CHAPTER 1

Area Between Curves; Distance/Velocity/Acceleration

1. Area Between Curves

Suppose that we are interested in the area enclosed by two curves $y = f_1(x)$ and $y = f_2(x)$ between the two vertical lines $x = a$ and $x = b$, then we can proceed exactly as before. Recall that we first created an area function:

$$A(u) = \text{area caught between } y = f_1(x) \text{ and } y = f_2(x) \text{ between } x = a \text{ and } x = u.$$

As before we can demonstrate the derivative of the function to be $|f_1(x) - f_2(x)|$ and as long as this function is continuous, the Fundamental Theorem of Calculus will guarantee that the desired area is given by

$$A(b) = \int_a^b |f_1(x) - f_2(x)|dx.$$

Note that in our old formulation of area under a curve $y = f(x)$, we did not have the absolute value sign, but for proper calculations, we had to make sure that the curve $y = f(x)$ was never below the x-axis or the curve $y = 0$. In other words, we needed $f(x) \geq 0$ in the desired interval $[a, b]$. So $f(x) = |f(x)|$ in the interval $[a, b]$ and we did not need to write the absolute value sign. Similarly, we can omit it here, if we can ensure that $f_1(x) \geq f_2(x)$ during the interval $[a, b]$.

On the other hand, we did interpret $\int_a^b f(x)dx$ as an algebraic area between the curve $y = f(x)$ and the x-axis and it is possible to make a similar algebraic interpretation of $\int_a^b (f_1(x) - f_2(x))dx$ as well.

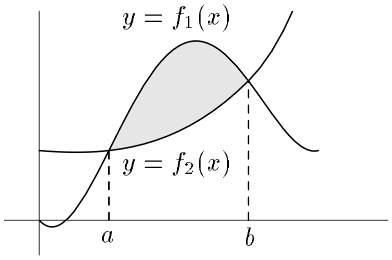
To work out concrete problems, it is better to remember the area formula in words as follows:

The area enclosed between $y = f_1(x)$ and $y = f_2(x)$ above the interval $x = a$ to $x = b$ is computed by integrating from a to b the expression $l(x)dx$. The expression $l(x)$ is the length of a typical vertical section of the area at x.

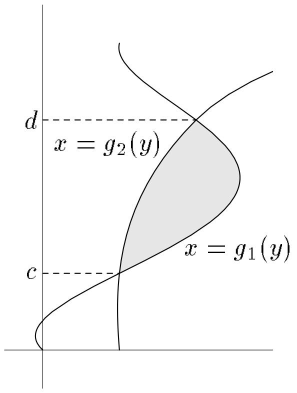
Typically, $l(x) = f_1(x) - f_2(x)$ provided you have $f(1) \geq f_2(x)$ during the interval $[a, b]$.

If only the two functions $f_1(x)$ and $f_2(x)$ are given, then the values of a, b will be the x-coordinates of common points of the two curves $y = f_1(x)$ and $y = f_2(x)$.

As before some shapes can best be integrated by considering exchanging the roles of x, y. In other words by letting $x = g_1(y)$ be the curve on the right and $x = g_2(y)$ the curve on the left, and computing Area$= \int_c^d g_1(y)dy - \int_c^d g_2(y)dy$, where c and d are the y-coordinates of the intersection points.



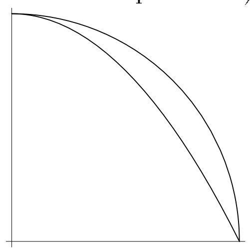
Area= $\int_a^b (f_1(x) - f_2(x))dx$



Area= $\int_c^d (g_1(y) - g_2(y))dy$

EXAMPLE 1.1. Find the area in the first quadrant between the unit circle and the parabola $y = 1 - x^2$.

