CHAPTER 1

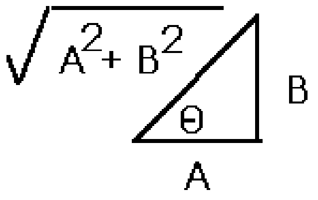
Techniques of Integration IV: Inverse Trig Substitution

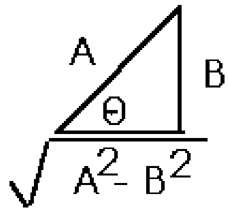
The technique of “ inverse trig substitution” is used when you want to integrate a function involving the square root of a quadratic polynomial, such as $\sqrt{25-x^2}$, $\sqrt{x^2-2x}$, $\sqrt{x^2+2x+2}$, etc. It is an “inverse” substitution because, instead of writing a new variable u as a function of x (i.e., $u = u(x)$), you write x in terms of a new variable θ (i.e., $x = x(\theta)$), where the new variable θ stands for an angle in a right triangle. Here is the procedure:

1. Take the quadratic which is under the radical and write it either as a sum of squares $A^2 + B^2$ or as a difference of squares $A^2 - B^2$, where one of the terms A or B is a constant and the other is either x or a multiple of x (or at worst an expression of the form $ax + b$). In the three examples above, we select A and B as follows:

$$25 - x^2 = 5^2 - x^2, \qquad x^2 - 2x = (x-1)^2 - 1^2, \qquad x^2 + 2x + 2 = (x+1)^2 + 1^2.$$

2. Draw a right triangle, and label two of the sides A and B. If the quadratic under the radical is $A^2 + B^2$, then take the two sides to be the legs of the right triangle. If the quadratic is $A^2 - B^2$, then take A to be the hypotenuse and B to be one of the legs. In this way the radical itself (i.e., $\sqrt{A^2+B^2}$ or else $\sqrt{A^2-B^2}$) will be the third side.

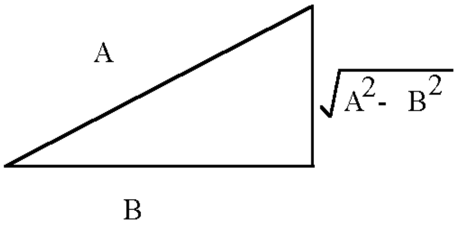




3. Let θ be one of the acute angles of the right triangle (either one will do). Then write everything in the integral — x, dx, and the square root — in terms of the new variable θ (also change the limits of integration, if you're evaluating a definite integral). Hopefully, the new integral with θ as the variable of integration will be relatively easy to evaluate.

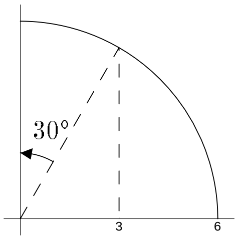
We now give some examples of the use of this method.

EXAMPLE 1.1. Find the area in the first quadrant that is bounded by the x- and y-axes, the line $x = 3$, and the circle of radius 6 centered at the origin.



To evaluate $\int_0^3 \sqrt{36-x^2}dx$, we go through the three steps above. (1) Set $A = 6$ and $B = x$, so that the radical is $\sqrt{A^2 - B^2}$. (2) Draw a right triangle with the sides and angle labeled as shown (see the figure at the right)
(3) Write $x = 6\sin\theta$, so that $dx = 6\cos\theta\, d\theta$. In addition, the square root is the adjacent side, so that $\sqrt{36-x^2} = 6\cos\theta$. Since $\theta = \arcsin(x/6)$, the limits of integration become $\arcsin(0) = 0$ and $\arcsin(3/6) = \pi/6$. Thus,

$$\begin{aligned}\int_0^3 \sqrt{36-x^2}\,dx &= \int_0^{\pi/6} (6\cos\theta)6\cos\theta\,d\theta = 36\int_0^{\pi/6} \cos^2\theta\,d\theta \\ &= 36\int_0^{\pi/6} \frac{1+\cos(2\theta)}{2}\,d\theta = 36\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/6} \\ &= 3\pi + 9\sin(\frac{\pi}{3}) = 3\pi + \frac{9}{2}\sqrt{3}.\end{aligned}$$

