CHAPTER 1

Differential Equations I: Separation of Variables

The differential equations in this section will be of a slightly more complicated form than in the section on exponential growth and decay, where we treated the differential equation $y' = \pm ky$ ($+k$ leading to exponential growth, and $-k$ leading to exponential decay). Namely, we shall deal with differential equations of the form $y' = \pm k(y-b)$, where b is some constant. Before discussing the applications, let's solve the differential equation $\frac{dy}{dt} = k(y-b)$ by the same method — separation of variables — that we used before. We first multiply both sides by dt and divide through by $y-b$, and we then put an integral sign in front of both sides:

$$\begin{aligned} \int \frac{dy}{y-b} &= \int k\,dt \\ \ln|y-b| + C_1 &= kt + C_2 \\ \ln|y-b| &= kt + C \quad (\text{where } C = C_2 - C_1) \\ |y-b| &= e^{kt+C} = e^C e^{kt} \\ y - b &= \pm e^C e^{kt}. \end{aligned}$$

To find the constant $\pm e^C$ in front of e^{kt}, we suppose that we have the initial information $y(0) = y_0$; then substituting $t = 0$ on both sides gives us: $y_0 - b = \left(\pm e^C\right) e^0 = \pm e^C$, and so we can write $y - b = (y_0 - b)e^{kt}$, i.e.,

The differential equation $\frac{dy}{dt} = k(y-b)$ has solution $y(t) = b + (y_0 - b)e^{kt}$.

If our differential equation is of the form $\frac{dy}{dt} = -k(y-b)$ with a negative constant of proportionality $-k$, then the solution is exactly the same with k replaced by $-k$. Thus,

The differential equation $\frac{dy}{dt} = -k(y-b)$ has solution $y(t) = b + (y_0 - b)e^{-kt}$.

1. Installment Buying

Large purchases (cars, houses, furniture, etc.) are often made according to the following system. When you make the purchase (time $t = 0$), you pay only a small fraction of the price (called a "downpayment") and agree to pay the remaining amount — which we shall denote y_0 — over a certain period of time. In essence, you are borrowing the y_0 dollars from the seller (or a mortgage company in the case of a house). We shall let T denote the term of this loan (in years), i.e., the time during which you continue to make payments, and we shall let a denote the rate in dollars per year at which you pay money during this time T. (If you make monthly payments, then a equals 12 times the amount you pay each month.) If the seller is willing to loan you the y_0 interest free, then the situation is very simple: $a = y_0/T$. But in practice you must pay back not only y_0 to make up the purchase price, but also interest on the money you owe. Let k be the instantaneous rate of interest.

In order to treat this problem by a differential equation, we shall make two simplifying assumptions:

1. The interest is compounded continuously. (This is the same simplifying assumption that we made in the section on exponential growth.)
2. Your payments are being made continuously (rather than at the end of the month, as usually happens in practice). For example, in the case of monthly payments of \$300, this would mean, roughly speaking, that instead of waiting until the end of the month, you arrange to have your bank's computer transfer one penny from your account to the seller's account every 86.4 seconds.

Even though both of these simplifying assumptions seem to diminish the realism of our treatment of installment buying, in practice the inaccuracy introduced is very small — an error of considerably less than 1%.

From a practical point of view, perhaps the most useful thing to figure out is the monthly payment $a/12$, which depends on y_0 (which is purchase price minus downpayment), the term T, and the interest rate k. But there are other questions which might also interest us. In any case, the way to approach the matter is to introduce the function

$$y(t) = \text{the amount you still owe at time } t.$$

This amount $y(t)$ starts out at $y(0) = y_0$, and steadily decreases until at time $t = T$ you've paid off the loan, i.e., $y(T) = 0$. This function $y(t)$ satisfies a differential equation, which we first state in words: *the instantaneous rate of change of the amount $y(t)$ that you owe is the result of two contributions: (1) a positive contribution caused by interest, which accumulates at a rate equal to k times the current amount you owe, and (2) a negative contribution of a dollars/year, as you make payments to decrease your debt.*