By sketching a graph of the two curves, we can see that they intersect at (0,1) and (1,0). Alternatively, we can find the points of intersection algebraically by setting the two functions equal: $1 - x^2 = \sqrt{1 - x^2}$. Squaring both sides and moving everything to the right of the equality gives $0 = x^2 - x^4 = x^2(1 - x^2)$, so that $x = 0, \pm 1$ (we ignore $x = -1$, since that is not in the first quadrant). This gives us the two points. Which is the top curve? The simplest way to answer this is to take an intermediate point, say $x = 0.5$: the y-coordinate of the corresponding point on the circle is 0.866, while the y-coordinate of the point of the parabola is 0.75; thus, the circle is the top curve. Finally, the area we want is:



$$
\begin{aligned}
&\text{area under circle} - \text{area under parabola} \\
&= \pi/4 - \int_0^1 (1 - x^2)\,dx \\
&= (\pi/4) - (2/3) = 0.11873.
\end{aligned}
$$

Another way to make sure that you have the right formula is as follows. We have determined the limits of integration to be $0, 1$ already. For any typical x-value in between, the circle has the point $(x, \sqrt{1 - x^2})$ and the parabola has the point (x, x^2). The desired length function is the $|\sqrt{1 - x^2} - x^2|$. How do we know that the expression inside the $||$ is already nonnegative? As above, we can check it at some value like $x = 0.5$ and see that it is positive. To finish the argument, we need to know that it stays nonnegative in $[0, 1]$. But if it did become negative somewhere during this interval, then it will become 0 during the interval also. *What Theorem are we using?* But such a value of x must be the x-coordinate of a common point of the circle and the parabola *Why?*. Hence we are done!

What is the purpose of such abstract and complicated arguments, when we can just "see" which curve is above and which is below? The idea is to double check the sketching for messy curves and to possibly bypass the sketching altogether when it might be too complicated.

Note that we did not use the full formula:

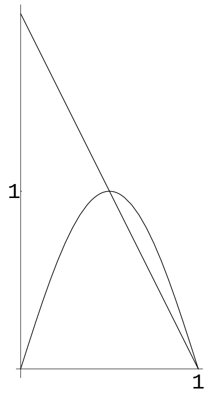
$$\int_0^1 |\sqrt{1-x^2} - x^2| dx = \int_0^1 (\sqrt{1-x^2} - x^2) dx = \int_0^1 \sqrt{1-x^2} dx - \int_0^1 x^2 dx$$

in our calculations. We short-circuited the first difficult integral by interpreting it as the area of a quarter circle! You should use such shortcuts whenever possible!

EXAMPLE 1.2. Find the area in the first quadrant bounded by $2x + y = 2$ and $y = \sin(\pi x)$.

Here one cannot find the intersection points algebraically by solving the equation $2 - 2x = \sin(\pi x)$. However, if we graph both $y = 2 - 2x$ and $y = \sin(\pi x)$ carefully, we see that they both cross the x-axis at (1,0) and they have another point of intersection at (0.5,1). Since the sine curve is above the line joining these two points, it follows that the area we want is

$$\int_{0.5}^1 \sin(\pi x) dx - \int_{0.5}^1 (2 - 2x)\, dx =$$
$$- (1/\pi)\cos(\pi x)|_{0.5}^1 - (2x - x^2)|_{0.5}^1$$
$$= (1/\pi) - 0.25 = 0.0683.$$



As above, let us give indirect arguments to verify our work. We have somehow guessed the limits to be $0.5, 1$ as above and we need to show that $l(x) = \sin(\pi x) - (2 - 2x) \geq 0$ on the interval $[0.5, 1]$. Note that $h(0.5) = h(1) = 0$. We claim that $h(x)$ does not have another root in $[0.5, 1]$. This will give us what we want. (**Why?**).

Suppose, if possible, $h(c) = 0$ for some $0.5 < c < 1$. Then $h'(x)$ must have at least two roots in $[0.5, 1]$; at least one in $(0.5, c)$ and at least one in $(c, 1)$ (**Why?**). We show that this is not possible! This will finish our claim.

The derivative $h'(x) = \pi\cos(\pi x) + 2$ and its derivative $h''(x) = -\pi^2 \sin(\pi x)$. Clearly $h''(x) < 0$ on the interval $[0.5, 1]$ (*Why?*) and so $h'(x)$ steadily decreases from $h'(0.5) = 2$ to $h'(1) = 2 - \pi < 0$. Hence, $h'(x)$ can have exactly one root in $[0.5, 1]$.

