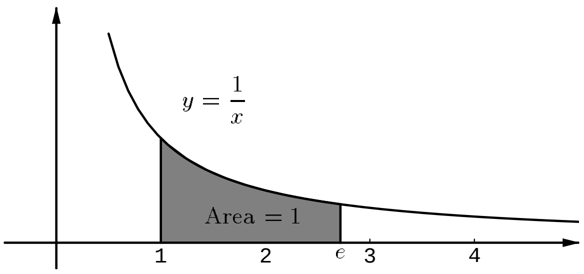
CHAPTER 1

The Number e and the Exponential Function

The number e is a fundamental constant of mathematics, every bit as important as π. It is defined as follows: **e is the number whose natural logarithm is exactly 1**. That is: $\ln e = 1$. In geometric terms, this says that you look at the area under the reciprocal function $y = 1/x$ between 1 and a, and move the upper limit of integration a to the right until you get an area of 1. (See the drawing below). It is not hard to see that e is somewhere between 2 and 3 (see Problem 1 at the end of the section). The value of e to several decimal places is: $e = 2.71828\cdots$. (The exact value cannot be given in decimal form, since e, like π, is an irrational number.)



In Chapter 8, we obtained the following relationship between the natural logarithm and the log-to-the-base-a: $\ln x = \ln a \cdot \log_a x$. In the special case $a = e$, since $\ln e = 1$ this gives us

$$\ln x = \log_e x,$$

in other words, *the natural logarithm is the same as the logarithm to the base e*. That is, $\ln x$ is equal to the power to which e must be raised to get x. This can be expressed in the form $e^{\ln x} = x$, and also in the form $\ln(e^x) = x$. In other words, *the functions* $\ln x$ *and* e^x *are inverse functions*. We shall return to this later.

There is another way to come up with the number e, which we shall explain using bank interest. Suppose that you put an initial amount of money P (called the "principal") in the bank at time $t = 0$. Suppose that the bank pays exactly 100% annual interest (this is not as unreasonable as it may seem, if you happen to live in a country with high inflation), compounded n times per year. If $n = 1$, then the money simply sits in the bank for one year without increasing, at which point you get your 100% interest; that is, you have $2P$ in the bank after the first year is up. If $n = 2$, then after 6 months the money in the bank increases from P to $1.5\,P$ (that is, after half a year you get 50%), and after another 6 months the money in the bank increases from $1.5\,P$ to $1.5 \cdot 1.5\,P = 2.25\,P$ (that is, you get another 50%, this time of a greater amount). In general, if the interest is compounded n times in the year, then every $1/n$ years whatever sum you have in the bank is multiplied by $\left(1+\frac{1}{n}\right)$. After 1 year the amount in the bank is

$$\left(1+\frac{1}{n}\right)^n P.$$

For example, if $n = 12$ (monthly compounding), then you have $2.613\,P$ after 1 year; if $n = 365$ (daily compounding), then you have $2.71457\,P$ after 1 year; if $n = 31536000$ (compounding every second), then you have $2.71828\,P$ after 1 year; and the larger n is (i.e., the more frequently the compounding occurs) the more money you have. However, this amount does not increase without bound. Rather, as $n \longrightarrow \infty$ the amount you have in the bank after 1 year becomes closer and closer to $e\,P = 2.71828\cdots P$. (In the next paragraph we explain why the value gets closer and closer to $e\,P$.) Taking the limit as n goes to infinity is called "continuous compounding." For all practical purposes, compounding every second is equivalent to continuous compounding, and even daily compounding is close to continuous compounding.

