CHAPTER 1

Taylor's Theorem

The dominant theme in this chapter is approximation and the main result, an extension of the Mean Value Theorem called Taylor's Theorem, is of fundamental importance because it provided a method of approximating functions by polynomials. Thus, such problems as computing e^x, sin x, or ln x to a desired degree of accuracy can be solved by simply evaluating a polynomial. The idea behind Taylor's theorem also explains the technique for evaluating certain limits called indeterminate forms and provides a way of estimating the rate of convergence of Newton's method for approximating roots of equations.

1. Taylor Polynomials

We know that if the function f is differentiable (has a derivative) at $x = a$, then the tangent line

$$P_1(x) = f(a) + f'(a)(x - a)$$

at the point $(a, f(a))$ is a good local approximation to the graph of f, that is, $P_1(x) \approx f(x)$ if $x \approx a$. For example, the tangent to the function $f(x) = \sqrt{x}$ at $x = 4$ is

$$P_1(x) = 2 + 0.25(x - 4).$$

Now choose a value of x reasonably near 4, say 4.3. Then $P_1(4.3) = 2.075$ and $\sqrt{4.3} = 2.0736...$ so that the error in the approximation $\sqrt{4.3} \approx 2.075$ is less than 0.0014 in absolute value.

The proper generalization of this notion of approximation by the tangent stems from the observation that $P_1(x)$ is a polynomial of degree one that agrees with f and f' at $x = a$:

$$P_1(a) = f(a),\ P_1'(a) = f'(a).$$

The idea is to construct polynomials $P_n(x)$ of higher degree whose value and first n derivatives match those of f at x. Thus, for $n = 2$, we seek a polynomial $P_2(x)$ such that

$$P_2(a) = f(a),\ P_2'(a) = f'(a),\ P_2''(a) = f''(a). \tag{1}$$

Since $P_1(x)$ has the form $c_0 + c_1(x - a)$, it is reasonable to expect that $P_2(x)$ will have the form

$$P_2(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

for some choice of the coefficients c_0, c_1 and c_2. The condition $P_2(a) = f(a)$ is satisfied by choosing $c_0 = f(a)$ and since $P_2'(x) = c_1 + 2c_2(x - a)$, the condition $P_2'(a) = f'(a)$ is satisfied if $c_1 = f'(a)$. And, finally, from $P_2''(x) = 2c_2$ it follows that $P_2''(a) = f''(a)$ if $c_2 = \frac{1}{2}f''(a)$. Hence

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 \tag{2}$$

is a polynomial that satisfies the conditions in (1). And, as we shall see in the next section, there is no other choice, that is, there is exactly one polynomial of degree two such that the value of the polynomial and its first two derivatives agree with the value of f and its first two derivatives at $x = a$. Similarly,

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

is the unique polynomial of degree three such that

$$P_3(a) = f(a),\ P_3'(a) = f'(a),\ P_3''(a) = f''(a),\ P_3'''(a) = f'''(a).$$

The following definition extends this idea to polynomials of degree n. We remind you that $0! = 1,\ 1! = 1,\ n! = 1 \cdot 2 \cdot 3 \cdots n,\ f^{(0)} = f$, and $f^{(n)}$ denotes the nth derivative of f.

Definition of Taylor Polynomials. If the first n derivatives of f exist at a, then the nth *degree Taylor polynomial*[1] for f at a is defined by

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{2}$$

or, in summation notation,

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{3}$$

The nth-degree Taylor polynomials is structured so that the value of $P_n(x)$ and its first n derivatives match those of f at a. Having thus constructed $P_n(x)$ like f at a, we expect $P_n(x)$ to closely approximate $f(x)$ if x is near a. It does, and in the next section we will estimate the accuracy of that approximation.

Example 1. Write the third degree Taylor polynomial for $f(x) = \sqrt{x}$ at $a = 81$ and use it to approximate $\sqrt{85}$.

Solution. According to formula (2)

$$P_3(x) = f(81) + f'(81)(x-81) + \frac{f^{(2)}(81)}{2!}(x-81)^2 + \frac{f^{(3)}(81)}{3!}(x-81)^3$$

with $f(x) = \sqrt{x}$. From the computations

$$f(x) = \sqrt{x},\ f'(x) = \frac{1}{2}x^{-1/2},\ f^{(2)}(x) = -\frac{1}{4}x^{-3/2},\ f^{(3)}(x) = \frac{3}{8}x^{-5/2},$$

$$f(81) = 9,\ f'(81) = \frac{1}{2 \cdot 9}x^{-1/2},\ f^{(2)}(81) = -\frac{1}{4 \cdot 9^3},\ f^{(3)}(81) = \frac{3}{8 \cdot 9^5},$$

we obtain the Taylor polynomial

$$P_3(x) = 9 + \frac{1}{2 \cdot 9}(x-81) - \frac{1}{2 \cdot 4 \cdot 9^3}(x-81)^2 + \frac{1}{2 \cdot 8 \cdot 9^5}(x-81)^3.$$

Notice that the linear part, the first two terms on the right, correspond to the tangent to $f(x) = \sqrt{x}$ at the point $(81, 9)$. At the $x = 85$,

$$P_3(85) = 9 + \frac{1}{2 \cdot 9}(4) - \frac{1}{2 \cdot 4 \cdot 9^3}(4)^2 + \frac{1}{2 \cdot 8 \cdot 9^5}(4)^3 = 9 + \frac{2}{9} - \frac{2}{9^3} + \frac{4}{9^5} = 9.2195464... .$$

Clearly, this is a good approximation to $\sqrt{85} = 9.2195444...$. Actually, $|P_3(85) - \sqrt{85} < .21 \times 10^{-5}$. ‡

[1] After one of the contributors to the subject, English mathematician Brook Taylor (1685-1731)

Example 2. Write the 3rd-degree Taylor polynomial for $f(x) = x^3 - x^2 + x - 1$ at $a = 2$.