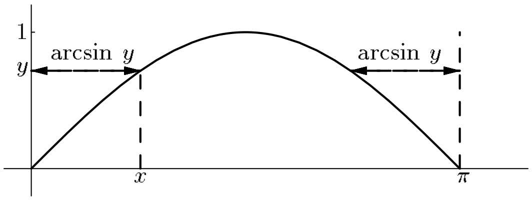


We could have actually obtained the same result by some simple geometry. The region in question is made up of a 30° sector of a circle (i.e., 1/12 of a circle of radius 6) and a triangle of base 3 and altitude $3\sqrt{3}$ (see the figure at the right). So the area is

$$\frac{1}{12}\pi\cdot 6^2 + \frac{1}{2}3\cdot 3\sqrt{3} = 3\pi + \frac{9}{2}\sqrt{3}\,.$$

Example 1.2. Find the center of mass of a uniform flat plate in the shape of one hump of the sine curve.

By considerations of symmetry, $\overline{x}$ is clearly $\pi/2$. To find $\overline{y}$, we must find the distance across from left to right at height y. From the diagram at the right, it should be clear that this distance is $\pi - 2\arcsin(y)$. The bottom integral in $\overline{y}$ is just the area under $y = \sin x$ from 0 to π, which is 2. So it remains to compute the top integral



$$\int_0^1 y(\pi - 2\arcsin y)\,dy = \frac{\pi}{2} - 2\int_0^1 y\arcsin y\,dy\,.$$

Using integration by parts, we transform the last integral as follows:

$$\int_0^1 y\arcsin y\,dy = \frac{1}{2}y^2\arcsin y\,\Big|_0^1 - \int_0^1 \frac{1}{2}y^2\frac{1}{\sqrt{1-y^2}}dy = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{y^2dy}{\sqrt{1-y^2}}.$$

We now use an inverse trig substitution, drawing our triangle so that 1 is the hypotenuse and y is the leg opposite the angle θ. We have $y = \sin\theta$, $dy = \cos\theta\,d\theta$, $\sqrt{1-y^2} = \cos\theta$, and the limits of integration become 0 to $\pi/2$. Putting this all together, we find that the top integral in $\overline{y}$ is equal to

$$\begin{aligned}\frac{\pi}{2} - 2\int_0^1 \arcsin\, y\,dy &= \frac{\pi}{2} - 2\left(\frac{\pi}{4} - \frac{1}{2}\int_0^1 (y^2dy/\sqrt{1-y^2})\right)\\ &= \int_0^1 (y^2dy/\sqrt{1-y^2}),\end{aligned}$$

which with the substitution $y = \sin\theta$ becomes

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta} &= \int_0^{\pi/2} \sin^2\theta\,d\theta\\ &= \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2}d\theta = \left(\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/2} = \pi/4\,.\end{aligned}$$

We conclude that $\overline{y} = \pi/8 = 0.3927$.

Using this type of procedure — integration by parts followed by inverse trig substitution — one can integrate $x^n \arcsin x$ for any positive integer n.

EXAMPLE 1.3. Find the antiderivative of $1/\sqrt{x^2-2x}$.

As shown above, we can write x^2-2x as a difference of two squares:

$$x^2 - 2x = (x-1)^2 - 1^2\,.$$

So we choose the hypotenuse of the right triangle to be $x-1$ and the leg opposite θ to be 1 (see the drawing at the right). Then $\sin\theta = 1/(x-1)$ and so $x = 1 + (1/\sin\theta)$, $dx = -(\cos\theta/\sin^2\theta)\,d\theta$, and the radical is $\cot\theta = \cos\theta/\sin\theta$. Hence,

$$\int \frac{dx}{\sqrt{x^2-2x}} = \int \frac{-(\cos\theta/\sin^2\theta)d\theta}{\cos\theta/\sin\theta} = -\int \frac{d\theta}{\sin\theta}.$$

This integral is very similar to Problem 3 in the last homework set (just reverse the roles of sin and cos). The result (taking into account the minus sign in front of the integral) is: $\frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} + C$. Finally, we can write the answer in terms of x by reading $\cos\theta$ off the triangle: $\cos\theta = \sqrt{x^2-2x}/(x-1)$. (Note: The answer can be simplified algebraically, using the properties of ln; the result is: $\ln|x-1+\sqrt{x^2-2x}| + C$.)