To see why the limiting value as $n \longrightarrow \infty$ is eP, we compute the limit as $n \longrightarrow \infty$ of $(1 + (1/n))^n$. Instead of looking at this power, let us take its natural log and find the limit of that:

$$\lim_{n\longrightarrow\infty} \ln\left((1+(1/n))^n\right) = \lim_{n\longrightarrow\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}.$$

But this is the slope of the chord on the graph of $y = \ln x$ joining the points $(1, 0)$ and $(1 + \frac{1}{n}, \ln(1 + \frac{1}{n}))$. As n increases, this chord approaches the tangent line to the curve at $x = 1$, and so this slope approaches the derivative at $x = 1$, which is 1 (since $\frac{d}{dx}\ln x = \frac{1}{x}$). Thus, as $n \longrightarrow \infty$, the ln of $(1+(1/n))^n$ approaches 1. Thus, $\lim_{n\longrightarrow\infty}(1+(1/n))^n$ is equal to the number whose ln is 1, namely e:

$$\lim_{n\longrightarrow\infty}\left(1+\frac{1}{n}\right)^n = e.$$

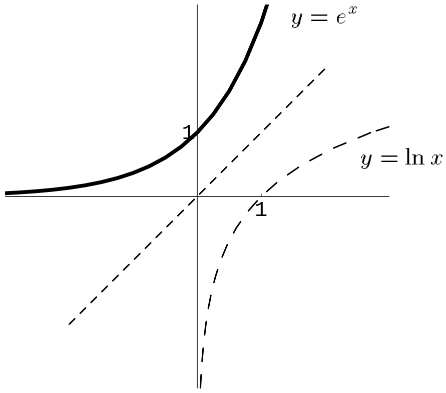
Let us return to the bank interest problem, and ask how much money is in the bank after t years. In t years the interest is compounded nt times, so that the amount in the bank is

$$\left(1+\frac{1}{n}\right)^{nt} P = \left(\left(1+\frac{1}{n}\right)^n\right)^t P.$$

If the money is "compounded continuously," i.e., we take the limit as $n \longrightarrow \infty$, then this amount is simply $e^t P$.

(In practice, most banks compound money daily: $n = 365$. We shall always assume, however, that the money is compounded "continuously," i.e., $n = \infty$. As a practical matter, there is very little difference. For example, even at the high interest rate of 100%, with daily compounding \$1000 grows to \$2714.57 in a year, while with continuous compounding it grows to \$2718.28. With lower interest rates, the difference between daily compounding and continuous compounding is even less.)

The function e^x is called the **exponential function**. It is the inverse function of $\log_e$, which we saw is the natural log function ln. In other words, to say that $y = e^x$ is equivalent to saying that $x = \ln y$. In the graph at the right we show the graphs of e^x (solid line) and $\ln x$ (dotted line). As always with inverse functions, these two graphs are the mirror images of one another with respect to the line $y = x$.



The function $y = f(x) = e^x$ has a very basic property, which we discover when we find its derivative y'. To find this derivative, we start with the relation $x = \ln y$, and take d/dx of both sides:

$$1 = \frac{1}{y}\frac{dy}{dx}, \qquad \text{so that} \qquad \frac{dy}{dx} = y, \qquad \text{i.e.,} \qquad \frac{d}{dx}e^x = e^x.$$

That is, the function $y = e^x$ has the very special property that *its derivative is the very same function.* It turns out that there is no other function — except for constant multiples Ce^x — that has this property.

1. Some Properties of the Exponential Function

By the law of exponents, if A and B are any numbers or expressions, then

(Property 1) $$e^A \cdot e^B = e^{A+B}; \qquad e^A/e^B = e^{A-B}; \qquad \left(e^A\right)^B = e^{AB}.$$

Because ln and the exponential function are inverse functions,

(Property 2) $$\ln(e^A) = A; \qquad e^{\ln A} = A.$$

By the chain rule,

(Property 3) $$\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x).$$

Since $\frac{d}{dx}e^{kx} = ke^{kx}$ (by (3) with $u(x) = kx$), it follows that

(Property 4) $$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C.$$

EXAMPLE 1.1. Find the center of mass of a uniform plate bounded by the lines $x = 0$, $x = 2$ and the curves $y = 1/2^x$ and $y = -1/2^x$.