Notice how knowledge of higher derivatives of a function yields useful information about the function itself. This technique is often used.

2. Distance/Velocity/Acceleration

We next recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If $F(u)$ is an anti-derivative of $f(u)$, then $\int_a^b f(u)\, du = F(b) - F(a)$. Suppose that we want to let the upper limit of integration vary, i.e., we replace b by some variable x. We think of a as a fixed starting value x_0. In this new notation the last equation (after adding $F(a)$ to both sides) becomes:

$$F(x) = F(x_0) + \int_{x_0}^x f(u)\, du.$$

(Here u is the variable of integration, called a "dummy variable," since it is not the variable in the function $F(x)$. In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $\int_{x_0}^{x} f(x)dx$ is bad notation , and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the distance an object travels (say, along the x-axis) between time t_0 and time t. Let $s(t)$ denote the position of the object at time t (its distance from a reference point, such as the origin on the x-axis). Then the displacement of the object during the interval is $s(t) - s(t_0)$. If the object moves in a single direction along a line, then the displacement is equal to the distance traveled.

Since $s(t)$ is an anti-derivative of the velocity function $v(t)$, we can write

$$s(t) - s(t_0) = \int_{t_0}^{t} v(u)du.$$

Similarly, since the velocity is an anti-derivative of the acceleration function $a(t)$, we have

$$v(t) = v(t_0) + \int_{t_0}^{t} a(u)du.$$

As an example, let us take the case when an object is acted upon by a constant force F. By Newton's laws of motion, the acceleration is the constant F/m, where m is the mass. Then we first have

$$v(t) = v(t_0) + \int_{t_0}^{t} F/m\, du = v_0 + (F/m)u|_{t_0}^{t} = v_0 + \frac{F}{m}(t - t_0);$$

and then we have

$$\begin{aligned} s(t) &= s(t_0) + \int_{t_0}^{t} \left(v_0 + \frac{F}{m}(u - t_0)\right) du = s_0 + (v_0 u + \frac{F}{2m}(u - t_0)^2)|_{t_0}^{t} \\ &= s_0 + v_0(t - t_0) + \frac{F}{2m}(t - t_0)^2. \end{aligned}$$

For instance, when $F/m = -g$ is the constant of gravitational acceleration, then this is the falling body formula (where we neglect air resistance).

3. Total vs. Net Distance

The integral of the velocity function gives the displacement or the **net** distance traveled. This is because the integral of negative velocity (area underneath the t-axis) counts negatively. If you want to know the **total** distance traveled, you must integrate the $|v(t)|$ or the speed function. To do this, the best method is to find out where the velocity function crosses the t-axis, integrate separately over the time intervals when $v(t)$ is positive and when $v(t)$ is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is $v(t) = -9.8t + 19.6$. This is a straight line which is positive for $t < 2$ and negative for $t > 2$. The net distance traveled in the first 4 seconds is $\int_0^4 (-9.8t + 19.6)dt = 0$, while the total distance traveled in the first 4 seconds is $\int_0^2 (-9.8t + 19.6)dt + \left|\int_2^4 (-9.8t + 19.6)dt\right| = 39.2$ meters (i.e., 19.6 meters up and 19.6 meters down).

Example 1.3. The acceleration of an object is given by $a(t) = \cos(\pi t)$, and its velocity at time $t = 0$ is $1/2\pi$. Find the displacement (net distance) as well as the total distance traveled in the first 1.5 sec.

