Contents

CHAPTER 9

Derivative of Trig Functions

1. The Derivative of $\sin x$

To find the slope of the tangent line to the curve $y = f(x) = \sin x$, we must go back to the definition of the derivative as the limit of the difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \longrightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \longrightarrow 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}.$$

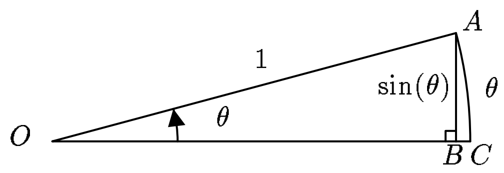
Recall the procedure used to find the derivative of the much simpler function x^2: expand $(x+\Delta x)^2$, simplify algebraically, and then take the limit as Δx gets closer and closer to 0. We do essentially the same thing for $\sin x$, but we need the following three basic facts in order to do this:

$$\text{Fact 1. } \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$$

$$\text{Fact 2. } \lim_{\theta \longrightarrow 0} \frac{\sin\theta}{\theta} = 1;$$

$$\text{Fact 3. } \lim_{\theta \longrightarrow 0} \frac{1-\cos\theta}{\theta} = 0.$$

Fact (1) was proved in the last section (Problem 4 of the homework). Facts (2) and (3) can be seen from the diagram below, where θ is a small angle in radians. By the definition of the radian measurement of an angle, θ is the distance along the circumference from A to C. Meanwhile, $\sin\theta = \overline{AB}$ and $1 - \cos\theta = \overline{BC}$.



Thus, Fact (2) says that although the line $\overline{AB}$ is smaller than the arc AC, the ratio of the two lengths gets closer and closer to 1 as θ gets smaller. Fact (3) says that the ratio of the line $\overline{BC}$ to the arc AC gets closer and closer to 0 as θ gets smaller. We are now ready to simplify the above difference quotient.

We have:

$$\frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin x\cos(\Delta x) + \cos x\sin(\Delta x) - \sin x}{\Delta x}$$

(by Fact (1) with $\alpha = x$ and $\beta = \Delta x$)

$$= \frac{\cos x\sin(\Delta x) - \sin x(1 - \cos\Delta x)}{\Delta x}$$

(putting $\sin x\cos(\Delta x)$ last on top and factoring out $-\sin x$)

$$\begin{aligned}&= \cos x\frac{\sin(\Delta x)}{\Delta x} - \sin x\frac{1-\cos(\Delta x)}{\Delta x}\\ &\longrightarrow (\cos x)\cdot 1 - (\sin x)\cdot 0 = \cos x \qquad \text{as} \qquad \Delta x \longrightarrow 0.\end{aligned}$$

The last step comes from Facts (2) and (3) with θ replaced by Δx. We therefore conclude the following basic derivative formula:

$$\frac{d}{dx}\sin x = \cos x.$$

EXAMPLE 9.1. You know the exact value $\sin(30^\circ) = 0.5$. Find $\sin(31^\circ)$ using the tangent line approximation, and compare with the exact value (found by calculator).

In the tangent line approximation we choose $f(x) = \sin x$, $x = 30^\circ = \pi/6$ and $\Delta x = 1^\circ = \pi/180$. Here it was important that we changed Δx from degrees to radians. By the tangent line approximation,

$$\begin{aligned}\sin(31^\circ) &= f(\frac{\pi}{6} + \frac{\pi}{180}) \approx 0.5 + \frac{\pi}{180}f'(\frac{\pi}{6})\\ &= 0.5 + \frac{\pi}{180}\cos(\frac{\pi}{6}) = 0.5 + \frac{\pi}{180}\cdot\frac{\sqrt{3}}{2} = 0.515115;\end{aligned}$$

the exact value is $0.515038\ldots$.

When we have a derivative formula for a new function, we can combine it with other rules to find the derivative of a more complicated function that is built up from the basic function.

EXAMPLE 9.2. Find the derivative of $\sin^2(\sqrt{x})$.