By symmetry, the center of mass is on the x-axis. Its x-coordinate is

$$\overline{x} = \frac{\int_0^2 \frac{x}{2^x}dx}{\int_0^2 \frac{dx}{2^x}} = \frac{\int_0^2 xe^{-(\ln 2)\,x}dx}{\int_0^2 e^{-(\ln 2)\,x}dx},$$

where we have used properties (1) and (2) to write $1/2^x$ as $e^{-(\ln 2)\,x}$. We next use property (4) with $k = -\ln 2$ to find the bottom integral, and then the trapezoid rule (with $n = 4$) to find the top integral:

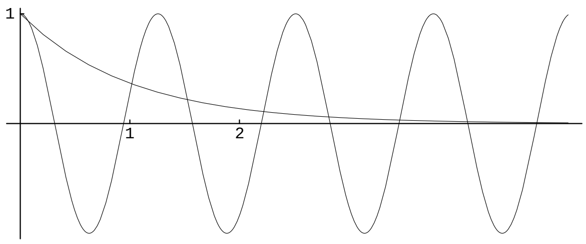
$$\int_0^2 e^{-(\ln 2)\,x}dx = -\frac{1}{\ln 2}e^{-(\ln 2)\,x}\Big|_0^2 = -\frac{1}{\ln 2}\frac{1}{2^x}\Big|_0^2 = \frac{1-(1/4)}{\ln 2} = 1.08202;$$

$$\int_0^2 \frac{x}{2^x}dx \approx \frac{1}{2}\left(0 + \frac{0.5}{\sqrt{2}} + \frac{1}{2} + \frac{1.5}{2\sqrt{2}} + \frac{1}{2}\cdot\frac{2}{2^2}\right) = \frac{5}{16}\sqrt{2} + \frac{3}{8} = 0.81694.$$

(Note: this last integral can be evaluated precisely by the method of integration by parts, which we will get to in about a week; it turns out that the exact value is $-1/(2\ln 2)+3/(2\ln 2)^2 = 0.83968$.) Finally, $\overline{x} = 0.83968/1.08202 = 0.77603$.

EXAMPLE 1.2. Sketch the curve $f(t) = e^{-t/T}\cos(\omega t)$ for $t > 0$ (where ω and T are constants). For simplicity, take $T = 1$ sec.

To get an idea of what to expect, let's sketch the two functions whose product is $f(t)$. The graph of $y = e^{-t}$ is obtained by reflecting the graph of $y = e^t$ (shown above) about the y-axis. The graph of $\cos(\omega t)$ is obtained by squeezing the graph of $\cos t$ by a factor of ω toward the y-axis (e.g., horizontal contraction by a factor of 5 if $\omega = 5$, horizontal expansion by a factor of 5 if $\omega = 1/5$). The graphs of $y = e^{-t}$ and $y = \cos(5t)$ are shown below:



The effect of multiplying $\cos(\omega t)$ (which oscillates between 1 and -1) by e^{-t} (which steadily decreases from 1 to 0) is to obtain a function which oscillates between maximum and minimum values (*local maxima and minima*) that are steadily decreasing. This type of function $f(t)$, called "damped harmonic oscillation," arises in practice, for example, if you want to describe the motion of an object at the end of a spring which is bouncing back and forth with amplitude that decreases because of friction and air resistance.

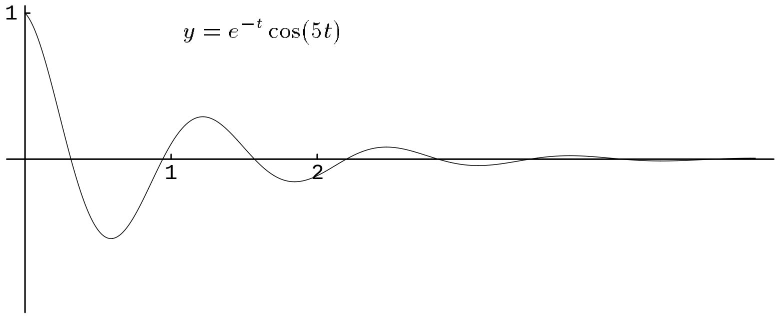
Now let us determine the t-values for which our function $f(t)$ has max/min points and points of inflection. To find the max/min points, we set $f'(t) = 0$:

$$0 = \frac{d}{dt}e^{-t}\cos(\omega t) = -e^{-t}\cos(\omega t) + e^{-t}(-\omega\sin(\omega t)) = -e^{-t}(\cos(\omega t) + \omega\sin(\omega t)).$$