We compute

$$v(t) = v(0) + \int_0^t \cos(\pi u) du = \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi u)|_0^t = \frac{1}{\pi}\big(0.5 + \sin(\pi t)\big).$$

The *net* distance traveled is then

$$\begin{aligned} s(1.5) - s(0) &= \int_0^{1.5} \left(\frac{1}{\pi}(0.5 + \sin(\pi t)\right) dt \\ &= \frac{1}{\pi}\left(0.5t - \frac{1}{\pi}\cos(\pi t)\right)\Big|_0^{1.5} = \frac{0.75}{\pi} + \frac{1}{\pi^2} = 0.340. \end{aligned}$$

To find the *total* distance traveled , we ask when $(0.5 + \sin(\pi t))$ is zero, and when it is positive and negative. This function is 0 when $\sin(\pi t)$ is -0.5, i.e., when $\pi t = 7\pi/6$, $11\pi/6$, etc. The value $\pi t = 7\pi/6$, i.e., $t = 7/6$, is the only value in the range $0 \le t \le 1.5$. We see that $v(t) > 0$ for $t < 7/6$ and $v(t) < 0$ for $t > 7/6$. Hence, the total distance traveled is

$$\begin{aligned} &\int_0^{7/6} \left(\frac{1}{\pi}(0.5 + \sin(\pi t)\right) dt + \left|\int_{7/6}^{1.5} \left(\frac{1}{\pi}(0.5 + \sin(\pi t)\right) dt\right| \\ &= \frac{1}{\pi}\left(\frac{3.5}{6} - \frac{1}{\pi}\cos(7\pi/6) + \frac{1}{\pi} + \left|0.75 - \frac{3.5}{6} + \frac{1}{\pi}\cos(7\pi/6)\right|\right) = 0.409. \end{aligned}$$

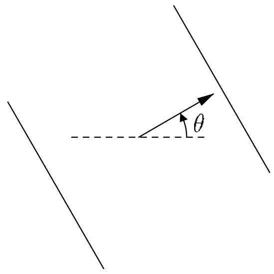
HOMEWORK 1.

1. Find the area between each of the following pairs of curves:
(a) $y = x^4 - x^2$ and $y = x^2$ (the part on the right of the y-axis);
(b) $x = y^3$ and $x = y^2$;
(c) $y = x^n$ and $x = y^n$ (in the first quadrant), where $n > 1$;
(d) $x = 1 - y^2$ and $y = -x - 1$;
(e) $x = 3y - y^2$ and $x + y = 3$;
(f) $y = \cos(\pi x/2)$ and $y = 1 - x^2$ (in the first quadrant);
(g) $y = \sin(\pi x/3)$ and $y = x$ (in the first quadrant).

2. Find the "triangular-looking" area bounded by the three curves:
(a) $y = \sin x$, $y = \cos x$, and $x = 0$ (in the first quadrant);
(b) $y = \sqrt{x}$, $y = 20 - x$, and $y = 2$.

3. For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph $v(t)$ to determine when it's positive and when it's negative):
(a) $v = \cos(\pi t)$, $0 \le t \le 2.5$; (b) $v = -9.8t + 49$, $0 \le t \le 10$;
(c) $v = 3(t-3)(t-1)$, $0 \le t \le 5$; (d) $v = \sin(\pi t/3) - t$, $0 \le t \le 1$.

4. Two parallel charged plates create a force of constant magnitude F on a charged particle of mass m. Suppose that the plates are rotating at the constant angular velocity of ω radians/sec, so that the *direction* of the force is *not* constant (see the figure at the right). Assume that the particle is at rest at the origin of the xy-plane at time $t = 0$, and that the force pulls in the positive x-direction at time $t = 0$ and rotates counterclockwise. (Also assume that ω is small enough so that the effects of the magnetic field created by the varying electric field can be neglected.) To describe the motion of the particle, consider the x- and y-directions separately.



(a) Express the x-component of the force vector in terms of the angle θ through which the plates have rotated, and express the latter angle in terms of t. Then find the x-acceleration function and the y-acceleration function (by dividing the force by the mass m). Here F, m, and ω are constants.

(b) Find the horizontal component of velocity and the vertical component of velocity as functions of time.

(c) Find formulas for the x-coordinate of the particle at time t and for the y-coordinate of the particle at time t.

(d) Set the constants all equal to 1, and sketch the parameterized curve in part (c).

(e) How is the trajectory affected if the particle has an initial velocity v_0 in the y-direction? For what value of v_0 (expressed in terms of F, m, and ω) will the particle travel in a circle?