Written mathematically, the first contribution is ky and the second contribution is $-a$. Thus, the function $y(t)$ satifies the differential equation

$$\frac{dy}{dt} = ky - a.$$

If we rewrite the right side in the form $k(y - \frac{a}{k})$, we see that this is the same type of differential equation that we already solved at the beginning of the section. We can simply take the solution we already found, where we substitute $b = a/k$. The result is:

$$y(t) = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{kt}.$$

We now want to interpret this result to obtain useful information. First, let us find the rate of payment a (or the monthly payment rate $a/12$) as a function of y_0, T, and k. The basic fact we use is that the amount you owe must be reduced to 0 when $t = T$. That is,

$$0 = y(T) = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{kT}.$$

To solve this for a, we do a little algebra. We start out by dividing through by e^{kT}, to make things look slightly simpler (no parentheses needed):

$$0 = \frac{a}{k}e^{-kT} + y_0 - \frac{a}{k}.$$

(Recall that the reciprocal of e^{kT} is e^{-kT}.) Since a is the unknown, we move all the terms with a over to the left, and factor out a:

$$\frac{a}{k}\left(1 - e^{-kT}\right) = y_0.$$

Now it is easy to solve, by multiplying both sides by k and dividing both sides by $1 - e^{-kT}$:

$$a = \frac{ky_0}{1 - e^{-kT}}.$$

EXAMPLE 1.1. A few years ago the State of Washington had a 12% ceiling on interest rates, while neighboring Oregon had no such limit. During this time an ad for a \$450 dinette set running in Pacific Northwest newspapers said: no money down; residents of Washington take 32 months to pay at 12% interest; residents of Oregon make the same monthly payments as in Washington, although their interest rate will be 18%. Find the monthly payment that residents of both states make, and also find out how much beyond the 32 months must Oregonians continue to make payments because of their higher interest rate.

In Washington, $T = 32/12$ years, $k = 0.12$, and $y_0 = 450$. Hence,

$$\frac{a}{12} = \frac{1}{12} \cdot 0.12 \cdot 450 \Big/ \left(1 - e^{-0.12 \cdot 32/12}\right) = \$16.43.$$

In Oregon, we now know the value of a (namely, $12 \cdot 16.4323 =\$197.19$), and we want to find T. So we return to the equation $y(T) = 0$ above, and instead of solving for a we solve for T. Moving the term with T over to the left, we have

$$\left(\frac{a}{k} - y_0\right) e^{kT} = \frac{A}{k}.$$

We divide both sides by $((a/k) - y_0)$, simplify the fraction on the right by multiplying by k/k, and then take the ln of both sides. The result is

$$kT = \ln\left(\frac{a}{a - ky_0}\right), \quad \text{i.e.,} \quad T = \frac{1}{k}\ln\left(\frac{a}{a - ky_0}\right).$$

Substituting $k = 0.18$, $a = 197.19$, and $y_0 = 450$, we obtain: $T = 2.9386$ years $= 35.26$ months. That is, Oregonians must continue paying for about 3 1/4 months more.

The next example deals with home mortgages. By a "mortgage" we shall always mean a fixed-rate, fixed-payment mortgage, i.e., the standard kind. Adjustable-rate mortgages and other variants will not be examined here.

EXAMPLE 1.2. To buy a house for \$71,500.00, you make a downpayment of 20% and take a 30-year mortgage for the remaining 80% at 11% interest. Find your monthly payments $a/12$, your yearly rate of payment a, and the total amount you end up paying for the house. Also compute how much of the money paid in the first year is tax-deductible, and how much of the money paid in the 30th year is tax-deductible. (Note: money paid toward interest is deductible from U.S. federal income tax, while money that goes toward reducing the amount you owe is not.)

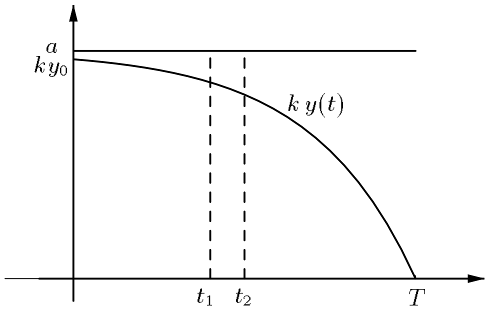
Here $y_0 = 0.8 \cdot 71500 = 57200$, and so

$$a = \frac{0.11 \cdot 57200}{1 - e^{-0.11 \cdot 30}} = \$6532.96; \qquad \frac{a}{12} = \$544.41.$$

