CHAPTER 1

Exponential Growth and Decay

1. Continuously Compounded Interest

Let's return to the bank interest problem and suppose a more realistic annual interest rate of k (this is a percent written in decimal form, for example 8%= 0.08 — in this problem and many that follow, the units of k are reciprocal time, i.e., $0.08\frac{1}{\text{yr}}$ annual interest rate). We again want to find out how our principal P grows in t years. We again assume continuous compounding. But this time we shall use a different method to derive the function $y(t)$ that tells the amount of money in the bank after t years.

The method is to set up and solve a certain " differential equation." Recall that a **differential equation** is an equation for finding an unknown function in which the derivative of the unknown function (and perhaps its second derivative or higher derivatives) appears. For example, (1) the differential equation $y' = f(x)$, where $f(x)$ is a given function, is satisfied by an antiderivative of $f(x)$; (2) the differential equation $\ddot{y} = -g$ is satisfied by the falling body formula $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$.

Differential equations are fundamental in applications, because often we have to get at a function $y(t)$ indirectly, using some information about the rate of change of $y(t)$ that comes from translating some basic principle (of physics or finances, for instance) into mathematics. For example, the differential equation $\ddot{y} = -g$ is simply the statement that gravity near the surface of the earth imparts a constant acceleration.

In the case of bank interest continuously compounded at rate k, the basic principle is that *at any instant interest is accumulating at a rate in dollars per year equal to k times the current amount in the bank.* For example, if $k = 8\% = 0.08$ and you put \$1000 in the bank, then at first your money increases at the rate of \$80.00 per year; but a little while later, when you have (say) \$1010 in the bank, at that instant it is increasing at the rate of \$80.80 per year. That is, you are "earning interest on the interest" as well as on the original principal. The basic principle in italics can be written mathematically as follows:

$$\frac{dy}{dt} = ky,$$

i.e., the function $y(t)$ telling the number of dollars in the bank at time t years increases at a rate equal to k times the present value of this function.

Our task now is to solve this equation to find an expression for y in terms of t. A useful technique in this and many other differential equations is called " separation of variables." This means that we get everything involving y on one side of the equation and everything involving t on the other. Here the dy and dt are handled just like algebraic quantities. (When discussing the chain rule and also the method of substitution in integrals, we saw that it often makes sense to split up expressions like dy/dx or dy/dt, treating dy, dx, dt separately.) In the above differential equation we separate variables by multiplying both sides by dt and dividing both sides by y. After that the second step is to put an integral sign on both sides:

$$\int \frac{dy}{y} = \int k\, dt.$$

On the left we have an indefinite integral with respect to the variable of integration y, and on the right we have an indefinite integral with respect to t. On the left we get $\ln|y| + C_1$. Since $y(t)$ is a positive function, we drop the absolute value sign. Also, we move the constant of integration to the right of the equality, subtracting it from the constant of integration we get on the right from integrating $\int k\, dt = kt + C_2$. That is, we set $C = C_2 - C_1$. Thus, we have:

$$\ln y = kt + C.$$

Finally, to solve for y we exponentiate both sides (this means: raise e to the expression on each side of the equality). Since $e^{\ln y} = y$, this gives us a formula for y:

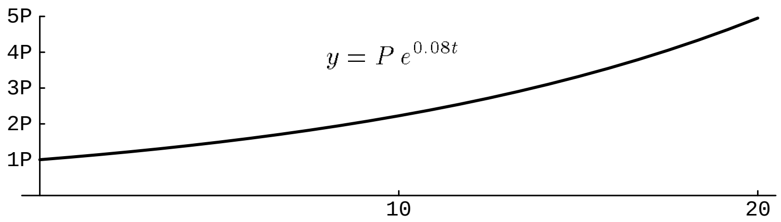
$$y(t) = e^{kt+C} = e^C e^{kt}.$$

We are not done, however, because we must determine the constant e^C in front of e^{kt}. As in acceleration–velocity–distance problems, we do this by using initial information. In the bank interest problem, the initial information is that we had P dollars in the bank at time $t = 0$: $y(0) = P$. Substituting $t = 0$ in the equation $y(t) = e^C e^{kt}$ gives the result: $P = y(0) = e^C e^0 = e^C$. Thus, we conclude that

$$y(t) = Pe^{kt}.$$

(The special case of this formula when $k = 100\% = 1$ was obtained in the last section by a different method.)

A function of the form $y = Pe^{kt}$ is called **exponential growth**. Its graph is shown below (in the case $k = 8\% = 0.08$).



