d) the velocity of a weight attached to the end of a spring, which is extended and released at time $t = 0$, if the weight reaches its maximum velocity of 10 cm/sec after 0.1 sec;

(e) the height above the ground of a pebble that is picked up in the tread of a bicycle tire at time $t = 0$, if the tire has radius 40 cm and the bicycle is traveling at 10 m/sec.