The amount you end up paying for the house is equal to the downpayment plus aT, i.e., $\$14300 + 30 \cdot \$6532.96 = \$210289$, which is about three times the price of the house! (This isn't as bad as it seems, for at least two reasons: (1) a significant part of the \$138789 that goes for interest is being paid by the government, in the form of reduced income tax, and (2) the dollars that you pay 20 or 30 years later are dollars whose real value has been reduced by inflation.) Next, we want to find out how much of the \$6532.96 paid between $t = 0$ and $t = 1$ is tax-deductible, and how much of the \$6532.96 paid between $t = 29$ and $t = 30$ is tax-deductible. The simplest way to do this is to ask how much is *not* tax-deductible, and that is the amount which has gone to decrease your indebtedness. Thus, the amount in the first year that is not tax-deductible is

$$\begin{aligned}
y(0) - y(1) &= \left(\frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{k\cdot 0}\right) - \left(\frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{k\cdot 1}\right) \\
&= \left(y_0 - \frac{a}{k}\right)\left(1 - e^k\right) = \left(\frac{a}{k} - y_0\right)\left(e^k - 1\right) \\
&= \left(\frac{6532.96}{0.11} - 57200\right)\left(e^{0.11} - 1\right) = \$254.71.
\end{aligned}$$

The remaining $6532.96 - 254.71 = \$6278.25$ is tax-deductible. In the 30th year, on the other hand, the non-tax-deductible amount is $y(29) - y(30) = y(29) - 0 = y(29) = \$6185.64$, and so only \$347.32 is deductible in the last year.



What is happening is that over the term of the mortgage the way your a dollars per year payment divides up is changing. At the beginning, when you still owe a lot of money, most of the a goes toward interest payment, and only very little goes toward reducing the principal. As time goes on, more and more of the a goes toward reducing the principal, and less and less of it goes toward interest payment. This can be represented visually by a graph (see the figure at the right) that depicts what our original differential equation is saying. The graph shows the horizontal line a along with the curve $ky(t)$. The differential equation says that at any instant, of the a dollars/year that you pay, $ky(t)$ of it must go toward interest, and the remaining $a - ky$ goes toward reducing the amount you owe. In symbols, $dy/dt = -(a - ky)$ (the minus sign appears because we are *decreasing* y). Over any time interval from t_1 to t_2, the amount you pay is $a(t_2 - t_1)$, which is the area under the horizontal line from t_1 to t_2. Of that area, the area under the curve $ky(t)$ is the amount that goes toward interest (the tax-deductible amount), and the area between the curve and the horizontal line is the amount that goes toward paying off the principal.

Recall that we made two simplifying assumptions at the very beginning. Since Example 16.2 is based on an actual purchase, we can show the actual values computed by the mortgage company, based on daily compounding and payments made at the end of each month. Our error turns out to be quite small: the true monthly payments for this mortgage were \$545.12. We were off by \$0.71, which is only 0.13% of the total.

Below we show the table summarizing a statement sent us by the mortgage company for the 8th year. This statement gives the same type of information as the above graph between $t = 7$ and $t = 8$. The first column has the constant value $a/12$ that is paid each month. The fourth column shows how much of this has gone toward the principal, and the sixth column shows the difference, which has gone toward interest. The sum of the amounts in the sixth column is the "interest total" given at the bottom — the amount that is tax-deductible.

ANNUAL STATEMENT ON YOUR MORTAGAGE LOAN #5-03-00016965 INTEREST RATE 11.000%						
TOTAL PAYMEMT	DATE PAID	INTEREST PAID TO	PRINCIPAL ACCOUNT Transactions	Balance	INTEREST PAYMENT	
	01-01-87			54,542.50	←BEGINNING BALANCE	
545.12	01-12-87	01-01-87	45.15-	54,497.35	499.97	
545.12	02-11-87	02-01-87	45.56-	54,451.79	499.56	
545.12	03-11-87	03-01-87	45.98-	54,405.81	499.14	
545.12	04-13-87	04-01-87	46.40-	54,359.41	498.72	
545.12	05-11-87	05-01-87	46.83-	54,312.58	498.29	
545.12	06-11-87	06-01-87	47.25-	54,265.33	497.87	
545.12	07-13-87	07-01-87	47.69-	54,217.64	497.43	
545.12	08-11-87	08-01-87	48.12-	54,169.52	497.00	
545.12	09-11-87	09-01-87	48.57-	54,120.95	496.55	
545.12	10-13-87	10-01-87	49.01-	54,071.94	496.11	
545.12	11-12-87	11-01-87	49.46-	54,022.48	495.66	
545.12	12-11-87	12-01-87	49.91-	53,972.57	495.21	
TOTAL AS OF DECEMBER 31, 1987					5971.51	

