1. Absolute Convergence

As we have seen, the alternating series

$$(1) \qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3}\right) + \left(-\frac{1}{4}\right) + \cdots$$

is convergent, but the harmonic series

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \left|-\frac{1}{4}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

obtained from it by taking the absolute value of each term, is not. Thus tampering with its signs may destroy the convergence of a series – but not necessarily as we see by examining the series

$$(2) \qquad 1 - \tfrac{1}{2} + \tfrac{1}{2^2} - \tfrac{1}{2^3} + \cdots .$$

This is a convergent geometric series $\sum_{n=0}^{\infty} x^n$ with $x = -1/2$ and the series of absolute values

$$|1| + \left|-\frac{1}{2}\right| + \left|\frac{1}{2^2}\right| + \left|-\frac{1}{2^3}\right| + \cdots = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

is also a convergent geometric series. The point is that some convergent series having both positive and negative terms persist in their convergence after each term is replaced by its absolute value – others do not. The underlying concept is called *absolute convergence.*

Definition. Absolute Convergence

The series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is said to be *absolutely convergent* if the series

$$(3) \qquad \sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots$$

is convergent.

According to his definition and the preceding discussion, the series (2) is absolutely convergent; the series (1) is not. Of course, if $a_n \geq 0$ for all n, then

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots = a_1 + a_2 + a_3 + \cdots .$$

Hence a convergent series of positive terms is always absolutely convergent.

Example 1. The series

$$(a) \qquad 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots + \frac{(-1)^{n+1}}{(2n-1)^2} + \cdots$$

is absolutely convergent because

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots + \frac{1}{(2n-1)^2} + \cdots$$

is convergent. For instance, apply the integral test to $f(x) = \dfrac{1}{(2x-1)^2}$. But the series

(b)
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n+1}}{(2n-1)} + \cdots$$

is not absolute convergent because

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{(2n-1)} + \cdots$$

is a divergent series, as may be seen by applying the integral test to $f(x) = \dfrac{1}{2x-1}$.

It is traditional to give a name to a series which converges but does not converge absolutely. **Definition** The series $\sum a_n$ is called *conditionally convergent* if it is convergent but $\sum |a_n|$ diverges.

This definition simply divides all convergent series into two groups – those that converge absolutely and those that do not. The series

$$\ln\, 2 - \ln\, 3/2 + \ln\, 4/3 - \ln\, 5/4 + \cdots$$

and

$$1 - 1/\sqrt{2} - 1/\sqrt{3} + 1/\sqrt{4} + 1/\sqrt{5} - 1/\sqrt{6} - 1/\sqrt{7} + \cdots$$

are conditionally convergent: each series converges but the corresponding series, with all connecting signs positive, diverge. In such series you should think of convergence as being the result of delicately balanced internal cancellation, a balance that is shattered by making each connecting sign positive. On the other hand, in an absolutely convergent series like

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots,$$

you may change the signs any way you please and the resulting series remains convergent.

We close this section with an important test for absolute convergence.

Theorem 15. The Ratio Test

In the series $\sum a_n$ suppose that $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = L$. Then

(a) if $0 \le L < 1$, the series converges absolutely;

(b) if $L > 1$, the series diverges; and

(c) if $L = 1$, the test gives no information concerning convergence.

Proof. We begin with (a). The assertion is that the rations $|\dfrac{a_{n+1}}{a_n}|$ tend to some number L in the interval $0 \le L < 1$. Since the number $r = \dfrac{L+1}{2}$ satisfies the inequality $L < r < 1$, it is possible to choose N large enough so that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \text{ if } n \ge N.$$