Solution. First, $f(2) = 5$, and from the derivatives

$$f'(x) = 3x^2 - 2x + 1,\ f''(x) = 6x - 2,\ f^{(3)}(x) = 6$$

we have

$$f'(2) = 9,\ f''(2) = 10,\ f^{(3)}(2) = 6.$$

Hence,

$$\begin{aligned} P_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 \\ &= 5 + 9(x-2) + 5(x-2)^2 + (x-2)^3. \ \ddagger \end{aligned}$$

In Example 2, it important to observe that $P_3(x)$ is more than just a good approximation to $f(x)$ near $a = 2$; indeed, $P_3(x) = f(x)$ for *all* x. In other words, $P_3(x) = 5 + 9(x-2) + 5(x-2)^2 + (x-2)^3$ is simply another way of writing $f(x) = x^3 - x^2 + x - 1$. One way to see this is to expand and collect terms in $P_3(x)$. In the next section we shall establish the general fact that if $f(x) = c_0 + c_1 x + \cdots c_n x^n$ is a polynomial of degree n and $P_n(x)$ is the nth-degree Taylor polynomial for f at a, then $P_n(x) = f(x)$ for all x.

2. Taylor's Theorem

Now we turn to the problem of estimating the accuracy of the approximation $f(x) \approx P_n(x)$. Our method will be to find a bound for the quantity

$$R_n(x) = f(x) - P_n(x),$$

called the *error* of the nth *remainder term*. The key step is the following important result known as Taylor's Theorem.

Theorem 1. Taylor's Theorem If the function f has derivatives of all orders up to and including $n+1$ on an open interval $(a-r, a+r)$ about a and x is any number between $a-r$ and $a+r$ other than a, then there is a number c between a and x such that

$$f(x) = f(a) + f'(x-a) + \cdots + \frac{f^{(n)}(a)}{n!} + \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}, \tag{1}$$

that is, $f(x) = P_n(x) + R_n(x)$ and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{2}$$

Further, if $|f^{(n+1)}(t)| \leq M$ on the interval from a to x,

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}. \tag{3}$$

We shall refer to formula (1) as *Taylor's formula with remainder.* Once a and n are chosen, it expresses $f(x)$ as the sum of a Taylor polynomial of degree n in powers of $x - a$ plus a remainder (2) which we will routinely estimate by (3).

Proof. The proof rests on a shrewd choice of a constant K followed by successive applications of Rolle's Theorem to the function g defined by

$$g(t) = f(t) - P_n(t) - K(t-a)^{n+1} \tag{4}$$

in which t varies over the closed interval from a to x. First, because $f(a) = P_n(a)$, $g(a) = f(a) - P_n(a) - K(a-a)^{n+1} = 0$ and, also,

$$g(x) = f(x) - P_n(x) - K(x-a)^{n+1} = 0$$

provided

$$K = \frac{f(x) - P_n(x)}{(x-a)^{n+1}}.$$

Remember: x is fixed, t is the variable. Using this value of K, we may apply Rolle's Theorem on the interval from a to x to conclude that $g'(c_1) = 0$ for at least one choice of c_1 between a and x.
$f'(a) = P_n'(a)$, the derivative $g'(t) = f'(t) - P_n'(t) - (n+1)(t-a)^n$ is also zero at $t = a$. Hence $g'(a) = g'(c_1) = 0$, and a second application of Rolle's Theorem shows that there is a number c_2 between c_1 and a at which $g''(c_2) = 0$.

The idea is to continue attacking with Rolle's Theorem, obtaining numbers $c_1 \leq c_2 < \cdots c_k$ at which $g'(c_1) = 0,\ g'(c_2) = 0, ..., g'(c_k) = 0$. This is possible so long as $g^{(k)}(a) = 0$. Since $f^{(k)}(a) = P_n^{(k)}(a)$ for $k = 0, 1, ..., n$ and

$$g^{(k)}(t) = f^{(k)}(t) - P_n^{(k)}(t) - (n+1)(n)(n-1)\cdots(n-k+2)K(t-a)^{n+1-k},$$

it follows that $g^{(k)}(a) = 0$ if $k = 0, 1, ..., n$ but

$$\begin{aligned} g^{(n+1)}(t) &= f^{(n+1)}(t) - 0 - (n+1)!K \\ &= f^{(n+1)}(t) - (n+1)!K \end{aligned}$$

need not be zero at $t = a$. Therefore, after n applications, we find that $g^{(n)}(a) = g^{(n)}(c_n) = 0$. A final approximation of Rolle's Theorem yields a number $c_{n+1} = c$ between c_n and a such that $g^{(n+1)}(c) = 0$, that is

$$\begin{aligned} g^{(n+1)}(c) &= f^{(n+1)}(c) - (n+1)!K = 0 \\ K &= \frac{f^{(n+1)}(c)}{(n+1)!}. \end{aligned}$$

Substitution of this new expression for K in (4) gives

$$g(t) = f(t) - P_n(t) - \frac{f^{(n+1)}(c)}{(n+1)!}(t-a)^{n+1}.$$

But $g(x) = 0$. Hence

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

This establishes formula (1).

Estimate (3) follows directly from the remainder formula (2). For if $|f^{(n+1)}(t)| \leq M$ everywhere between a and x, then, in particular $|f^{(n+1)}(c)| \leq M$ and

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right| \leq \frac{M}{(m+1)!}|x-a|^{n+1}.\ \ddagger$$

Example 1. (a) Write Taylor's formula with remainder for $n = 3$, $a = 0$ and $f(x) = \sqrt{1+x}$. (b) Use it to approximate $\sqrt{1.15}$ and estimate the error.

Solution. (a) The third degree Taylor polynomial for $f(x) = \sqrt{1+x}$ at $a = 0$ is

$$P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3,$$

by Example 3 of the last section and, since $f^{(4)}(x) = \frac{-3\cdot 5}{2^4}(1+x)^{-7/2}$,

$$R_3(x) = -\frac{3\cdot 5}{2^4\cdot 4!}(1+c)^{-7/2}x^4$$

for some c between 0 and x. Hence Taylor's formula (1) is

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{3\cdot 5}{2^4\cdot 4!}(1+c)^{-7/2}x^4.$$

(b) At $x = 0.15$, this formula gives the approximation

$$\sqrt{1.15} \approx 1 + \tfrac{1}{2}(0.15) - \tfrac{1}{8}(0.15)^2 + \tfrac{1}{16}(0.15)^3$$

rounded to six places.