2. Newton's Law of Cooling and Heating

Newton's law of cooling states that *if an object is brought into an environment, then the rate of cooling (or heating, if the object is colder than the environment) is proportional to the temperature difference between the object and the environment.* Let $y(t)$ be the temperature of the object at time t, let $y_0 = y(0)$ be the initial temperature of the object, and let b be the temperature of the environment. Then Newton's law can be translated into the mathematical equation $dy/dt = -k(y-b)$. Here k is the constant of proportionality, which in practice depends on the thermal properties of the object in the particular case at hand. k is positive, and the minus sign is necessary in front of k because dy/dt is negative when the object's temperature $y(t)$ is greater than b, and dy/dt is positive when $y(t)$ is less than b. This differential equation is of the form that was solved at the beginning of the section. Thus, we have $y(t) = b + (y_0 - b)e^{-kt}$. Notice that as t gets larger, the exponential term in this function gets smaller and smaller, so that $y(t)$ approaches b, i.e., the temperature of the object gets closer and closer to that of the environment. (In this section we are assuming that the ambient temperature b is constant, i.e., that it is not heated up by the object and does not change with the changing time of day. In the final section on differential equations, we will treat an example where b changes with time.)

EXAMPLE 1.3. A cup of coffee at 200°F is brought out into a room which is kept at 60°F. Two minutes later you measure the coffee's temperature to be 180°. Find a formula for the coffee's temperature at any time t.

Here we have $y_0 = 200$ and $b = 60$, but we do not know what k is. In the formula for $y(t)$, namely $b + (y_0 - b)e^{-kt} = 60 + 140e^{-kt}$, we can find k by using the information given about $y(2)$:

$$\begin{aligned} 180 = y(2) &= 60 + 140e^{-k\cdot 2} \\ 120 &= 140e^{-2k} \\ \ln(120/140) &= -2k \\ k &= -\frac{1}{2}\ln(120/140) = \frac{1}{2}\ln(7/6) = 0.077. \end{aligned}$$

Thus, $y(t) = 60 + 140e^{-0.077t}$.

3. Falling Bodies with Air Resistance

Now we shall find a formula for the velocity $v(t)$ of a falling body, no longer assuming that air resistance is negligible.

The vertical motion of a falling object is affected by two things: gravity and air resistance. Near the surface of the earth, gravity contributes a constant acceleration g. (Let us take downward to be the positive direction, in which case gravitational acceleration is g rather than $-g$.) The air resistance depends on the aerodynamic properties of the object and also on its speed. Generally, the faster the object is moving, the greater the force of air molecules pushing against it. In this section we shall assume that the force of air resistance is proportional to the velocity. Since acceleration is proportional to force, this also means that the acceleration (actually, deceleration) from air resistance is assumed to be proportional to $v(t)$. Thus, there are two contributions to dv/dt: g and $-kv(t)$ (here k is positive, and there's a minus sign in front of k because the air resistance works in the opposite direction to velocity). In other words, the unknown function $v(t)$ satisfies the differential equation $dv/dt = g - kv$, which we rewrite in the form

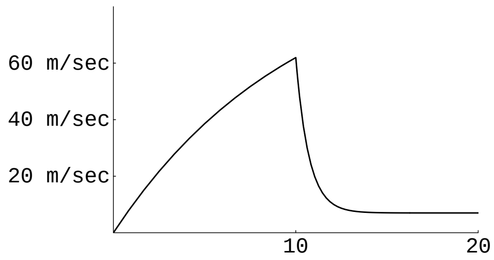
$$dv/dt = -k\left(v - \frac{g}{k}\right).$$

We see that this is in the form that was solved at the beginning of the section (with v instead of y and g/k in place of b). So our solution is the velocity function

$$v(t) = \frac{g}{k} + \left(v_0 - \frac{g}{k}\right)e^{-kt}.$$

Notice that as the time increases, the exponential term in the formula for $v(t)$ approaches zero, and so the velocity approaches the value g/k. We call g/k the **terminal velocity** and denote it v_∞. The terminal velocity does not depend on the initial velocity: no matter what initial velocity we give to the object, eventually its velocity will become essentially equal to the value g/k (in the sense that the difference between $v(t)$ and g/k cannot be detected in practice). In other words, the graph of $v(t)$ has a horizontal asymptote at height g/k. Another way to find the terminal velocity is to ask when the two forces on the object — gravity and air resistance — balance, i.e., when the right side of the differential equation equals zero: $g - kv = 0$. This happens when $v = g/k = v_\infty$.