In this inequality we give n, successively, the values $N,\ N+1,\ N+2, \ldots$ to obtain interlocking inequalities which, in tandem, imply that

$$|a_{N+1}| < |a_N|r,$$

$$|a_{N+2}| < |a_{N+1}|r < |a_N|r^2,$$

$$|a_{N+3}| < |a_{N+2}| < |a_N|r^3,$$

$$\vdots$$

At the kth level,

$$|a_{N+k}| < |a_N|r^k \ (k \geq 1).$$

These inequalities are the key because, recalling that $r < 1$, they show that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is dominated by the convergent geometric series

$$|a_N|r + |a_N|r^2 + |a_N|r^3 + \cdots = |a_N| \sum_{n=1}^{\infty} r^n.$$

Hence $\displaystyle\sum_{n=1}^{\infty} |a_{N+k}|$ converges by the Comparison Test. Now add on the first N terms, and we see that

$$\sum_{n=1}^{\infty} |a_n| = \{|a_1| + |a_2| + \cdots + |a_N|\} + \sum_{n=1}^{\infty} |a_{N+k}|$$

converges.

Turning to (b) we need only observe that if $L > 1$ or $L = \infty$ then the ratios ultimately become and remain greater than 1 – meaning that, for some N,

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \text{ if } n \geq N.$$

Again, by giving n the values $N,\ N+1,\ N+2, \ldots$, there results a string of inequalities

$$|a_N| < |a_{N+1}| < |a_{N+2}| < \cdots$$

from which it is clear that the positive numbers $|a_n|$ increase for $n \geq N$ and, in particular, $\lim\limits_{n\to\infty} a_n \neq 0$. Hence the series $\sum a_n$ must diverge.

The proof of part (c) is left as a problem. ‡

In a series of positive terms, the Ratio Test is applied without absolute values, in the form

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then $\sum a_n$ converges if $L < 1$, diverges if $L > 1$, and the test is inconclusive if $L = 1$.

Example 2. Use the Ratio Test to determine the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!} = 1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \cdots$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1} = \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots$

Solution. (a) Here $a_n = \dfrac{n}{2^n}$, $a_{n+1} = \dfrac{n+1}{2^{n+1}}$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}}\frac{2^n}{n} = \lim_{n\to\infty} \frac{1}{2}\frac{n+1}{n} = \frac{1}{2}.$$

Because $\dfrac{1}{2} < 1$, the series converges. ‡

(b) We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)n!}$$

and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Because $e > 1$, the series diverges. ‡

(c) In this series the Ratio Test is useless inasmuch as

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3n+1}{3(n+1)+1} = 1.$$

That the series is actually divergent may be seen by applying the Integral Test with $f(x) = \dfrac{1}{3x+1}$. ‡

Example 6. Use the Ratio Test to show that series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n)!} = \frac{1}{2} - \frac{1\cdot 3}{4!} + \frac{1\cdot 3\cdot 5}{6!} - \frac{1\cdot 3\cdot 5\cdot 7}{8!} + \cdots$$

is absolutely convergent.

Solution. In this more complicated example

$$a_n = (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n)!}, a_{n+1} = (-1)^{n+2}\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(2n+2)!}$$

and

$$\frac{a_{n+1}}{a_n} = -\frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{(2n+2)!}\;\frac{(2n)!}{1\cdot 3\cdot 5\cdots(2n-1)}$$

$$= \frac{2n+1}{(2n+1)(2n+2)}. \text{ Note that } [(2n+1)! = (2n+2)(2n+1)(2n)!]$$

Hence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{2n+2} = 0$$

and the series converges absolutely. ‡

At the elementary level, the Ratio Test is our best convergence test, especially for series in which a_n contains $n!$; $(2n)!$ or some other factorial expression.

CHAPTER 1

Power Series

The ideas of the foregoing sections culminate in the study of infinite series in which each term is a function. Chief among these are *power series*, series that are important primarily because they provide a natural and powerful method of defining and studying functions – the ordinary as well as the exotic.

1. Power Series

By a *power series in x* we mean a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{1}$$

in which the a_n denote real numbers called *coefficients* and x is considered a *variable.*

The first question in discussing a power series is this: For which values of x does the series converge? It is the nature of a power series in x to converge absolutely in some symmetric open interval of the form $|x| < r$ and to diverge if $|x| > r$ or else to converge absolutely for $-\infty < x < \infty$. This open interval of absolute convergence
is most conveniently found by applying the Ratio Test.