Estimation of the error

$$R_3(0.15) = -\frac{15}{384}(1+c)^{-7/2}(0.15)^4$$

begins with the admission that we simply don't know the value of c; the theory only tells us that $0 < c < 0.15$. The best we can hope for is to put an upper bound on the error. To do this, we argue that, since $0 < c$, $1 < 1 + c$ and, therefore, $(1+c)^{-7/2} = \dfrac{1}{(1+c)^{-7/2}} < 1$. This gives the bound

$$\begin{aligned} |R_3(0.15)| &= \left| -\frac{15}{384}(1+c)^{-7/2}(0.15)^4 \right| \le \frac{15}{384}(0.15)^4 \\ &= 0.00001977\ldots . \end{aligned}$$

Consequently, $|R_3(0.15)| < 0.00002$ and the approximation $\sqrt{1.15} \approx 1.072398$ is off by no more than 2 units in the fifth decimal place. Thus

$$\sqrt{1.15} \approx 1.0724\,,$$

meaning that 1.0724 is the best 4 decimal place approximation to $\sqrt{1.15}$. ‡

Some remarks on procedure are now in order. In the initial approximation $\sqrt{1.15} \approx 1.072398$, we rounded to six decimal places, our strategy being to keep more digits than would be justified in the final result because we were uncertain as to the magnitude of the remainder term. Then, once we obtained a bound for the error, we rounded further to reflect the accuracy justified. Also, we should observe that the process of bounding the remainder usually amounts to placing c at one end or the other of the interval connecting a to x. Although useful, the resulting estimate may be too great by several factors.

The use of absolute values in the error estimates (3) is natural because usually we are only interested in the magnitude of the error, not whether it is positive or negative. Nonetheless, it is frequently worthwhile to look at the sign of $R_n(x)$ when making the approximation $f(x) \approx P_n(x)$. Thus, in the preceding example, we can see that $\sqrt{1.15} < 1.072398$ because $R_3(0.15) < 0$.

Example 2. In applying Taylor's formula (1) as an approximation tool, it is essential that x, the point at which the approximation is desired, be as near as possible to the base point a, where the derivatives are to be computed. On the other hand, you want the derivatives to be easily computable. In practice, the choice of a a represents a compromise between these two goals. Take $f(x) = x^x$ so that $f^{(k)}(x) = e^x$; hence the only reasonable choice is $a = 0$ because $f^{(k)}(0) = 1$ (easy to compute) for $k = 0, 2, 3, \ldots$. Taylor's formula with remainder at $a = 0$ for $f(x) = e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1},$$

where c is between 0 and x. At $x = \frac{1}{2} = 0.5$, this formula reduces to

$$\sqrt{e} = e^{0.5} = 1 + (0.5) + \frac{(0.5)^2}{2!} + \cdots + \frac{(0.5)^n}{n!} + \frac{e^c}{(n+1)!}(0.5)^{n+1}$$

and the challenge is to find a small positive integer n such that

$$R_n(0.5) = \frac{e^c}{(n+1)!}(0.5)^{n+1} < 0.5 \times 10^{-6}.$$

First we find a bound for e^c. Since $e < 3$ and $0 < c < 0.5$, it follows that $e^c < 3^{0.5} < 2$. Therefore

$$R_n(0.5) < \frac{2}{(n+1)!}(0.5)^{n+1}.$$

If, now, n is chosen so that the right side is less than 0.5×10^{-6}. The smallest such n, found by examining the following table, is $n = 7$ where $R_7(0.5) < 0.2 \times 10^{-6}$.

TABLE 13.1

n	$\frac{2}{(n+1)!}(0.5)^{n+1}$
1	0.25
2	0.04166667
3	0.00520833
4	0.00052083
5	0.00004340
6	0.00000310
7	0.00000019

Hence the approximation

$$\begin{aligned}\sqrt{e} &\approx 1 + (0.5) + \frac{(0.5)^2}{2} + \cdots + \frac{(0.5)^7}{7!} \\ &\approx 1.6487212\end{aligned}$$

has an error of at most 2 units in the seventh decimal place and, therefore, $\sqrt{e} \approx 1.648721$. ‡

In this example we obtained the best (rounded) six decimal place approximation by forcing the error under 0.5×10^{-6}. This suggests that five place accuracy will be obtained by forcing the error under 0.5×10^{-5}, four place accuracy if the error is less than 0.5×10^{-4}, and so on. This often is the case and the rule is worth using, but it does not always work as may be seen in the problem of approximating

$$\ln\ 1.25 = 0.2231435...$$

to four place accuracy. Suppose we obtain the approximation 0.22314 with a maximum error of 0.5×10^{-4}, meaning that ln 1.25 is greater than or equal to $0.22314 - 0.00005 = 0.22309$

and less than or equal to $0.22134 + 0.00005 = 0.22319$. However, approximation is obtained by rounding down to 0.2231 or up to 0.2232.

Example 3. (a) If $f(x) = \ln x$, write Taylor's formula with remainder for $n = 5$ at $a = 1$. (b) Use the formula to approximate ln 0.8 and estimate the error.

Solution. (a) Based on the derivatives

$$\begin{array}{ll} f(x) = \ln\,,\ f(1) = 0 & f^{(4)}(x) = -\dfrac{3!}{x^4},\ f^{(4)}(1) = -3! \\ f'(x) = \dfrac{1}{x},\ f'(1) = 1 & f^{(5)}(x) = \dfrac{4!}{x^5},\ f^{(5)}(1) = 4! \\ f^{(2)}(x) = \dfrac{1}{x^2},\ f^{(2)}(1) = -1 & f^{(6)}(x) = -\dfrac{5!}{x^6},\ f^{(6)}(c) = -\dfrac{5!}{c^6} \\ f^{(3)}(x) = \dfrac{2}{x^3},\ f^{(3)}(1) = 2 & \end{array}$$

the fifth degree Taylor's formula is

$$\ln\ x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6c^6}(x-1)^6.$$

In the remainder

$$R_6(x) = -\frac{1}{6c^6}(x-1)^6,$$

c denotes a number between 1 and x.
(b) The error in the approximation

$$\begin{aligned} \ln\ 0.8 &\approx -0.2 - \tfrac{1}{2}(-0.2)^2 + \tfrac{1}{3}(-0.2)^3 - \tfrac{1}{4}(-0.2)^4 + \tfrac{1}{5}(-0.2)^5 \\ &\approx -0.223131 \end{aligned}$$

is

$$R_6(0.8) = -\frac{1}{6c^6}(-0.2)^6.$$

It is worth recording that $R_6(0.8) < 0$ so that $\ln\ 0.8 < -0.223131$. Because $0.8 < c < 1$, $\dfrac{1}{c} < \dfrac{1}{0.8}$ and $\dfrac{1}{c^6} < \dfrac{1}{(0.8)^6}$; therefore,

$$|R_6(0.8)| = -\frac{1}{6c^6}(-0.2)^6| = \frac{1}{6c^6}(0.2)^6 < \frac{1}{6}\left(\frac{0.2}{0.8}\right)^6 = 0.0000406\ldots.$$