Example 1.4. A parachutist drops from a plane at time $t = 0$. She freefalls for 10 seconds, during which time the constant of air resistance is $k = 0.1$. After her parachute opens, the air resistance becomes much greater: $k = 1.4$. Take downward to be the positive direction, and take $g = 9.8$ m/sec^2. Find formulas for her velocity for $0 \le t \le 20$, and draw a graph of this velocity function. Since the parachutist dropped from the plane, rather than throwing herself downward, the initial velocity at time $t = 0$ is 0. Thus, for the first 10 seconds we have $v(t) = \frac{g}{k}\left(1 - e^{-kt}\right) = 98\left(1 - e^{-0.1t}\right)$. Starting at $t = 10$ we have a new situation. The new initial velocity is $v(10) = 98\left(1 - e^{-0.1 \cdot 10}\right) = 61.95$ m/sec, and the new g/k is $9.8/1.4 = 7.0$. In using our falling body formula, we must be careful, since the starting time is now 10 rather than 0. Thus, $t - 10$ is the time elapsed since the parachute opened, and so we use $t - 10$ in place of t in the formula. The result is:



$$v(t) = \begin{cases} 98\left(1 - e^{-0.1t}\right), & \text{for } 0 \le t \le 10; \\ 7 + 54.95e^{-1.4(t-10)}, & \text{for } 10 \le t \le 20. \end{cases}$$

A graph of the velocity function is shown above.

4. Mixing Problems

Suppose we have a container which holds a certain volume V of water and at time $t = 0$ has a certain amount y_0 of dissolved impurity (such as salt, pollutant, etc.). A pipe carries water into the container at a rate r, and another pipe carries water out of the container at the same rate r (so that the volume V of water in the container remains the same). The incoming water has a concentration k of impurity (measured in gm/liter, lb/ft^3, gm/m^3, etc.). Let $y(t)$ be the amount of impurity in the water in the container at time t. We make the assumption that the impurity brought into the water by the first pipe immediately spreads uniformly throughout the container, so that the water in the container always has the same concentration of impurity throughout (i.e., there is no pileup of impurity near the mouth of the pipe). As a result, the concentration of impurity in the water taken up by the exit pipe is equal to the overall concentration in the container, namely, amount of impurity per volume $= y(t)/V$.

We are now ready to set up a differential equation for $y(t)$. The basic principal is that *the rate at which impurity enters the container is equal to the concentration of impurity (amount per volume) times the rate of the incoming water (volume per time), and the rate at which impurity leaves the container is equal to the concentration of impurity in the exit pipe times the rate of the water going out.* Thus, dy/dt has a positive contribution of kr and a negative contribution of y/V times r, i.e.,

$$\frac{dy}{dt} = kr - \frac{y}{V}r = kV\frac{r}{V} - \frac{r}{V}y = -\frac{r}{V}(y - kV).$$

This differential equation is of the form that was solved at the beginning of the section, except that the k here has nothing to do with the $-k$ that we had earlier. Now it is r/V that is playing the role of k, and kV that is playing the role of b. So we have the solution

$$y(t) = kV + (y_0 - kV)e^{-rt/V}.$$

It is also sometimes useful to have a formula for the concentration in the container at time t. This is $y(t)/V$, which we get by dividing the above formula by V:

$$\frac{y(t)}{V} = k + \left(\frac{y_0}{V} - k\right)e^{-rt/V}.$$

Notice that as $t \longrightarrow \infty$, the concentration of impurity in the container approaches k, which is the concentration in the incoming water. This makes sense: roughly speaking, the incoming water with its concentration k eventually "displaces" the water that was originally there.

The special case $k = 0$. If $k = 0$, then our formula for $y(t)$ becomes $y(t) = y_0 e^{-rt/V}$, which is an exponential decay function. (In this case our original differential equation was simply $dy/dt = -\frac{r}{V}y$.) Thus, if you remove impurities from the water by pumping in pure water, you find that the amount of impurity decreases exponentially, so that eventually the amount of impurity is too small to be measured. What is the "half-life" of the impurities, i.e., the length of time required to cut the amount of impurity in half? We use the formula $H = (\ln 2)/k$ that we derived before, where k is replaced by r/V. The result is $H = (\ln 2)V/r$.

This concludes our discussion of four applications of differential equations of the form $y' = \pm k(y - b)$. We now make some final observations.

5. The Constant Solution

There is one and only one constant function $y(t)$ =const that satisfies a differential equation of the form $dy/dt = \pm k(y - b)$. Namely, since a constant function has derivative zero, it follows that a constant function satisfies the differential equation if and only if $0 = \pm k(y - b)$, i.e., if and only if $y(t) = b$. In all four of the practical applications discussed above, the constant solution has a special meaning.