Example 1. Find the values of x at which the power series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n \cdot 3^n} = \frac{1}{3}x - \frac{1}{2 \cdot 3^2}x^2 + \frac{1}{3 \cdot 3^3}x^3 + \cdots$$

converges.

Solution. In this power series $a_0 = 0,\ a_1 = \frac{1}{3},\ a_2 = -\frac{1}{2 \cdot 3^2}$, etc. To apply the Ratio Test, we put

$$u_n = (-1)^{n+1} \frac{x^n}{n\, 3^n},\quad u_{n+1} = (-1)^{n+2} \frac{x^{n+1}}{(n+1)3^{n+1}}$$

and calculate

$$\frac{u_{n+1}}{u_n} = (-1)^{n+2} \frac{x^{n+1}}{(n+1)3^{n+1}} \frac{n\, 3^n}{(-1)^n 3^{n+1} x^n} = -\frac{n}{3(n+1)}x.$$

For fixed x,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| -\frac{n}{3(n+1)}x \right| = \lim_{n\to\infty} \frac{n}{3(n+1)}\, |x| = \frac{|x|}{3}.$$

According to the Ratio Test, the series converges absolutely if $\dfrac{|x|}{3} < 1$ and diverges of $\dfrac{|x|}{3} > 1$. Thus the series converges absolutely for x in the open interval

$$|x| < 3,\ -3 < x < 3$$

and diverges if

$$x < -3 \text{ or } x > 3.$$

At $x = 3$, the power series reduces to the conditionally convergent alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

and at $x = -3$, it reduces to the divergent series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^n}{n\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = -1 - \frac{1}{2} - \frac{1}{3} - \cdots .$$

To sum up: This power series converges for every x in the interval $-3 < x \leq 3$ and diverge otherwise. In the open interval $-3 < x < 3$, the convergence is absolute, at the endpoint $x = 3$ the convergence is conditional. ‡

Example 2. Find the values of x at which the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges.

Solution. To properly interpret the summation, it is necessary to remember that $0! = 1,\ 1! = 1$ and $x^0 = 1$.

Applying the Ratio Test with $u_n = x^n/n!$ gives

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0,$$

no matter what the values of x. Conclusion: the series converges absolutely for all x, for $-\infty < x < \infty$. Later we shall show that this series has sum e^x. ‡

The following theorem provides a precise description of the possible situations than can occur in analyzing the convergence of a power series in x. A proof is outlined in the problems.

Theorem 18. Given a power series in x, exactly one of the following hold:

(a) The series converges only at $x = 0$.

(b) For some positive number r, the series converges absolutely in the open interval $|x| < r$ and diverges if $|x| > r$.

(c) The series converges absolutely for all x.

Several remarks are in order. First, the situation in (a) is extreme and uninteresting because to be useful a power series must converge throughout some interval. The series

$$1 + x + 2!x^2 + 3!x^3 + 4!x^4 + \cdots + n!x^n + \cdots$$

illustrates the possibility. Like every power series in x, it converges at $x = 0$, but if $x \neq 0$ the Ratio Test gives

$$\lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)\,|x| = \infty$$

and the series diverges. The problem is caused by the rapid rate at which the coefficients $a_n = n!$ increase.

Second, the endpoint r and $-r$ of the open interval $-r < x < r$ of absolute convergence set forth in part (b) must be analyzed separately. At these two points the series may diverge or converge, either absolutely or conditionally.

Thus the convergence set may be a single point, any of the intervals $(-r, r)$, $(-r, r]$, $[-r, r)$, $[-r, r]$, or $(-\infty, \infty)$. In all cases we shall refer to the set of x where a power series converges as its *interval of convergence* and to r as its *radius of convergence*. The possibilities corresponding to parts (a), (b), and (c) respectively are that r is 0, r is finite, and r is infinite.