Hence $-0.000041 < R_6(0.8) < 0$ and

$$-0.223172 < \ln\ 0.8 < -0.223131.$$

From this we can conclude that $\ln 0.8 = -0.2231...$, meaning that the first four decimal digits are the actual digits in the complete representation of ln 0.8, but only that $\ln 0.8 \approx -0.223$, not knowing whether the fifth digit is less than or greater than or equal to 5. Note that the error is less than 0.5×10^{-4}. ‡

In the Taylor's formula expansion of ln x in powers of $x-1$ plus a remainder,

$$\begin{aligned}\ln\ x &= (x-1)-\frac{1}{2}(x-1)^2+\cdots+\frac{(-1)^{n-1}}{n}(x-1)^n+\frac{(-1)^{n+1}}{n+1}\frac{1}{c^{n+1}}(x-1)^{n+1}\\ &= P_n(x)+R_n(x),\end{aligned}$$

the approximation $\ln x \approx P_n(x)$ is good for small n only so long as x is near 1. The following table illustrates this dependence for two values of x. The actual value ln $1.2 = 0.1823215...$ is approximated well by $P_n(1.2)$.

Table 13.2

n	$P_n(1.2)$	ACTUAL ERROR	$P_n(2)$	ACTUAL ERROR
1	0.2	− 0.017678...	1	−0.306852...
2	0.18	0.002321...	0.5	0.193147...
3	0.182266...	−0.000345...	0.833333...	−0.140186...
4	0.182266...	0.000054...	0.583333...	0.1098138...
5	0.182330...	−0.000009...	0.783333...	−0.090186...
6	0.1823200	0.000001...	0.616666...	0.076480...

for $n = 2, , 3, 4, 5, 6$. In fact $P_6(0.8) = 0.1823200$ is accurate through 5 decimal places. But at $x = 2$, the approximation of the actual value ln $2 = 0.693147...$ by $P_n(2)$ is worthless for small n. At $n = 6$ the best estimate we can make for the remainder $R_6(2) = \frac{1}{7}c^{-7} = \frac{1}{7}c^{-7}$ is $R_6(2) < \frac{1}{7} \approx 0.14$, while the absolute value of the actual error in the approximation $P_n(2) \approx$ ln 2 is about half the remainder, that is, $|tlj\ 2 - P_n(2)| \approx \dfrac{1}{2(n+1)}$. This means it would take an n of about 10,000 to reduce the error to 0.5×10^{-4}. Such computations dramatically illustrate why we insist that $x - a$ be small in Taylor's formula when the goal is the approximation of $f(x)$. In general, x is given and the initial problem is to find a nearby point a at which f and its derivatives can be easily calculated.

Example 4. Approximate cos 68° to four decimal place accuracy.

Solution. The point nearest to 68° at which $f(x) = \cos\ x$ and its derivatives are easily evaluated is $a = 60° = \frac{\pi}{3}$ radians. For this reason our decision is to expand cos x in powers of $x - \frac{\pi}{3}$ plus a remainder. Since $|\sin\ c| \le 1$ and $|\cos\ c| \le 1$, we have the estimate

$$|rn(x)| = \left|\frac{f^{n+1}(c)}{(n+1)!}\left(x-\frac{\pi}{3}\right)^{n+1}\right| \le \frac{1}{(n+1)!}\,|x-\frac{\pi}{3}|^{n+1}.$$

To obtain the desired accuracy, we let $x = \dfrac{68\pi}{180}$ in the remainder and increase n until

$$\frac{1}{(n+1)!}\left|\frac{68\pi}{180}-\frac{\pi}{3}\right|^{n+1} < 0.5\times 10^{-4}.$$

This inequality is first satisfied at $n = 3$ where $R_3\left(\dfrac{68\pi}{180}\right) < 0.000012$. From the derivatives

$$f(x) = \cos\ x,\ f\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}, \qquad f^{(3)}(x) = \sin\ x,\ f^{(3)}\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2},$$

$$f'(x) = -\sin\ x,\ f'\left(\tfrac{\pi}{3}\right) = -\tfrac{\sqrt{3}}{2}, \qquad f^{(4)}(x) = \cos\ x,\ f^{(4)}(c) = \cos\ c;$$

$$f^{(2)}(x) = -\cos\ x,\ f^{(2)}\left(\tfrac{\pi}{3}\right) = -\tfrac{1}{2},$$

it follows that Taylor's formula with remainder at $a = \frac{\pi}{3}$ and with $n = 3$ is

$$\cos\ x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{x}{3}\right)^3 + \frac{\cos\ c}{4!}\left(x - \frac{x}{3}\right)^3.$$

Hence, in the approximation

$$\cos\ 68° = \cos\ \tfrac{68\pi}{180} \quad \approx \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(\tfrac{68\pi}{180}\right) - \tfrac{1}{2}\tfrac{1}{2!}\left(\tfrac{68\pi}{180}\right)^2 + \tfrac{\sqrt{3}}{2}\tfrac{1}{3!}\left(\tfrac{68\pi}{180}\right)^3$$

the error is no more than two units in the fifth decimal digit and, therefore, $\cos 68° \approx 0.3746$ rounded to four decimal places. ‡

Our final example on approximation illustrates the power of Taylor's formula in a different setting.

Example 5. Approximate the integral $\displaystyle\int_0^{0\cdot 2} \sqrt{x}\ \sin\ x\ dx$ by making a substitution based on the approximation $\sin\ x \approx x - \dfrac{x^3}{3!}$. Estimate the error.