Installment buying. The constant solution $y(t) = a/k$ means that you have chosen your yearly rate of payment a to equal ky_0 (the interest rate times the amount you borrowed). That is just enough to pay the interest, without diminishing the amount you owe. This constant solution separates the two cases: (1) $a > ky_0$, in which case the amount you owe $y(t) = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{kt} = \frac{a}{k} - \left(\frac{a}{k} - y_0\right) e^{kt}$ decreases and becomes zero in a finite amount of time T; and (2) $a < ky_0$, in which case you are not even paying enough to take care of the interest, and the amount you owe grows without bound.

Law of cooling/heating. The constant solution $y(t) = b$ means that you have brought an object into the room whose temperature is the same as that of the room, in which case, of course, the temperature of the object remains constant. This constant solution separates the two cases: (1) $y_0 > b$, in which case you are bringing into the room a hotter object, which then cools, approaching the ambient temperature b as $t \longrightarrow \infty$; and (2) $y_0 < b$, in which case you have a colder object, which heats up, approaching the ambient temperature as $t \longrightarrow \infty$. Note that in all cases the constant solution b is also the value that $y(t)$ approaches as $t \longrightarrow \infty$.

Falling bodies with air resistance. The constant solution $v(t) = g/k$ means that you have thrown the object downward at an initial velocity v_0 exactly equal to the terminal velocity $v_\infty = g/k$. In that case the object falls at constant speed. This constant solution separates the two cases: (1) $v_0 < g/k$ (for example, the object falls from rest), in which case the object accelerates downward, approaching $v_\infty = g/k$ as $t \longrightarrow \infty$; and (2) $v_0 > g/k$ (for example, a parachutist after the parachute opens), in which case the object decelerates, approaching $v_\infty = g/k$ as $t \longrightarrow \infty$. Again note that, whatever the initial velocity, the constant solution g/k is also the value that $v(t)$ approaches as $t \longrightarrow \infty$.

Mixing problems. The constant solution $y(t) = kV$ means that you start with a concentration $y_0/V = k$ which is equal to the concentration of impurity in the incoming water; thus, the incoming water does not change the amount of impurity in the container. This constant solution separates the two cases: (1) $y_0 < kV$, when the incoming water increases the amount of impurity until the concentration $y(t)/V$ approaches k in the limit as $t \longrightarrow \infty$; and (2) $y_0 > kV$, when the incoming water decreases the amount of impurity, and again the concentration approaches k as $t \longrightarrow \infty$. Once again, in all cases the constant solution kV is also the limiting value of $y(t)$ as $t \longrightarrow \infty$.

6. A Final Remark

In all four of these applications, we made some simplifying assumptions, which introduce errors. That is, our formulas will not be an exact description of what happens in practice. The amount of error introduced depends on the particular situation. Generally speaking, in installment buying problems the error is very small, while in the other three applications our numerical values for $y(t)$ or $v(t)$ might be way off. However, even if we cannot always rely upon our *quantitative* results, we have obtained some useful *qualitative* information. For example, the role of the constant solution of the differential equation — as the limiting value of the function as $t \longrightarrow \infty$ — is actually observed in practice. For example, the terminal velocity of a falling body is the speed at which air resistance exactly balances (is equal to) the force of gravity.

Homework 1.

1. Suppose you buy an \$800 stereo system, taking 3 years to pay at 16% interest with no money down. What are your monthly payments, and how much will you end up paying for the stereo in all?

2. Suppose you want to buy a \$20000 boat but have only \$5000 for a downpayment. You can get a 5-year loan for the rest at 18%.

(a) What are your monthly payments, and how much will you end up paying for the boat?

(b) If you can only afford \$300 monthly payments, how long will it take you to pay off the loan?

(c) Suppose you arrange a loan where you are paying \$300 per month as in part (b). How much of the \$3600 that you pay in the first year is tax-deductible? How much of the \$3600 that you pay in the fifth year is tax-deductible?

3. Suppose that a mortgage company will give you a 30-year mortgage at 12% interest, but they consider you an acceptable credit risk only if you make a 20% downpayment and if your monthly payments do not exceed one fourth of your \$1600 monthly salary. What is the most expensive house you can buy, and how much cash on hand do you need to buy it?

4. You buy a house for \$90000 and make a \$20000 downpayment, using a 30-year mortgage for the rest.

(a) Suppose that between the time you agree to buy the house and the time the arrangements are finalized, the interest rate jumps from 11% to 12%. What increase in the price of the house would have been equivalent to the increase in interest rate (i.e., would have led to the same increase in your monthly payments, assuming your downpayment remains the same)?