The notation $\sin^2(\text{something})$ means $\Big(\sin(\text{something})\Big)^2$. To evaluate this derivative, we have to use the chain rule twice:

$$\begin{aligned}\frac{d}{dx}\Big(\sin\sqrt{x}\Big)^2 &= 2\sin\sqrt{x}\,\frac{d}{dx}\sin\sqrt{x} = 2\sin\sqrt{x}\cos\sqrt{x}\,\frac{d}{dx}\sqrt{x}\\ &= 2\sin\sqrt{x}\cos\sqrt{x}\,(1/2\sqrt{x}) = \frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}.\end{aligned}$$

2. Other Trig Derivatives

All of the other trig functions can be expressed in terms of the sine, and so their derivatives can easily be calculated. The most important ones are:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin(x+\frac{\pi}{2}) = \cos(x+\frac{\pi}{2}) = \sin((x+\frac{\pi}{2})+\frac{\pi}{2}) = \sin(x+\pi) = -\sin x;$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\Big(\cos x\Big)^{-1} = (-1)\Big(\cos x\Big)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \tan x\sec x.$$

HOMEWORK 9.

1. You know the exact value $\tan(45°) = 1$. Find $\tan(44°)$ using the tangent line approximation, and compare with the exact value (found by calculator).

2. Find the derivative of (a) $y = \cot x$, (b) $y = \csc x$ (csc is the abbreviation of cosecant).

3. Use the tangent line approximation to find (a) $\cos(62°)$, (b) $\sin(29°)$, (c) $\cot(47°)$.

4. Find the derivative of each of the functions in Problem 8(a)–(e) of the last homework set. In each case find the first positive time t for which the derivative is 0 and the first positive t for which the derivative reaches its maximum value; then explain in practical terms what is happening at each of these moments.

5. Find the derivative of (a) $y = \sin x \cos x$, (b) $y = \sin(\cos x)$, (c) $y = \sqrt{x \tan x}$, (d) $y = \dfrac{\tan x}{1 + \sin x}$.

6. Suppose that the point $(\pi/3, \pi/4)$ is on the implicitly defined curve $\dfrac{\sin x}{x} + \dfrac{\sin y}{y} = \text{const.}$ Use the tangent line approximation to determine the y-coordinate of the point on the curve with x-coordinate $\dfrac{\pi}{3} + \dfrac{\pi}{180}$.

7. Suppose that the point $(1, 1.5)$ is on the curve $x \sin(\pi y/3) + y \sin(\pi x/4) = $const. Use the tangent line approximation to determine the y-coordinate of the point on the curve with x-coordinate 1.1.

8. A rope is tied at its left end, and its right end is moved up and down to create a sinusoidal wave of period 6 ft. The equation of the shape of the rope ("standing wave") is then

$$y = A \sin\left(\frac{2\pi}{6} x\right) = A \sin\left(\frac{\pi}{3} x\right),$$

where we are taking the origin to be the point where the rope is fastened, and A is the amplitude. Now actually, the amplitude A varies with time as the rope oscillates. Suppose that the maximum value of A is 2 ft, and it varies sinusoidally with time t with period 2 sec:

$$A = 2 \sin(\pi t).$$

Thus, the vertical coordinate y of a point on the rope depends on both its x-coordinate and the time t:

$$y = 2 \sin(\pi t) \cdot \sin\left(\frac{\pi}{3} x\right).$$

(a) Sketch the appearance of the rope at quarter-second intervals from $t = 0$ to $t = 2$ sec (all on the same graph).

(b) Suppose your vision is blocked by a horizontal ledge at a height of 6 inches above the x-axis, so you see only the tips of the waves. Write an equation that gives the x-value where the wave emerges above the ledge implicitly as a function of t (i.e., write an equation of the form $F(x, t) = $const).

(c) What is the first x-value where the rope emerges above the ledge when $t = 1/6$ sec?

(d) How rapidly is that first x-value moving leftward at the instant $t = 1/6$ sec?

(e) Approximately where will that first point where you see the rope be located when $t = 0.2$ sec? (Use parts (c) and (d) and the tangent line approximation.)