Solution. Since the integral is one which cannot be evaluated by finding an antiderivative, we simply replace $\sin\ x$ by $x - \dfrac{x^3}{3!}$ and write

$$\begin{aligned}\int_0^{0\cdot 2} \sqrt{x}\ \sin\ x\ dx \quad &\approx \int_0^{0\cdot 2} \sqrt{x}\left(x - \tfrac{x^3}{3!}\right)\ dx \int_0^{0\cdot 2} \left(x^{3/2} - \tfrac{1}{6}x^{7/2}\right)\ dx \\ &= \left[\tfrac{2}{5}x^{5/2} - \tfrac{1}{27}x^{9/2}\right]_0^{0\cdot 2} = \tfrac{2}{5}(0.2)^{5/2} - \tfrac{1}{27}(0.2)^{9/2} \\ &\approx 0.00712892\,.\end{aligned}$$

Of course, the error committed depends on the size of the remainder term in the expansion

$$\begin{aligned}\sin\ x \quad &= x = 0 + x - 0 \cdot \tfrac{x^2}{2!} - \tfrac{x^3}{3!} + 0 \cdot \tfrac{x^4}{4!} + \cos\ c \cdot \tfrac{x^5}{5!} \\ &= x - \tfrac{x^3}{3!} + + \cos\ c \cdot \tfrac{x^5}{5!}\,.\end{aligned}$$

In the remainder terms $R_4(x)$, c is between 0 and x. (We could have chosen $n = 3$, but the term involving x^4 is absent and $R_4(x)$ is smaller than $R_3(x)$ in absolute value on the interval $[0, 0.2]$.) It follows from the last equation that the difference between the function and its Taylor approximation is given by

$$\sqrt{x}\ \sin\ x - \sqrt{x}\left(x - \frac{x^3}{3!}\right) = \sqrt{x}\ \cos\ c$$

and hence that

$$|\sqrt{x}\ \sin\ x - \sqrt{x}\left(x - \frac{x^3}{3!}\right)| = |\sqrt{x}\ \cos\ c\ \frac{x^5}{5!}| \le \frac{x^{11/2}}{5!}$$

on the interval $[0, 0.2]$. This inequality provides a concrete statement about how well $\sqrt{x}\left(x - \frac{x^3}{3!}\right)$ approximates $\sqrt{x}\sin x$. Next we use the integral inequality $\left|\int_a^b ((u(x) - v(x))\,dx\right| \le \int_a^b |u(x) - v(x)|\,dx$ to obtain the following estimate:

$$\left|\int_0^{0.2} \sqrt{x}\ \sin\ x\ dx - \int_0^{0.2} \sqrt{x}\ x - \tfrac{x^3}{3!}\ dx\right| \le \int_0^{0.2} |\sqrt{x}\ \sin\ x - \sqrt{x}\ \left(x - \tfrac{x^3}{3!}\right)|\ dx \le \int_0^{0.2} \tfrac{x^{11/2}}{5!}\ dx$$

$$= \tfrac{1}{120}\,\tfrac{2}{13}\,(0.2)^{13/2} = 0.000000036\ldots.$$

To sum up, the error in the approximation $\int_0^{0.2} \sqrt{x}\ \sin\ x\ dx \approx 0.00712892$ is no more than 0.00000004, accordingly, $\int_0^{0.2} \sqrt{x}\ \sin\ x\ dx \approx 0.007129$" rounded to six decimal places. ‡

3. L'Hospital's Rule: The Indeterminate Form $\frac{0}{0}$

In our previous work we have been frequently asked to find the limit of a quotient $f(x)/g(x)$. The method introduced in Section 2.5 asserted that if $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, and $B \ne 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

However, nothing was said there about the case $B = 0$. Of course, one possibility is $B = 0$ and $A \ne 0$. If, for instance, $f(x) = x$, $g(x) = x - 4$, and $a = 4$, then

$$A = \lim_{x\to 4} f(x) = 4,\ B = \lim_{x\to 4} g(x) = 0$$

and

$$\lim_{x\to 4} \frac{f(x)}{g(x)} = \lim_{x\to 4} \frac{x}{x-4}$$

fails to exist because the ratio increases to ∞ as $x \to 4^+$ and decreases to $-\infty$ as $x \to 4^-$. The more interesting situation is $B = 0$ and $A = 0$. In this case the limit of the quotient $f(x)/g(x)$ is said to be "indeterminate at a" because it assumes the form "0/0" as x approaches a. The actual limit may or may not exist and, if it does any real number is a possible limit. Consider $f(x) = x$ and $g(x) = x^2$. Then $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$ and

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \text{ does not exist);}$$

but the functions $f(x) = k\ \sin\ x$ (k a real number) and $g(x) = x$ both have limit zero at $x = 0$ and, as we saw in Section ,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{k\ \sin\ x}{x} = k \lim_{x\to 0} \frac{\sin\ x}{x} = k \cdot 1 = k.$$

Taylor's formula suggests a method for evaluation limits which have the indeterminate form 0/0.

Let f and g be two functions which have continuous second derivatives in some open interval about a. Then, using Taylor's formula, we may write

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(z_1)}{2}(x-a)^2$$

for some z_1 between a and x, and

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(z_2)}{2}(x-a)^2$$

for some z_2 between a and x. Now assume that

$$\lim_{x\to a} f(x) = f(a) = 0 \text{ and } \lim_{x\to a} g(x) = g(a) = 0.$$

Thus for x in the indicated interval and $x \neq a$,

$$\begin{aligned}\frac{f(x)}{g(x)} &= \frac{f(a) + f'(a)(x-a) + \frac{f''(z_1)}{2}(x-a)^2}{g(a) + g'(a)(x-a) + \frac{g''(z_2)}{2}(x-a)^2} = \frac{f'(a)(x-a) + \frac{f''(z_1)}{2}(x-a)^2}{g'(a)(x-a) + \frac{g''(z_2)}{2}(x-a)^2} \\ &= \frac{f'(a) + \frac{f''(z_1)}{2}(x-a)}{g'(a) + \frac{g''(z_2)}{2}(x-a)}.\end{aligned}$$

As $x \to a$, z_1 and z_2 must also tend to a and the second term in both the numerator and denominator of this last quotient tend to zero. Hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

provided that $g'(a) \neq 0$. This analysis suggests that, under certain conditions, a quotient $f(x)/g(x)$ that has the indeterminate form 0/0 at $x = a$ and the quotient of derivatives $f'(x)/g'(x)$ have the same limit as x tends to a. The various conditions that permit this procedure are commonly known as L'Hospital's rules after the 17th century French mathematician who popularized them.