(b) Suppose that the interest rate is 12%. How much more would you have to pay each month if you want to pay off the mortgage in 25 years rather than 30 years?

(c) What is the shortest term mortgage for which your monthly payments do not exceed \$750?

5. An item is bought using a loan of y_0 dollars, which is paid back at the steady rate of a dollars/year. The instantaneous interest rate is 10% (assume continuous compounding), and the term of the loan is 20 years. At what instant t is exactly half of the money going for interest (i.e., half of the a dollars per year being paid is going for interest at that instant) and half for principal?

6. Suppose that on your 16th birthday a wealthy distant relative dies and leaves you \$30000 in a trust fund that works as follows. The money in the fund always receives 8% interest compounded continuously. You receive no money until you turn 21. Then for the next 30 years you receive a fixed amount each month, which is determined in such a way that the fund is precisely used up on your 51st birthday. (Make the simplifying assumption that you receive money continuously throughout the month rather than in a lump payment once each month.)

(a) How much money is in the trust fund when you turn 21?

(b) How much will you receive each month between your 21st and 51st birthdays?

(c) If you decide to take only \$300 per month and leave the rest in the fund, how long will the trust fund last?

(d) How much can be withdrawn per month if you want the trust fund to last forever?

7. The table below is based on data published in the June 15, 1981 issue of The Seattle Times. It shows the monthly payments and total cost for a 42-month \$6000 loan for a new car, depending on the interest rate demanded by various lending institutions.

Basic new car loan \$6,000, 42 months

LENDER	INTEREST RATE	MONTHLY PAYMENT	TOTAL COST
Boeing Credit Union	13.75%	\$180.78	\$7,792.76
Auto Dealer No.1	16%	\$187.50	\$7,875.00
First Interstate Bank	18%	\$193.59	\$8,130.78
Auto Dealer No.2	18.25%	\$194.35	\$8,162.72
Rainer National Bank	19.25%	\$197.45	\$8,292.90
Seattle-First National Bank	20.5%	\$201.35	\$8,456.70

Use the procedure in this section to compute the monthly payments in each of the six cases, and see if your answers agree with the values given. They will not be exactly the same, because of our simplifying assumptions. However, you should not be more than about a dollar off.

8. You make a pudding for a dinner party, and put it in the refrigerator at 6 p.m. ($t = 0$). Your refrigerator maintains a constant temperature of 40°. The pudding will gel when it cools to 45°. When you put the pudding in the refrigerator you measure its temperature to be 190°, and when the first guest arrives at 7 p.m. you measure it again and get a temperature reading of 100°. When is the earliest you can serve dessert?

9. Water at 55° is pumped into a swimming pool on a 90° summer day. After 2 hours the temperature of the water is 60°. In how many hours (assuming the outside temperature remains 90°) will the water reach a comfortable swimming temperature of 70°?

10. A detective discovers a murder victim in a room at 8:00 a.m. one morning. The temperature of the body is 80.0°F. Two hours later (at 10:00 a.m.) the temperature of the body is 75.0°F. Assuming that the room is kept at the constant temperature 70.0°F, determine the time at which the murder was committed. (Assume that the murder victim had a normal temperature of 98.6°F at the time of death.)

11. The detective in Problem 10 discovers another murder victim at noon the next day in another 70.0°F room, and immediately has the body transported to a morgue, where the temperature is kept at 50.0°F. She measures the body's temperature right before it is taken from the room, and again at 4 p.m., after it has been 4 hours in the morgue. The former temperature is 85.0°F, and the latter temperature is 65.0°F. When was the murder committed? (Assume that the same constant of proportionality in Newton's law applies in the room and in the morgue.)

12. Suppose you drop an object from a great height, so that it reaches very nearly terminal velocity by time it hits the ground. By measuring the impact you determine that this terminal velocity is 49 m/sec.

(a) Find k.

(b) Graph the velocity function for the following values of v_0: 0 (dropped from rest), 20 m/sec downward, 100 m/sec downward, 49 m/sec downward, 20 m/sec upward.

(c) At what velocity must the object be thrown upward if you want it to reach its peak height after 3 sec?

(d) If v_0 is 20 m/sec upward, how long does the object take to reach its peak height?