Effective annual interest rate. By the **effective** interest rate we mean the percent by which the money actually increases if left in the bank 1 year. The increase is from P to $Pe^{k\cdot 1}$, i.e., $P(e^k - 1)$. Thus, the percent increase is $e^k - 1$ (which we usually convert to a percent by multiplying by 100):

$$\text{effective annual interest rate} = e^k - 1.$$

For example, if $k = 0.08$, then the effective rate is $e^{0.08} - 1 = 0.0833 = 8.33\%$. If $k = 100\%$, then the effective rate is $e - 1 = 171.83\%$. In general, if k is small, then the effective rate is only slightly more than k; while if k is large, then the effective rate is much greater, because it takes into account the interest earned on the interest. Sometimes people say "annual interest rate" when they really mean "effective annual interest rate." Sometimes k is called the **instantaneous interest rate**.

Note. *Whenever we say simply "interest rate" we shall mean the instantaneous rate k, not the effective rate $e^k - 1$.*

Doubling time. At time t we have Pe^{kt} in the bank. Let us ask how long it will take for the money to double. This time period, which we call D, must satisfy the equation $Pe^{k(t+D)} = 2Pe^{kt}$. Dividing both sides by Pe^{kt} and then taking the ln of both sides, we obtain $kD = \ln 2$, i.e., $D = (\ln 2)/k$. Notice that the doubling time D does not depend on the starting time t or on the principal P: it depends only on the interest rate. For example, at 8% interest rate the doubling time is $(\ln 2)/0.08 = 8\ 2/3$ years.

2. Other Examples of Exponential Growth.

Any time we have something whose rate of increase is proportional to its present value (i.e., $\frac{dy}{dt} = ky$), we obtain the exponential function $y(t) = y_0e^{kt}$, where k is the constant of proportionality between the rate of increase and the present value, and y_0 is the initial value of our function. Among the examples of this situation are:

Inflation. Rate of increase in cost of living is measured as an annual percent of the current cost of living. If the rate of inflation remains constant, then this cost of living follows the law of exponential growth.

Example 1.1. In the six-month period from June to December 1990 in Nicaragua, the córdoba deflated from an exchange rate of about 200000 per dollar to 2000000 per dollar. Find both the effective annual rate and the instantaneous rate k of inflation of the price in córdobas of an item whose dollar cost remains constant.

Here the inflation was by a factor of 10 in 1/2 year, and hence by a factor of 100 in 1 year. So the cost increased by $99 = 9900\%$. Thus, the effective rate is $99 = e^k - 1$. To find k, we add 1 to both sides and take ln, obtaining $k = \ln 100 = 4.605 = 460.5\%$.

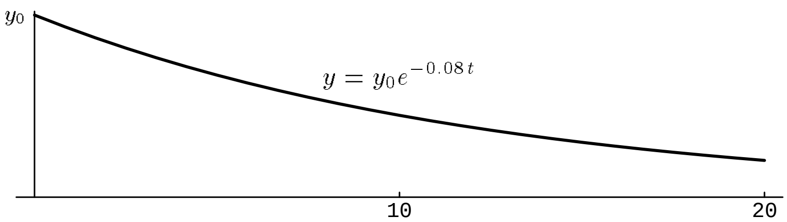
Population. If the birth and death rates remain constant (and there is no immigration or emigration), then the population as a function of time follows the law of exponential growth, in which k is the difference: $k =$ birth rate $-$ death rate.

Example 1.2. If a bacteria culture has a 4% per hour instantaneous birth rate and a 2.5% instantaneous death rate, what is its doubling time?

Here $k = 4\% - 2.5\% = 1.5\%$, and so $D = (\ln 2)/0.015 = 46.2$ hours.

3. Exponential Decay

Here we have a function $y(t)$ which **decreases** at a rate proportional to its current value, with proportionality constant k: $\frac{dy}{dt} = -ky$. Mathematically speaking, nothing much changes: to solve this differential equation we simply replace k by $-k$ in our earlier derivation, thereby obtaining the function $y(t) = y_0e^{-kt}$. However, the graph of y_0e^{-kt} as t increases looks very different from the graph of y_0e^{kt}, namely, it ends up decreasing rapidly toward 0:



Half-life. An exponential decay function $y(t) = y_0 e^{-kt}$ has a " half-life" rather than a doubling time. This is the length of time H that it takes to decrease to half its current value. To find H, we set $y_0 e^{-k(t+H)}$ equal to half of $y_0 e^{-kt}$. Dividing this equation through by $y_0 e^{-kt}$ and then taking ln gives: $-kH = \ln(1/2) = -\ln 2$, and so $H = (\ln 2)/k$. Note that this is the same as the doubling time formula in the case of exponential growth.

4. Examples of Exponential Decay.

Radioactive decay.

EXAMPLE 1.3. Some future visitors to Earth find a planet that was rendered uninhabitable by a catastrophe that occurred many thousands of years before. By reconstructing the accident, they determine how much radioactive Plutonium 239 was released at that time. They also measure the amount of Plutonium 239 remaining, and find that only 1/500 of the radioactive Plutonium is still present. Using the value 24400 years for the half-life of Plutonium 239, determine how much time elapsed between the accident and the arrival of the visitors.