Since e^{-t} is never 0, this derivative vanishes when $\cos(\omega t)+\omega\sin(\omega t) = 0$, i.e. (if we divide through by $\omega\cos(\omega t)$), when $\tan(\omega t) = -1/\omega$. Thus, our solutions t are all positive values (in radians) of $\arctan(-1/\omega)$, divided by ω. For example, if $\omega = 5$, then we successively obtain a minimum at $t = 0.589$, a maximum at $t = 1.217$, a minimum at $t = 1.845$, a maximum at $t = 2.474$, and so on. To find the inflection points, we set $f''(t) = 0$, i.e. (after a little simplification):

$$e^{-t}\big((1-\omega^2)\cos(\omega t) + 2\omega\sin(\omega t)\big) = 0.$$

Thus, the points of inflection occur when $t = \frac{1}{\omega}\arctan\left(\frac{\omega^2-1}{2\omega}\right)$. For example, if $\omega = 5$, then $t = 0.2\arctan 2.4 = 0.235, 0.864, 1.492, 2.120$, etc. The curve $y = e^{-t}\cos(5t)$ is sketched at the top of the next page.



$y = e^{-t}\cos(5t)$
1
1
2

HOMEWORK 1.

1. (a) Let l_1 be the tangent line to the reciprocal function at the point $(2, 0.5)$. Find the area under l_1 between $x = 1$ and $x = 3$.

(b) Let l_2 be the chord joining $(1, 1)$ to $(2.4, 1/2.4)$ on the graph of the reciprocal function. Find the area under l_2.

(c) Explain how your answers to parts (a) and (b) imply that the number e must be between 2.4 and 3.

2. Simplify the following expressions:

(a) $e^{\ln 3+\ln x}$; (c) $e^{\ln x-2\ln y}$; (e) $\ln\left(\sqrt{e^{x^2}}\right)$.

(b) $\ln\left(1/e^{x^2}\right)$; (d) $e^{-\ln(1/x)}$;

3. Solve for y in terms of x:

(a) $e^{0.1y} = x^2$; (c) $x = \ln\sin y$; (e) $\ln(1-y) - \ln y = 0.01x$.

(b) $\ln(y - x) = 2\ln x$; (d) $e^{y^2} = x^2$;

4. Find the derivative of

(a) $e^{2x}\text{Arctan}\, x$; (b) $\dfrac{x}{e^{\sin(\pi x)}}$; (c) e^{e^x}; (d) $x^2e^{-\cos x}$; (e) $e^{\text{Arcsin}\, x}$.

5. The function obtained by averaging the function e^x and the function e^{-x}, i.e., $(e^x+e^{-x})/2$, comes up often enough in applications to have earned the right to have its own name: "hyperbolic cosine." The function $(e^x - e^{-x})/2$ is called the "hyperbolic sine." These are abbreviated "cosh" and "sinh" ("sinh" is pronounced "sinch"):

$$\cosh x = \frac{e^x + e^{-x}}{2}; \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Find derivative formulas for cosh and sinh similar to the derivative formulas for cos and sin. (b) Simplify $\cosh^2 x - \sinh^2 x$. (c) Sketch the hyperbolic sine and cosine.

6. Evaluate:

(a) $\displaystyle\int_{\ln 0.04}^{\ln 4} e^{x/2}dx$, (b) $\displaystyle\int_0^1 xe^{-x^2}\,dx$, (c) $\displaystyle\int_0^{\pi} e^{\cos x}\sin x\,dx$, (d) $\displaystyle\int_0^{\infty}\frac{dt}{10^t}$.

7. Find the volume of the solid obtained by rotating the shape in Example 1.1 in the text around the x-axis.

8. Find the x- and y-coordinates of the maxima/minima and points of inflection, and sketch:

(a) $y = e^{\frac{2}{3}\sin(3x)}$, $0 \le x \le \pi/3$; (b) $y = x^2\ln(1/x)$, $0 < x < 1$.