Theorem 2. Suppose that f and g are continuous functions on the closed interval $[a, b]$, differentiable on the open interval (a, b), and that

$$\lim_{x\to a^+} f(x) = f(a) = 0, \quad \lim_{x\to a^+} g(x) = g(a) = 0. \tag{1}$$

Suppose also that

$$g'(x) \neq 0 \text{ for } a < x < b. \tag{2}$$

Then,

$$\text{if } \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = \infty \text{ [or } -\infty\text{], it follows that } \lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty \text{ [or } -\infty\text{].} \tag{3}$$

Proof. Our proof skirts Taylor's formula, depending rather on an approach which avoids assuming that f is differentiable at $x = a$. To begin, we choose a number x, $a < x < b$, and consider the new function

$$H(t) = f(x)g(t) - g(x)f(t), \quad a \leq t \leq x. \tag{5}$$

As a function of t, H is continuous on $[a, x]$ and differentiable for $a < t < x$. Further, $H(x) = f(x)g(x) - g(x)f(x) = 0$, $H(a) = f(x)g(a) - g(x)f(a) = 0$ by (1) and, since x is fixed,

$$H^{'}(t) = f(x)g^{'}(t) - g(x)f^{'}(t).$$

According to Rolle's theorem this derivative must be zero at least once between 0 and x. Hence there is a number z, $a < z < x$, such that $H^{'}(z) = f(x)g^{'}(z) - g(x)f^{'}(z) = 0$ or, since $g^{'}(z) \neq 0$ by (2),

$$\frac{f(x)}{g(x)} = \frac{f^{'}(z)}{g^{'}(z)}.$$

Because $a < z < x$, we see that $z \to a^+$ if $x \to a^+$ and, therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f^{'}(z)}{g^{'}(z)} = \lim_{z \to a^+} \frac{f^{'}(z)}{g^{'}(z)}.$$

Clearly, if $\lim\limits_{z \to a^+} \dfrac{f^{'}(z)}{g^{'}(z)} = L$, then $\lim\limits_{x \to a^+} \dfrac{f(x)}{g(x)} = L$. On the other hand, if $\dfrac{f^{'}(z)}{g^{'}(z)}$ increases without bound as $z \to a^+$, that is if $\lim\limits_{z \to a^+} \dfrac{f^{'}(z)}{g^{'}(z)} = \infty$, then $\lim\limits_{x \to a^+} \dfrac{f(x)}{g(x)} = \infty$ also. The same analysis applies if $\lim\limits_{z \to a^+} \dfrac{f^{'}(z)}{g^{'}(z)} = -\infty$. ‡

Although our result applies only to right-hand limits, the same argument establishes the result for left-hand limits. In combination, the result for right- and left-hand limits establishes the same result for two-sided limits. Thus the results of Theorem 2 are valid if $\lim\limits_{x \to a^+}$ is replaced everywhere by $\lim\limits_{x \to a^-}$ or everywhere by $\lim\limits_{x \to a}$ and the assumptions on f and g are modified accordingly.

Example 1. Find $\lim\limits_{x \to 0^+} \dfrac{e^x - 1}{\sqrt{x}}$.

Solution. Since $\lim\limits_{x \to 0^+} (e^x - 1) = \lim\limits_{x \to 0^+} \sqrt{x} = 0$, the given limit has the indeterminate form $0/0$. Further, both numerator and denominator are continuous for $0 \leq x$, differentiable for $0 < x$ and $\frac{d}{dx}(\sqrt{x})$ is never 0. Thus, L'Hospital's rule,

$$\lim_{x \to 0^+} \frac{e^x - 1}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\frac{d}{dx}[e^x - 1]}{\frac{d}{dx}[\sqrt{x}]} = \lim_{x \to 0^+} \frac{e^x}{\frac{1}{2\sqrt{x}}} = \lim_{x \to 0^+} \frac{1}{2}\sqrt{x}e^x = 0.$$

Note that $\sqrt{x}$ does not have a derivative at $x = 0$. ‡

Remark. Strictly speaking, the first equality in the calculation is not justified until the limit on the right is shown to exist or to tend to ∞ or $-\infty$. However, it is common practice in applying L'Hospital's rule to streaming the computations as shown.

Example 2. Find $\lim\limits_{x \to 0} \dfrac{\sin 4x}{3x}$. **Solution**. This quotient also has the indeterminate form $0/0$ because $\lim\limits_{x \to 0} \sin 4x = \lim\limits_{x \to 0} \sin 3x = 0$. Hence we may apply L'Hospital's rule to find that

$$\lim_{x\to 0}\frac{\sin\ 4x}{\sin\ 3x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin\ 4x]}{\frac{d}{dx}[\sin\ 3x]}=\lim_{x\to 0}\frac{4\ \cos\ 4x}{3\ \cos\ 3x}=\frac{4}{3}.\ \ddagger$$

Example 3. Find $\lim\limits_{x\to 0}\dfrac{xe^x-\sin\ x}{\sin\ (x^2)}$.

Solution. First we check to see that $\lim\limits_{x\to 0}(xe^x-\sin\ x)=\lim\limits_{x\to 0}\sin\ (x^2)=0$. Then, by L'Hospital's rule,

$$\lim_{x\to 0}\frac{xe^x-\sin\ x}{\sin\ (x^2)}=\lim_{x\to 0}\frac{\frac{d}{dx}[xe^x-\sin\ x]}{\frac{d}{dx}[\sin\ (x^2)]}=\lim_{x\to 0}\frac{xe^x+e^x-\cos\ x}{2x\ \cos\ (x^2)}.$$

But the last limit tells us nothing because $\lim\limits_{x\to 0}(xe^x+e^x-\cos\ x)=\lim\limits_{x\to 0}2x\ \cos\ (x^2)=0$, that is, we are again faced with an indeterminate form $0/0$. Our response is to apply L'Hospital's rule a second time:

$$\lim_{x\to 0}\frac{xe^x+e^x-\cos\ x}{2x\ \cos\ (x^2)}=\lim_{x\to 0}\frac{\frac{d}{dx}[xe^x+e^x-\cos\ x]}{\frac{d}{dx}[2x\ \cos\ (x^2)]}=\lim_{x\to 0}\frac{2e^x+xe^x+\sin\ x}{2\ \cos\ (x^2)-4x^2\ \sin\ (x^2)}=\frac{2}{2}=1.$$

Therefore

$$\lim_{x\to 0}\frac{xe^x-\sin\ x}{\sin\ (x^2)}=1.\ \ddagger$$

Example 4. Find $\lim\limits_{\theta\to 0}\dfrac{2\sin\ \theta-\sin\ 2\theta}{\theta-\sin\ \theta}$.