Example 3. Find the interval of convergence and the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} x^n = 2x + x^2 + \frac{8}{9} x^3 + \cdots .$$

Solution. The Ratio Test with

$$u_n = \frac{2^n}{n^2} x^n \text{ and } u_{n+1} = \frac{2^{n+1}}{(n+1)^2} x^{n+1}$$

gives

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2} x^{n+1} \cdot \frac{n^2}{2^n x^n} \right| = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} |x| = 2\,|x|.$$

Hence the power converges absolutely in the open interval $2\,|x| < 1$ or $|x| < 1/2$, that is, for

$$-1/2 < x < 1/2;$$

and diverges if

$$x < -1/2 \text{ or } x > 1/2.$$

Hence the radius of convergence is $r = 1/2$.

The series is also absolutely convergent at $x = 1/2$, where it reduces to the convergent p-series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{2^n 3} + \cdots ,$$

and at $x = -1/2$ where it reduces to the absolutely convergent alternating series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2} \left(-\frac{1}{2}\right)^n == \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = -\frac{1}{2} + \frac{1}{2^2} - -\frac{1}{3^2} + \cdots .$$

Consequently, the interval of convergence is $-1/2 \le x \le 1/2$ and the convergence is absolute throughout. ‡

The series

$$\sum_{n=1}^{\infty} a_n (x-a)^n = a_0 + a_1 (x-a) + a_2 (x-a)^2 + \cdots + a_n (x-a)^n + \cdots , \tag{2}$$

obtained from (1) by replacing x by $x - a$, is called a *power series in* $x - a$. Its interval of absolute convergence is symmetric about a.

Example 4. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}{2^{2n}n!}(x-1)^n = 1+\frac{1\cdot 3}{2^2}(x-1)+\frac{1\cdot 3\cdot 5}{2^4\cdot 2!}(x-1)^2+\frac{1\cdot 3\cdot 5\cdot 7}{2^6\cdot 3!}(x-1)^3+\cdots.$$

Solution. We apply the Ratio Test with

$$u_n = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{2n}n!}(x-1)^n \text{ and } u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{2^{2n+2}(n+1)!}(x-1)^{n+1}.$$

Note: The next odd integer after $2n+1$ is $2(n+1)+1 = 2n+3$ and $(n+1)! = (n+1)n!$. Thus

$$\frac{u_{n+1}}{u_n} = \frac{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{2^{2n+2}(n+1)!}(x-1)^{n+1}\frac{2^{2n}n!}{1\cdot 3\cdot 5\cdots(2n+1)}\,\frac{1}{(x-1)^n} = \frac{2n+3}{2^2(n+1)}(x-1)$$

and

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{2n+3}{4(n+1)}\,|x-1| = \frac{1}{2}\,|x-1|.$$

Consequently, there is absolute convergence where $\frac{1}{2}\,|x-1|<1$ or $|x-1|<2$. This means that the radius of convergence is $r=2$ and that the interval $-1<x<3$, with center 1 and radius 2, is an interval of absolute convergence.

Of course, the series diverges where $|x-1|>2$. It also diverges at the endpoints $x=3$ and $x=-1$ because the terms of the corresponding series do not tend to zero as n increases. For instance, at $x=3$ the original power series reduces to

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{2n}n!}(2)^n\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^n n!}$$

and

$$\sum_{n=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^n n!} = \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 4\cdot 6\cdots 2n} = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{2n+1}{2n} > 1$$

for all n. Therefore $-1<x<3$ is the interval of convergence. ‡

2. Power Series Calculus

On its interval of convergence a power series $\sum_{n=0}^{\infty} a_n x^n$ defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

and inside that interval of convergence f can be differentiated and integrated according to the following theorem whose proof we omit.

Theorem 19. Suppose the series (1) has a nonzero radius of convergence r. Then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2x^2 + 3a_3x^3 + \cdots \text{ if } |x|<r \tag{2}$$

and

$$\text{(3)}\qquad \begin{aligned}\int_0^x f(t)\,dt &= \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \\ &= a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots \text{ if } |x| < r.\end{aligned}$$

Further, the three series (1), (2), and (3) have the same radius of convergence.

The point is that inside its radius of convergence a power series is a continuous and differentiable function whose derivatives and integrals are found by operating on the series *t erm-by-term.* It is important to understand that the operations in (2) and (3) can be applied repeatedly. Thus the derivative of $f'(x)$ in (2) is

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)x^{(n-2)} \text{ if } |x| < r.$$

In other words, insofar as differentiation and integration are concerned, we operate on a power series exactly as if it were a polynomial. Of course, like any other convergent series, power series can be multiplied by a constant and added together term by term.

