CHAPTER 1

# More on Falling Bodies; Volumes of Revolution

If we know the initial velocity $v_0$ and the initial height $s_0$ of a falling body at time $t = 0$, then the formula

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

tells us the height above the ground of the falling body at any time $t$. The formula for the velocity is given by its derivative:

$$v(t) = -gt + v_0.$$

Note that we are neglecting air resistance and taking upward as the positive direction. We usually use the traditional value $g = 9.8$ m/sec$^2$ for the gravitational acceleration.

But in some situations we do not know $v_0$ and/or $s_0$, and must determine these values from other given information. Generally speaking, any two bits of information can be used to determine $v_0$ and $s_0$.

EXAMPLE 1.1. You see an object thrown downward from a building. It hits the ground 5 sec after it was thrown with an impact velocity of 60 m/sec. What was the height from which it was thrown, and with what initial velocity was it thrown?

We translate the information given into two equations. Since "hits the ground" means that $s = 0$, we have $s(5) = 0$; we also know that $v(5) = -60$ (i.e., 60 m/sec *downward*). We plug 5 into the $s(t)$ and $v(t)$ formulas to obtain two equations with the two unknowns $s_0$ and $v_0$:

$$\begin{aligned} 0 &= s(5) = -4.9 \cdot 5^2 + v_0 \cdot 5 + s_0 = -122.5 + 5v_0 + s_0; \\ -60 &= v(5) = -9.8 \cdot 5 + v_0 = -49 + v_0. \end{aligned}$$

The second equation immediately gives $v_0 = -60 + 49 = -11$ m/sec. Substituting this value for $v_0$ in the first equation and solving for $s_0$, we obtain:

$$s_0 = 122.5 - 5v_0 = 122.5 - 5 \cdot (-11) = 177.5.$$

Thus, the object was thrown downward at 11 m/sec from a height of 177.5 meters.

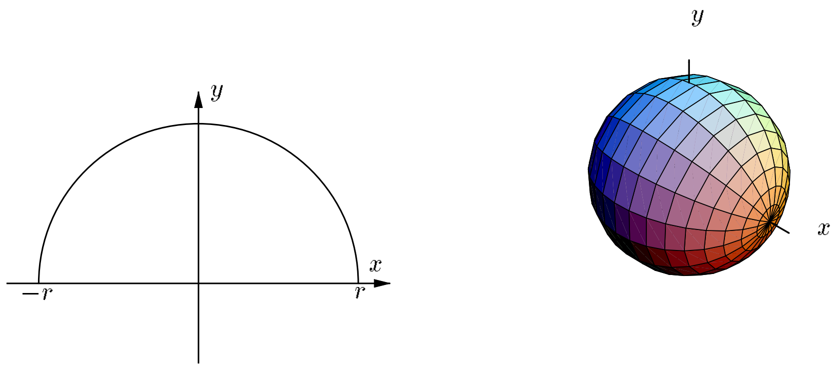
## 1. Volume of a Solid of Revolution (Slice Method)

A solid of revolution is a solid which is symmetrical with respect to a line called its *axis*. Such a solid can be obtained by taking some two-dimensional shape and revolving it around that axis. Familiar examples of solids of revolution include spheres, hemispheres, cylinders, and cones.

The basic situation to be dealt with first is the case when the axis of symmetry of the solid is the $x$-axis and the shape being revolved around the $x$-axis is the area under a curve $y = f(x)$ between $x = a$ and $x = b$. The familiar examples listed above can be obtained in this way.

Example 1.2. (Spheres and hemispheres)

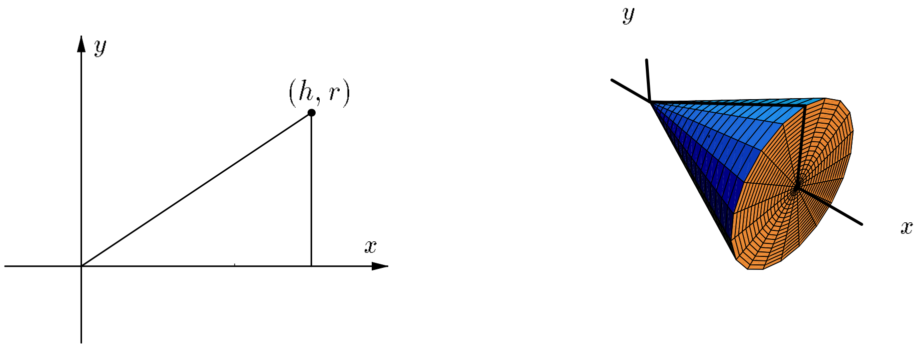
If we place the sphere of radius $r$ so that its center is at the origin of the $xy$-plane and take $f(x) = \sqrt{r^2 - x^2}$ (the upper semicircle of radius $r$), then we can get the sphere by rotating $f(x)$ between $x = -r$ and $x = r$ around the $x$-axis. To get a hemisphere of radius $r$, we situate the hemisphere so that the $x$-axis is its axis of symmetry, the $y$-axis between $r$ and $-r$ is a diameter of its circular base, and the "north pole" of the hemisphere is at the point $(r, 0)$. Then this hemisphere is the solid of revolution obtained by revolving $f(x) = \sqrt{r^2 - x^2}$ around the $x$-axis between $x = 0$ and $x = r$.