We first find k, using the relation $kH = \ln 2$: $k = \ln 2/24400 = 0.0000284$. Then we find t by setting $y(t) = \frac{1}{500}y_0$, i.e., $\frac{1}{500}y_0 = y_0 e^{-kt}$. Dividing by y_0 and taking ln gives $-kt = \ln(1/500) = -\ln 500$, so that $t = (\ln 500)/0.0000284 = 218800$ years.

Population when the death rate exceeds the birth rate.

"Real" salary (measured in inflation-adjusted dollars, i.e., buying power of what you earn) if the rate of salary increase is less than the inflation rate.

Finally, we should emphasize that real-world phenomena follow the law of exponential growth or decay only if the thing being measured satisfies the differential equation $y' = ky$ or $y' = -ky$. In practice, when we suppose this to be the case we are making a simplifying assumption that may be quite inaccurate. For example, most human or animal populations are influenced by many factors and cannot be described at all accurately by the exponential growth model.

HOMEWORK 1.

1. Suppose that interest on money in the bank accumulates at an annual rate of 7% compounded continuously. How much should be invested today so that 20 years from now it will be worth $20000? B

2. Suppose that a sum of money draws interest compounded continuously. How long will it take to double if the annual rate is (a) 6%?, (b) 10%?, (c) 18%? How long will it take to double if the effective annual rate is (a) 6%?, (b) 10%?, (c) 18%?

3. In an experiment to study the growth of bacteria, a medical student measured 5000 bacteria at time 0 and 8000 at time 10 minutes. Assuming that the number of bacteria grows exponentially, how many bacteria will be present after 30 minutes?

4. Suppose that the population of a certain species grows at the instantaneous rate of 2% per year. If the present population is 1000000, what will the population be in 1 year? In 20 years? How long will the population take to double?

5. Assume that the population of the Earth changes at a rate proportional to its current population. Assume further that in A.D. 1650 its population was 250 million, and in A.D. 1950 its population was 2.5 billion. Find an expression giving the population of the Earth at any time. If the greatest population the Earth can support is 25 billion, when will this limit be reached?

6. If you hear on the news that prices are rising exponentially at the effective yearly rate of 12%, what is the instantaneous annual rate of inflation? How long does it take prices to double at this rate?

7. Suppose that the average yearly cost of college in the U.S. (tuition and expenses) is increasing at the instantaneous rate of 10% per year. By what factor will the cost of college increase for a child born now in the 18 years before she goes to college?

8. Suppose that bank interest is compounded continuously at 6%, and inflation on food has instantaneous rate 13%. (a) If you put $10000 in the bank, how much will you have 100 years later? (b) If it now costs $100/week to feed an average family, write a formula for weekly food cost t years later. (c) What is the **total** cost of feeding a family over the next 10 years? (d) What will the weekly food bill be in 100 years? (e) If your great grandchild takes your money in part (a) out of the bank at that time, how long will it feed his or her family?

9. The half-life of Carbon 14 is 5568 years. Archeologists find an ancient table made from a tree which had y_0 grams of Carbon 14 when it was cut. Now the wood has only y_1 grams of Carbon 14. Express the age of the table in terms of y_0 and y_1.

10. A certain radioactive material has a half-life of 2 hours. Find the time needed for a given amount of this material to decay to one-tenth of its original mass.

11. A certain radioactive material is known to decay at a rate proportional to the amount present. A block of this material originally having a mass of 100 grams is observed after 20 years to have a mass of only 80 grams. Find an expression for the mass of the material at any time. Also find the half-life of the material.

12. Suppose that the average yearly real income (i.e., in inflation-adjusted dollars, say 1995-dollars) of a U.S. worker is now decreasing at the instantaneous rate of 1.5% per year, and that this trend continues indefinitely. If the current average salary is $18000, what will an average worker's yearly income (in 1995-dollars) be in 30 years?

13. You are flying in a plane at constant altitude y_0, constant speed v_0 and constant direction. You want to drop a container of relief supplies to a target that is straight ahead under your flight path. You will release the container at the instant when your line of sight to the target makes an angle α below the horizontal. The purpose of this problem is to determine α. Let $v(t)$ denote the horizontal component of the velocity of the container. Assume that air resistance causes a horizontal component of deceleration that is proportional to velocity, with proportionality constant k. Air resistance is the only force acting to change the velocity (since gravity does not affect horizontal motion).

(a) Write a differential equation for $v(t)$, and find a formula for $v(t)$.

(b) Find a formula for the horizontal distance traveled by the container between time 0 and time t.

(c) Suppose that after enough time has elapsed so that the exponential decay term in part (c) is negligible, a parachute opens and the container glides straight down until it hits the ground. Find α.