In this section we shall apply (2) and (3) to the problem of expressing some of the elementary functions of calculus as power series. That is, given a function $f(x)$ we will attempt to write it as a convergent power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

in some interval $|x| < r$. The problem is to find the coefficient a_n, and the process is called *expanding f in a power series* or *representing f by a power series.* Our initial efforts revolve around the geometric series

$$\text{(4)}\qquad 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}(|x| < 1).$$

Example 1. Find a power series representation for $\dfrac{1}{3-2x}$.

Solution. First we write the function in the form $\dfrac{1}{3}\dfrac{1}{1-\frac{2}{3}x}$, then apply the geometric series representation (4) with $\dfrac{2}{3}x$ in place of x. The result is

$$\begin{aligned}\frac{1}{3-2x} = \frac{1}{3}\,\frac{1}{1-\frac{2}{3}x} &= \frac{1}{3}\left[1 + \frac{2}{3}x + \frac{2^2}{3^2}x^2 + \frac{2^3}{3^3}x^3 + \cdots\right] \\ &= \frac{1}{3} + \frac{2}{3^2}x + \frac{2^2}{3^3}x^3 + \cdots \left(|x| < \frac{3}{2}\right). \ddagger\end{aligned}$$

Example 2. Find a power series representation for $\dfrac{1}{(1-x)^2}$.

Solution. Let $f(x) = \dfrac{1}{1-x}$. Then, from (4),

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \ (|x| < 1).$$

According to (2), the derivative $f'(x) = \frac{1}{(1-x)^2}$ is the term-by-term derivative of the series. Thus

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots \quad (|x| < 1). \ddagger$$

Example 3. Find a power series expansion for $\ln(1+x)$.

Solution. Because $1/(1+x)$ is the derivative of $\ln(1+x)$, we start with the series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + \cdots \quad (|t| < 1)$$

and integrate both sides. First,

$$\int_0^x \frac{1}{1+t}\, dt = \ln(1+t)]_{t=0}^x = \ln(1+x).$$

Hence, following (3),

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\, dt = \int_0^x \left(\sum_0^\infty (-1)^n t^n\right) dt = \sum_{n=0}^\infty (-1)^n \int_0^x t^n\, dt = \sum_{n=0}^\infty (-1)^n \frac{x^{n+}}{n+1},$$

that is,

(5) $$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \cdots + (-1)^n \tfrac{x^{n+1}}{n+} + \cdots$$

if $|x| < 1$. ‡

Remark 1. It is important to understand that the equality in (5) is good only so long as $|x| < 1$. This is because the starting point was the geometric series which itself converges only where $|x| < 1$. In contrast, the function $\ln(1+x)$ is defined and has derivatives of all orders where $x > -1$.

Remark 2. The series expansion in (5) provides a reasonable procedure for computing $\ln(1+x)$ if x is near zero. For example, being an alternating series we know that

$$\ln\frac{25}{24} = \ln\left(1 + \frac{1}{24}\right) \approx \frac{1}{2}4 - \frac{1}{2} - \frac{1}{2}\left(\frac{1}{24}\right)^2 + \frac{1}{3}\left(\frac{1}{24}\right)^3$$

$$\approx 0.04166667 - 0.00086806 + 0.00002411$$

$$\approx 0.04082272$$

with an error in absolute value less than $\frac{1}{4}\left(\frac{1}{24}\right)^4 < 0.0000008$. Hence $\ln\frac{25}{24} = 0.04082$, accurate to five decimal digits.

Example 5. Find a power series representation for $\ln\frac{1+x}{1-x}$.

Solution. In representation (5)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

we replace x by $-x$ to obtain

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and subtract the second series from the first. The resulting series representation is

$$\ln\frac{(1+x)}{(1-x)} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right), \tag{6}$$

valid if $|x| < 1$.

Example 6. Find a power series representation for $\tan^{-1} x$.