Rotate around $x$-axis to get sphere of radius $r$.

Example 1.3. (Right circular cones)

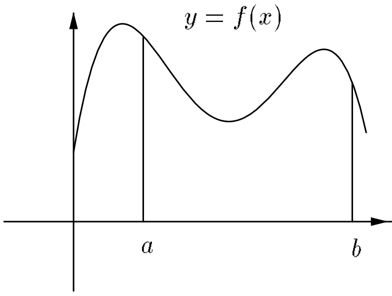
By " right circular cone " we mean a cone which is symmetric about an axis of symmetry, and for which any cross-section perpendicular to this axis is a circle. Such a cone is determined if we know its height $h$ and its base radius $r$. (In what follows, we shall always use the term "cone" to mean a right circular cone.) To describe this cone as a solid of revolution, we first choose an efficient way to situate it with respect to the $xy$-plane. We place it so that its vertex is at the origin and its axis of symmetry is the part of the $x$-axis from 0 to $h$. Then our cone is obtained by taking the line joining the origin to the point $(h, r)$ and rotating it around the $x$-axis. The advantage in situating the cone in this way is that the function we are revolving around the $x$-axis — the line from $(0, 0)$ to $(h, r)$ — has a very simple equation: $f(x) = mx$, where $m = r/h$ is the slope of the line.



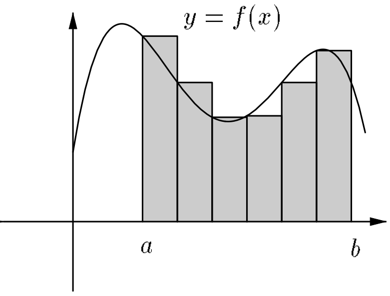

Rotate around $x$-axis to get cone of height $h$, base radius $r$.

We now return to the general situation of a solid of revolution obtained by revolving $y = f(x)$ around the $x$-axis between $x = a$ and $x = b$. We want to find its volume. To do this, we first take the region under the curve, and divide it up into Riemann sum rectangles as we did earlier. Of course, the rectangles do not exactly fill out the region: when the curve increases, the rectangles fall a little short, and when the curve decreases, the rectangles give you a little too much (see the second diagram below). But as the theory works out, the limit of the Riemann sums gives the correct answer.

Recall how the rectangles were constructed for the $n$-th left endpoint Riemann sum. First, the interval $[a, b]$ is divided into $n$ equal parts, each of length $\Delta x = (b - a)/n$. Next, above each subinterval we draw the rectangle whose height is the distance up to the curve at the left endpoint. The heights of the rectangles are successively $f(a)$, $f(a + \Delta x)$, $f(a + 2\Delta x), \ldots, f(a + (n - 1)\Delta x)$.




Graph of $y = f(x)$




Riemann sum rectangles

When we rotate our function $f(x)$ around the $x$-axis, what happens to the rectangles? As each narrow rectangle rotates through 360°, it forms a sideways cylinder whose "height" (actually, its distance across from left to right) is $\Delta x$ and whose base radius is the height of the rectangle. Since the volume of a cylinder is equal to its base area (which is $\pi \cdot \text{radius}^2$) times its height, it follows that the volume of the first cylinder is $\pi f(a)^2 \Delta x$; the volume of the second is $\pi f(a + \Delta x)^2 \Delta x$; the volume of the third is $\pi f(a + 2\Delta x)^2 \Delta x$, and so on. Thus, the total volume of all of the cylindrical slices is

$$\Big(\pi f(a)^2 + \pi f(a + \Delta x)^2 + \pi f(a + 2\Delta x)^2 + \cdots + \pi f(a + (n - 1)\Delta x)^2\Big)\Delta x,$$

which can be written more compactly using the summation notation, as follows:

$$\sum_{i=0}^{n-1} \pi f(a + i\Delta x)^2 \Delta x.$$

If $\Delta x$ is small (i.e., if $n$ is large), this volume is a very close approximation to the true volume of the solid of revolution. As $\Delta x \longrightarrow 0$ (i.e., as $n \longrightarrow \infty$), this sum gets closer and closer to the volume of the solid we want. As discussed in the Chapter on Riemann Sums, the limit of this sum will become the definite integral of $\pi f(x)^2$ (rather than the definite integral of $f(x)$). We conclude that

$$\text{Volume of solid of revolution} = \pi \int_a^b f(x)^2 dx.$$

(The constant $\pi$ that is next to $f(x)^2$ can be pulled out in front of the integral sign.)

One way to think of this formula is to look at the contribution to the volume that comes from the part of the region under $y = f(x)$ that lies between $x$ and $x + \Delta x$, where we suppose that $\Delta x$ is very small. This is a thin slice whose thickness is $\Delta x$ and whose circular base (at the left) has area $\pi f(x)^2$.

Since $\Delta x$ is small, the slice is very close to being a cylinder, and so its volume is very close to $\pi f(x)^2 \Delta x$. Now to find the volume of the entire solid of revolution we have to sum up the contributions as $x$ goes from $a$ to $b$. In the limit as $\Delta x$ goes to zero this sum becomes an integral, and we write $dx$ instead of $\Delta x$. That is, we get $\int_a^b (\pi f(x)^2) dx$, as before.

**More generally, we can think of this formula as follows. Suppose that we have a solid caught between the planes $x = a$ and $x = b$ and the cross sectional area corresponding to any value of $x$ is given by a function $F(x)$, then the total volume can be computed by the integral**

$$\int_a^b F(x)\, dx$$

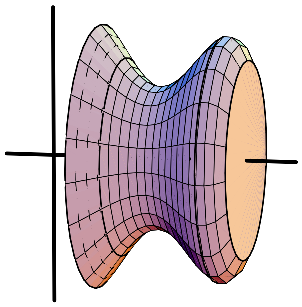
**In our current situation the cross sectional area happend to be a circle.**

**This formula can also be understood by creating a variable "volume" function**

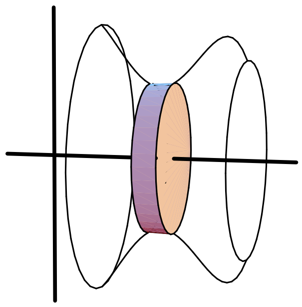
$$V(u) = \textbf{ volume caught between } x = a \textbf{ and } x = u$$

**and deducing as before that $V'(x) = g(x)$. Then we use the Fundamental Theorem of Calculus to deduce:**

$$V(b) = \int_a^b g(x)\, dx.$$



Volume of revolution.



Thin slice of solid.

EXAMPLE 1.4. Find the formula for the volume of a sphere of radius $r$.

We saw that the sphere is obtained by rotating the function $f(x) = \sqrt{r^2 - x^2}$ around the $x$-axis between $x = -r$ and $x = r$. Actually, it turns out to be a little simpler just to rotate the part between $x = 0$ and $x = r$ — thereby obtaining a hemisphere — and then doubling this volume. That is, using the formula we just derived, we see that the volume of the sphere is

$$2\pi \int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx = 2\pi \int_0^r (r^2 - x^2)\, dx = 2\pi \left(r^2 x - \frac{x^3}{3}\right) \Big|_0^r$$
$$= 2\pi(r^3 - r^3/3) = \frac{4}{3}\pi r^3.$$

EXAMPLE 1.5. Find the formula for the volume of a cone of height $h$ and base radius $r$.

According to our earlier discussion of how to get this cone as a solid of revolution, we have

$$\text{Vol} = \pi \int_0^h \left( \left( \frac{r}{h} \right) x \right)^2 dx = \pi r^2/h^2 \int_0^h x^2 dx = \pi (r^2/h^2)(h^3/3) = \frac{1}{3}\pi r^2 h.$$

Expressed in words, this formula says that *the volume of a cone is one third the volume of the cylinder of the same height and base radius.* By examining the computations in the last example, we see how the 1/3 arose: when we integrated $x^2$ to get $(x^3/3)|_0^h$.