Solution. This limit may be found by three applications of L'Hospital's rule:

$$\lim_{\theta\to 0}\frac{2\sin\ \theta-\sin\ 2\theta}{\theta-\sin\ \theta}=\lim_{\theta\to 0}\frac{2\cos\ \theta-\cos\ 2\cos\ 2\theta}{1-\cos\ \theta}=\lim_{\theta\to 0}\frac{-2\ \sin\ \theta+4\ \sin\ 2\theta}{\sin\ \theta}$$

$$=\lim_{\theta\to 0}\frac{-2\ \cos\ \theta+8\ \cos\ 2\theta}{\cos\ \theta}=\frac{6}{1}=6.\ \ddagger$$

CAUTION. Never apply L'Hospital's rule $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ without first verifying that $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=0$. For example, $\lim\limits_{x\to 1}\dfrac{\sin\ \frac{\pi}{6}x}{1+\ln\ x}=\dfrac{1}{2}$ because $\lim\limits_{x\to 1}(1+\ln\ x)=1$ and $\lim\limits_{x\to 1}\sin\ \frac{\pi}{6}x=\frac{1}{2}$. Clearly, it is not legitimate to apply L'Hospital's rule. Also, keep in mind that L'Hospital's rule replaces the quotient $f(x)/g(x)$ with the quotient of derivatives $f'(x)/g'(x)$, which is not at all the same as differentiating $f(x)/g(x)$ according to the quotient rule.

4. Other Indeterminate Forms

A second version of L'Hospital's rule applies to quotients $f(x)/g(x)$ in which both $f(x)$ and $g(x)$ tend to one of ∞ or $-\infty$ as $x\to a^+, x\to a^-, x\to a, x\to\infty$, or $x\to-\infty$. For example, it may be that $f(x) \to -\infty$ and $g(x) \to \infty$ as in

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x},$$

or that $f(x) \to \infty$ and $g(x) \to \infty$ as in

$$\lim_{x \to \infty} \frac{x^2}{e^x}.$$

We cover all possibilities by saying that $f(x)/g(x)$ has *the indeterminate form* $\frac{\infty}{\infty}$.

Theorem 3. L'Hospital's rule $\frac{\infty}{\infty}$.

Let f and g be differentiable functions in the open interval (a, b) and suppose that
(a) $\lim\limits_{x \to a^+} f(x) = \infty (\text{or } -\infty)$ and $\lim\limits_{x \to a^+} g(x) = \infty (\text{or } -\infty)$.
Suppose also that
(b) $g'(x) \neq 0$ for $a < x < b$.
Then, if
(c) $\lim\limits_{x \to a^+} \dfrac{f'(x)}{g'(x)} = L$ or is ∞ or $-\infty$,
it follows that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Exactly as in Theorem 2, this result remains valid if the limit $x \to a^+$ is replaced by any one of $x \to a^-$, $x \to a$, $x \to \infty$, or $x \to -\infty$. The proof, while following the pattern set in proving L'Hospital's rule for the case $0/0$, is tedious and we omit it. It can be found in any advanced calculus text.

Example 1. Find (a) $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$ and (b) $\lim\limits_{x \to \infty} \dfrac{x^2}{e^x}$.

Solution. Since $\lim\limits_{x \to \infty} x = \lim\limits_{x \to \infty} x^2 = \infty$ and $\lim\limits_{x \to \infty} e^x = \infty$, both quotients have the indeterminate form ∞/∞. Thus, by L'Hospital's rule,

(a) $$\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[e^x]} = \lim_{x \to \infty} \frac{1}{e^x} = 0$$

and, applying the rule twice,

(b) $$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^x]} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{\frac{d}{dx}[2x]}{\frac{d}{dx}[e^x]} = \lim_{x \to \infty} \frac{2}{e^x} = 0. \ ‡$$

More generally, we may choose any positive integer k and apply L'Hospital's rule k times to find that

$$\lim_{x \to \infty} \frac{x^k}{e^x} = \lim_{x \to \infty} \frac{k!}{e^x} = 0,$$

thereby showing that e^x increases much faster than x^k, for any positive integer k.

Example 2. Find $\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$.

Solution. Both numerator and denominator tend to ∞ as $x \to \infty$. Hence

$$\lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{\frac{1}{x}}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0.$$

Thus, $\ln x$ increases at a much slower rate than x^2. ‡

Example 3. Find $\lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{\ln x}$.

Solution. Since both numerator and denominator tend to $-\infty$ as $x \to 0^+$,

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln x} = \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x \cos x}{\sin x}.$$

To proceed, note that the last quotient has the indeterminate form $0/0$. Hence

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln x} = \lim_{x\to 0^+} \frac{x \cos x}{\sin x} = \lim_{x\to 0^+} \frac{\cos x - x \sin x}{\cos x} = 1.$$

or

$$\lim_{x\to 0^+} \frac{x \cos x}{\sin x} = \lim_{x\to 0^+} \frac{x}{\sin x} \cdot \lim_{x\to 0^+} \cos x = \lim_{x\to 0^+} \frac{1}{\cos x} \cdot \lim_{x\to 0^+} \cos x = 1 \cdot 1 = 1 \ \ddagger$$

We now turn our attention to some limit problems which can be treated by our technique only after some algebraic manipulations. Consider, for example, the product $f(x)g(x)$ in which $f(x) \to 0$ which $g(x) \to \infty$ or $-\infty$. We say that the resulting indeterminate form is of *type* $0.\infty$ and we handle it by rearranging $f(x)g(x)$ to

$$\frac{\frac{f(x)}{1}}{g(x)} \text{ or } \frac{\frac{g(x)}{1}}{f(x)},$$

thus reducing the product to one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 4. Find $\lim\limits_{x\to 0^+} x \ln x$.

Solution. This product is indeterminate of type $0\cdot\infty$ since $\lim\limits_{x\to 0^+} x = 0$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$. We change it into the form ∞/∞ and apply L'Hospital's rule as follows:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\frac{\ln x}{1}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0. \ \ddagger$$

PROBLEM SET 19.1

For each of the polynomials in Problems 1–3, write the Taylor polynomial of the same degree at the given value of a and show algebraically that the two polynomials are identical.