13. In Example 16.4, find formulas for the distance $s(t)$ fallen from the plane at time t for $0 \leq t \leq 10$ and for $t > 10$. Notice that the exponential decay term makes a negligible contribution when t is greater than about 15 sec. If the plane was at an altitude of 500 meters when the parachutist jumped, when does she reach the ground?

14. A dense heavy package is dropped from a plane. After 7 seconds the package releases a parachute that starts to slow its speed. Assume that the air resistance is negligible during the first 7 seconds, and that wind tunnel tests show that with the parachute open the air resistance balances gravity at a speed of 19.6 m/sec. Find formulas for the downward velocity $v(t)$ for $0 \leq t \leq 15$, and graph $v(t)$.

15. Water is flowing at 3 m^3/min in one end and at the same rate out the other end of a vat containing 100 m^3 of water. Let $y(t)$ be the amount (in grams) of impurity in the vat of water at time t min. The water flowing in contains 0.1 gm of impurity per cubic meter. Find $y(t)$ if the initial concentration of impurity in the container is

(a) 0.05 gm/m^3, (b) 0.20 gm/m^3, (c) 0.10 gm/m^3.

Graph $y(t)$ in each case (on the same xy-plane).

16. A tank originally contains 100 gal of fresh water. Water with 1/2 lb/gal of salt is poured into the tank at a rate of 2 gal/min, and the well-stirred mixture is allowed to leave at the same rate. After 10 minutes the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of 20 minutes.

17. Suppose that a room containing 1200 cubic feet of air is originally free of carbon monoxide. Beginning at time $t = 0$ cigarette smoke containing 4 percent carbon monoxide is introduced into the room at a rate of 0.1 ft^3/min, and the well-circulated mixture is allowed to leave the room at the same rate.

(a) Find an expression for the amount $y(t)$ of carbon monoxide (in cubic feet) in the room at time t.

(b) Extended exposure to a carbon monoxide concentration as low as 0.012% is harmful to the human body. Find the time at which this concentration is reached.

18. Consider a lake of constant volume V which at time t contains an amount $y(t)$ of pollutant, evenly distributed throughout the lake with a concentration $c(t)$, where $c(t) = y(t)/V$. Assume that water containing a concentration k of pollutant (measured in tons/km^3) enters the lake at a rate r (measured in km^3/yr), and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P (measured in tons/yr). Note that the given assumptions neglect a number of factors which may be important in some cases: the water added or lost by precipitation, absorption, and evaporation; the stratifying effect of temperature differences in a deep lake; the tendency of irregularities in the coastline to produce sheltered bays; and the fact that pollutants are not deposited evenly throughout the lake but rather at isolated points around its periphery. However, we shall neglect these factors in what follows. Thus, according to our assumptions, the rate of change of $y(t)$ is the sum of three contributions: rk tons/yr in the incoming water, P tons/yr entering additionally, and *minus* r times y/V leaving in the outgoing water.

(a) Write a differential equation for y.

(b) Solve the differential equation to find an expression for $y(t)$. The constants r, k, P, V, and y_0 should appear in your expression.

(c) What is the limiting value of $y(t)$ as $t \longrightarrow \infty$?

(d) If the addition of pollutants to the lake is terminated (i.e., $k = 0$ and $P = 0$ for $t > 0$), determine the time interval T which must elapse before the concentration of pollutants is reduced to 10% of its original value.

(e) The table below contains data for several of the Great Lakes. Use the data to determine the time T necessary to reduce the contamination of each of these lakes to 10% of the original value.

Lake Superior: $V = 12200\ km^3$, $r = 65.2\ km^3/yr$

Lake Michigan: $V = 4900$ km^3, $r = 158$ km^3/yr
Lake Erie: $V = 460$ km^3, $r = 175$ km^3/yr
Lake Ontario: $V = 1600$ km^3, $r = 209$ km^3/yr

19. Suppose that at time $t = 0$ a country's population is 300 million, and its death rate exceeds its birth rate by k. Normally, that would lead to exponential decay of the population given by the formula $y(t) = y_0e^{-kt}$. However, there is also a constant immigration of I million people per year.

(a) Write a differential equation for the population function.

(b) If $k = 1\%$ and $I = 4$ million people per year, find the population in 50 years, and also find the limiting value of the population.

20. The population of salmon in the Pacific Northwest, if not harvested, would grow by 10% per year (this is effective yearly rate, **not** instantaneous rate). Suppose that the current salmon population is 3 million, and that the salmon are being harvested at a constant rate of R per year. At what rate R should the salmon be harvested in order that the salmon population be 4 million in 4 years from now? What would happen if R were 300000 salmon per year?