## 2. Some Variants

Suppose we want our solid to be "right side up" with the $y$-axis as its axis of symmetry. Then the curve that is rotated around the $y$-axis should be viewed from the $y$-perspective as a function of the form $x = g(y)$ (rather than $y = f(x)$). Everything works the same with the roles of $x$ and $y$ reversed. Suppose that the solid is obtained by rotating $x = g(y)$ around the $y$-axis between $y = c$ and $y = d$. Then its volume is:

$$\pi \int_c^d g(y)^2 dy.$$

As another variant, suppose we want to rotate around the line $y = c$ which is parallel to the $x$-axis and a distance $c$ above it (if $c$ is negative, a distance $|c|$ below it). The only change is to replace $f(x)$ by $f(x) - c$, which is the distance from the axis of symmetry to the curve at the position $x$ on the $x$-axis. Thus, the volume is $\pi \int_a^b (f(x) - c)^2 dx$. A similar remark applies to the case when we take a function $x = g(y)$ and revolve it around the line $x = c$ rather than around the $y$-axis.

Example 1.6. Let $S$ be the shape in the $xy$-plane that is bounded by the curve $y = x^2$, the line $y = 4$ and the line $x = 1$, and which is to the right of the line $x = 1$. Find the volume of the shape obtained by revolving $S$ around (a) the $x$-axis, (b) the line $y = 4$, (c) the $y$-axis, (d) the line $x = 1$.

(a) Here we obtain a "hollowed out" shape. That is, to get our solid we first take the cylinder that is obtained by rotating the line $y = 4$ around the $x$-axis between $x = 1$ and $x = 2$, and from this cylinder we then carve out the solid obtained by rotating $y = x^2$ between $x = 1$ and $x = 2$. Thus,

$$\text{Vol} = \pi \cdot 4^2 \cdot 1 - \pi \int_1^2 (x^2)^2 dx = 16\pi - \pi \int_1^2 x^4 dx = (16 - (32 - 1)/5)\pi = 9.8\pi.$$

(b) Here we are taking the full solid of revolution obtained by rotating $y = x^2$ around the line $y = 4$. The result is

$$\text{Vol} = \pi \int_1^2 (x^2 - 4)^2 dx = \pi \int_1^2 (x^4 - 8x^2 + 16) dx = 53\pi/15.$$

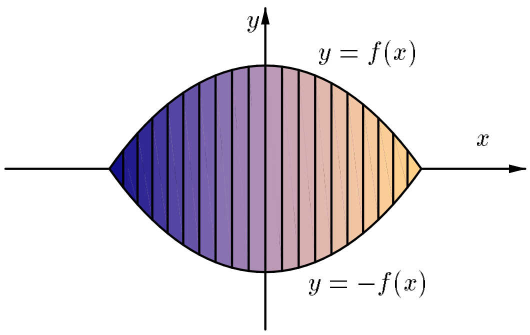
(c) Here we start with the solid of revolution obtained by rotating $x = \sqrt{y}$ around the $y$-axis between $y = 1$ and $y = 4$, and then hollow out a cylinder of height 3 and base radius 1 (obtained by rotating $x = 1$ around the $y$-axis between $y = 1$ and $y = 4$). The result is:

$$\text{Vol} = \pi \int_1^4 (\sqrt{y})^2 dy - \pi \cdot 1^2 \cdot 3 = \pi \int_1^4 y\, dy - 3\pi = 4.5\pi.$$

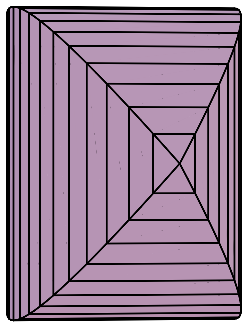
(d) Here we have the full solid of revolution where we rotate $x = \sqrt{y}$ around the line $x = 1$. Thus,

$$\text{Vol} = \pi \int_1^4 (\sqrt{y} - 1)^2 dy = \pi \int_1^4 (y - 2\sqrt{y} + 1) dy = 7/6\pi.$$

Example 1.7. Suppose that a solid has **square** cross-sections. The centers of the squares are on the $x$-axis, and the tops of the squares run through the line $y = f(x) = 4 - x^2$ between $x = -2$ and $x = 2$. (See the diagram below.) Find the volume of this solid.




Side view



Perspective view

Here the cross-section at $x$ is a square of side $2f(x)$. So the cross-sectional area is $\big(2f(x)\big)^2 = 4(4 - x^2)^2$. We integrate this from $-2$ to $2$. (Because of the symmetry — i.e., the fact that $f(x) = f(-x)$ — we can take the integral from $0$ to $2$ and then double the result.) Thus, the volume is

$$2 \int_0^2 4(4 - x^2)^2 dx = 8 \int_0^2 (16 - 8x^2 + x^4) dx = 136 \frac{8}{15}.$$