1. $p(x) = 3x + 5,\ a = 1$

2. $p(x) = x^5 - 1,\ a = 1$

3. $p(x) = (2x - 3)^3,\ a = -2$

4. Write the third degree Taylor polynomial for $f(x) = 1/\sqrt{x}$ at $x = 81$ and use it to approximate $\sqrt{83}$. Compared with the calculator value of $\sqrt{83}$; what is the error?

5. Write the fifth degree Taylor polynomial for $f(x) = \ln(1 + x)$ at $x = 1$. Use it to approximate $1/\sqrt{83}$. What is the error?

In Problems 6–9, calculate the Taylor polynomial of the indicated degree at a.

6. $f(x) = x \ln x;\ n = 3,\ a = 1$

7. $h(t) = \sin t;\ n = 4,\ a = \frac{\pi}{3}$

8. $f(t) = \sqrt{\frac{1+t}{1-t}};\ n = 2,\ a = 0$

9. $f(x) = \sin^2 x;\ n = 5,\ a = \pi$

10. If $p(x) = 3x^3 + 7x^2 + 10x + 2$ is the third degree Taylor polynomial at $a = -1$ for f, find $f''(-1)$.

11. Show that $e^x > 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$ if $x > 0$. [Hint: Show that the function F, obtained by subtracting the polynomial from e^x, and its first three derivatives are zero at $x = 0$ and that the fourth derivative is positive. Then integrate.]

12. On a single coordinate system plot $\sin x$ and its Taylor polynomials P_1, P_3, and P_5 at $a = 0$.

PROBLEM SET 19.2

For each of the functions in Problems 1–5, write Taylor's formula with remainder for the given values of a and n.

1. $\dfrac{1}{1-x}$; $a=0,\ n=6$

2. $(+x)^{-1/2}$; $a=0,\ n=4$

3. $\ln\dfrac{1+x}{1-x}$; $a=0,\ n=5$

4. $\sin\ x$; $a=0,\ n=6$

5. $\cos\ x3x$; $a=\frac{\pi}{2},\ n=4$

In Problems 6–8, use Taylor's formula with remainder at $a=0$ with $n=3$ to approximate the indicated quantity. Estimate the error and round your approximation to the number of decimal places justified.

6. $\sqrt{1.13}$

7. $e^{0.2}$

8. $\ln\ 0.98$

In Problems 9–11, use Taylor's formula with remainder to find the best (rounded) four decimal place approximation to the given quantity. As a first step, choose n so that $|R_n(x)| < 0.5\times 10^{-4}$.

9. $\sqrt{1.2}$

10. $\cos\ 29°$

11. $\ln\ 1.25$

12. The problem is to find the best four decimal place approximation to

$$\int_0^{0.3}\sin^4\ \theta\,d\theta.$$

Compare the effort in (a) using the approximation $\sin\ \theta\approx\theta$, (b) approximating with Simpson's method based on two parabolas, and (c) evaluating the integral analytically.

13. Approximate the integral $\displaystyle\int_0^{0\cdot 4} x\sqrt{\cos\ x}\,dx$ in two ways, first by using $\cos\ x\approx 1-\frac{x^2}{2}$ in the integrand and then with Simpson's method based on four parabolas.

PROBLEM SET 19.3

For Problems 1-9, write L'Hopital's Rule to find the indicated limits.

1. $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{2x^3 - 3x^2 - 4x + 4}$

2. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin\ x}$

3. $\lim\limits_{x \to \pi} \dfrac{\sin\ mx}{\sin\ nx}$; m, n integers

4. $\lim\limits_{x \to 0} \dfrac{1 - \cos\ 5x}{x}$

5. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin\ x}$

6. $\lim\limits_{x \to \infty} \dfrac{\frac{\pi}{2} - \text{Arctan}\ x}{\ln\ \left(1 + \frac{1}{x}\right)}$

7. $\lim\limits_{\alpha \to \pi} \dfrac{1 + \tan\ \frac{3\alpha}{4}}{\cot\ \frac{\alpha}{2}}$

8. $\lim\limits_{\alpha \to \frac{\pi}{2}} \dfrac{\sin^2\ 2\alpha}{1 + \cos\ 2\alpha}$

9. $\lim\limits_{\theta \to 0} \dfrac{\theta - \sin\ (\sin\ \theta)}{\theta - \sin\ \theta}$

In Problems 10 and 11, formulate and solve a limit problem that, to your mind, substantiates the claim.

10. Arctan $\approx x$ for small values of x.

11. The area of the region bounded by the tangent curve and the sine curve from 0 to a small positive number x is about three times the corresponding area of the region between the line $y = x$ and the sine curve.

12. Show that $\lim\limits_{x \to 0} \dfrac{\int_0^x \frac{\sin\ t}{t}\ dt}{x} = 1$.

PROBLEM SET 19.4

For Problems 1-10, find the limits.

1. $\lim\limits_{x\to\infty} \dfrac{x^4}{e^x}$

2. $\lim\limits_{x\to\infty} \dfrac{\ln\,(1+e^{-3x})}{x}$

3. $\lim\limits_{x\to\infty} x^3\left(\frac{\pi}{2} - \text{Arctan}\, x - \frac{1}{x}\right)$

4. $\lim\limits_{x\to 0^+} (1+x)^{\ln\, x}$

5. $\lim\limits_{x\to\infty} x^3\left(\frac{1}{2}\,\ln\,\frac{x+1}{x-1} - -\frac{1}{x}\right)$

6. $\lim\limits_{x\to\infty} \left(1+\frac{t}{x}\right)^x$

7. $\lim\limits_{x\to\infty} \left(\dfrac{1+\frac{1}{2x}}{1-\frac{1}{2x}}\right)^x$

8. $\lim\limits_{x\to\infty} \left(\frac{1}{x-1} - \frac{1}{\ln\, x}\right)$

9. $\lim\limits_{x\to\infty} \left(\frac{x+4}{x-4}\right)^x$

10. $\lim\limits_{x\to 0} \left[\frac{1}{\sin^2 x} - \frac{1}{x^2}\right]$

11. Translate each of the following statements into a limit problem involving ratios. Solve the problems.

(a) As $x\to\infty$, $\ln\, x$ increases at a much slower rate than any positive power of x.

(b) As $x\to\infty$, e^x increases at a much faster rate than any positive power of x.