EXAMPLE 1.4. Find $\int x\,dx/\sqrt{x^2+2x+2}$.

As mentioned earlier, we can write $x^2 + 2x + 2$ as a sum of two squares: $x^2+2x+2 = (x+1)^2+1^2$. So we choose $x+1$ to be one leg (say, the leg opposite θ) and 1 to be the other leg of the right triangle. Then $x+1 = \tan\theta$, and so $x = \tan\theta - 1$, $dx = \sec^2\theta\,d\theta$. In addition, $\sqrt{x^2+2x+2}$ is the hypotenuse of the triangle, which is $\sec\theta$. So our integral becomes

$$\int \frac{(\tan\theta - 1)\sec^2\theta\, d\theta}{\sec\theta} = \int \tan\theta \sec\theta\, d\theta - \int \sec\theta\, d\theta.$$

The function inside the first integral happens to be the derivative of $\sec\theta$. (If you didn't notice this, you could convert the integral to sines and cosines, i.e., to $\int \sin\theta\, d\theta/\cos^2\theta$, and use the substitution $u = \cos\theta$.) The second integral is Problem 3 of the last homework set. Putting this together (and using the fact that $\sec\theta = \sqrt{x^2+2x+2}$ and $\sin\theta = (x+1)/\sqrt{x^2+2x+2}$), we obtain the following answer:

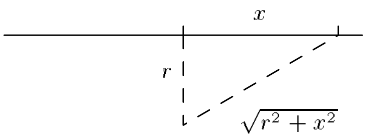
$$\int \frac{x\, dx}{\sqrt{x^2+2x+2}} = \sqrt{x^2+2x+2} + \frac{1}{2}\ln\frac{\sqrt{x^2+2x+2}+x+1}{\sqrt{x^2+2x+2}-(x+1)} + C.$$

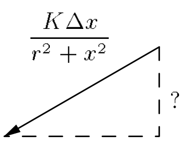
Example 1.5. (Force of repulsion from a line source of electric charge.)

Suppose that we have a long straight wire with an electric charge uniformly distributed along it. Our purpose is to determine how the force of repulsion exerted on a similarly charged particle drops off as the particle moves away from the wire.

The basic physical law needed here is: *the force of repulsion between two particles of charge* q_1 *and* q_2 *has magnitude* $k\, q_1 q_2/\textit{distance}^2$*, where* k *is a universal constant.* Before doing this example, we should review Example 9.7 in the section on improper integrals (noise from the freeway). There also we had an inverse square law. As in the noise example, we set up our wire as the x-axis, taking 0 to be the point at the foot of the perpendicular drawn from the charged particle. If c stands for the amount of charge per unit distance along the wire, then the contribution to the force of repulsion coming from the small segment of wire between x and $x + \Delta x$ is $k\, q_1(c\Delta x)/\text{distance}^2 = K\Delta x/(r^2+x^2)$, where we have set $K = kq_1c$. (See the diagram in the freeway example.)

However, there is one basic difference between the two examples. In the freeway problem we were dealing with a **scalar** (amount of noise) rather than force, which is a **vector** (it has both magnitude and direction). This means that the force contributed by the segment of the wire between x and $x+\Delta x$ points at an angle away from the vertical. What is that angle? You should convince yourself that the angle is $\arctan(x/r)$. The second diagram to the right below shows the force vector contributed by the part of the wire between x and $x + \Delta x$.





The horizontal component of this force (i.e., the component parallel to the wire) is canceled by the force contributed from the opposite segment (the segment between $-x$ and $-x-\Delta x$). So the contribution to the force of repulsion (the component pointing away from the wire) is only a portion of $K\Delta x/(r^2+x^2)$. The portion can be determined by similar triangles (see the diagram at the right). Comparing the two triangles, we find that the side "?" which we want to find has the same ratio to $K\Delta x/(r^2+x^2)$ as r does to $\sqrt{r^2+x^2}$. Thus, the contribution of the small segment is equal to:

$$\frac{r}{\sqrt{r^2+x^2}} \cdot \frac{K\Delta x}{r^2+x^2} = \frac{Kr\Delta x}{(r^2+x^2)^{3/2}}.$$

Proceeding as in the freeway example and making the simplifying assumption that the wire extends from $-\infty$ to ∞, we obtain the following definite integral for the force of repulsion: $\int_{-\infty}^{\infty} Kr\,dx/\left(r^2+x^2\right)^{3/2}$. This is an integral that can be evaluated by an inverse trig substitution, taking x and r to be the legs of the right triangle (say, x is the leg opposite θ). Then $x = r\tan\theta$, $dx = r\sec^2\theta\,d\theta$, $\sqrt{r^2+x^2} = r\sec\theta$. The limits of integration become $-\pi/2$ to $\pi/2$, so we obtain:

$$\int_{-\infty}^{\infty} \frac{Kr\,dx}{\left(\sqrt{r^2+x^2}\right)^3} = \int_{-\pi/2}^{\pi/2} \frac{Kr^2\sec^2\theta\,d\theta}{(r\sec\theta)^3} = \frac{K}{r}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \frac{2K}{r}.$$

Hence we conclude (as in the freeway example) that the force drops off proportionally to the distance (not the square of the distance).

1. A Final Word

Before leaving the subject of techniques of integration, we should point out that sometimes you have a choice of techniques, and your answer using one technique might at first glance seem to be different from your answer if you use a different technique.

Example 1.6. Find $\int \sin x\cos x\,dx$.

First method. Substitute $u = \sin x$, so that $du = \cos x\,dx$. We obtain: $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$.

Another method. Substitute $u = \cos x$, so that $du = -\sin x\,dx$. Then we get: $-\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$.

A third method. Use the trig identity for the sine of twice an angle: $\sin(2x) = 2\sin x\cos x$. We can thus rewrite our integral as $\int \frac{1}{2}\sin(2x)dx = -\frac{1}{4}\cos(2x) + C$.

Why three different answers? The point is that the three answers are not really different. Namely, the second answer can be written $-\frac{1}{2}(1-\sin^2 x)+C = \frac{1}{2}\sin^2 x + C - \frac{1}{2}$; and the third answer (if we use the cosine double angle formula $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$) can be written as follows: $-\frac{1}{4}(1-2\sin^2 x) + C = \frac{1}{2}\sin^2 x + C - \frac{1}{4}$. So the second and third answers differ from the first only by constants. But remember that the meaning of "$+C$" in the answer to an indefinite integral is that if you add any constant to the answer, you still have an antiderivative, i.e., the new function is still a correct answer.

HOMEWORK 1.

1. Use inverse trig substitution to find $\int dx/\sqrt{r^2 - x^2}$, where r is a constant (we evaluated this earlier using a u-substitution).

2. Find the anti-derivative of

(a) $1/\sqrt{x^2 + r^2}$,
(b) $1/\sqrt{x^2 - r^2}$.

3. Find the center of mass of the uniform flat plate bounded by the curve $y = x$ arcsin x, the curve $y = -x$ arcsin x, and the line $x = 0.5$.

4. Find the x-coordinate of the center of mass of the uniform flat plate bounded by the x- and y-axes, the line $x = 2$, and the curve $y = 1/\sqrt{x^2 + 6x + 13}$.

5. Find $\int_0^\infty dx/\left((x+2)\sqrt{x^2 + 4x + 5}\right)$.