Solution. Again the geometric series is involved because $1/(1+x^2)$ is the derivative of $\tan^{-1} x$. Integration of both sides of

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

from 0 to x gives

$$\int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\,dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Hence

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \tag{7}$$

if $|x| < 1$. ‡

An analysis of the remainder in (7) shows that the representation is valid at the endpoints $x = \pm 1$. Since $\tan^{-1} 1 = \pi/4$, this means that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots, \tag{8}$$

but its agonizingly slow convergence makes it a poor vehicle for computing π. It is better to observe that $\tan^{-1}(1/\sqrt{3}) = \pi/6$ and hence that

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5 - \cdots. \tag{9}$$

Another approach to computing π exploits the trigonometric identity

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$$

Using the first six terms in (7), we find that

$$\tan^{-1}\frac{1}{2} \approx 0.45363989 \text{ with error} < 0.0000094,$$

$$\tan^{-1}\frac{1}{3} \approx 0.32175051 \text{ with error} < 0.00000005$$

and hence that $\pi \approx 3.141565$ with error < 0.00004. If we note further that the seventh term in (7) is positive if $x > 0$, we see that our estimate is too small and that $\pi \approx 3.1416$ accurate to four decimal places.

3. Taylor Series

As we have just seen, special purpose techniques can be used to find power series representations for certain of the elementary calculus functions. For other such functions it is preferable to adopt a more theoretical approach based on Taylor's formula About some point $x = a$ that formula represents f in the form

$$f(x) = f(x) + f'(x)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + R_n(x) \tag{1}$$

where the remainder R_n is given by

$$R_n(x) = f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!} \tag{2}$$

for some c between a and x. Recall that we have frequently used the polynomial part to approximate $f(x)$ and it is natural to examine that approximation as $n \to \infty$. Specifically, suppose that f is a function that has derivatives of all orders (infinitely differentiable) in some interval about a and that

$$\lim_{n\to\infty} R_n(x) = 0 \tag{3}$$

for every x in that interval. Denoting the polynomial part in (1) by $P_n(x)$, we have

$$f(x) = P_n(x) + R_n(x)$$

and

$$\begin{aligned} f(x) &= P_n(x) + R_n(x) \\ &= \lim_{n\to\infty} [P_n(x) + R_n(x)] \\ &= \lim_{n\to\infty} P_n(x). \end{aligned}$$

This mean that the polynomials $P_n(x)$ form the partial sums of a series whose limit is $f(x)$.

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots. \tag{4}$$

Put differently, the series converges throughout some interval about a and its sum is $f(x)$. Note that the expansion (4) is a power series in $x - a$; it is called the *Taylor series for f at $x = a$*. If $a = 0-$ in (4) the expansion reduces to

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + \cdots. \tag{5}$$

This representation of f as a power series in x is called the *Maclaurin series for f*.

The crucial point is that the validity of (4) or (5) depends directly on the limit calculation in (3). We begin by rederiving the power series expansion for e^x.

Example 1. Find the Maclaurin series for e^x.

Solution. Since $f(x) = e^x$ is its own derivative, $f^{(n)}(x) = e^x$ for all n and $f^{(n)}(0) = e^0 = 1$. Consequently, the Maclaurin series (5) becomes

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

To justify the claim that this expansion is valid for all x, we must show that

$$R_n(x) = f^{(n+1)}(c)\frac{x^{n+1}}{(n+1)!} = e^c\frac{x^{n+1}}{(n+1)!}$$

tends to zero as $n \to \infty$. Since $0 < c < x$, we see that $e^c < e^x$ if $x > 0$, and $e^c < 0$ if $x < 0$. In either case $e^c < e^{|x|}$; hence

$$|R_n(x)| \leq e^{|x|}\ \frac{|x|^{n+1}}{(n+1)!}.$$

At this point, we need to recall that the ratio test shows the series $\sum \frac{x^n}{n!}$ to be a convergent series for all x. Hence

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \tag{6}$$

because the terms of a convergent series must tend to zero. Therefore,

$$0 \leq \lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} e^{|x|}\ \frac{|x|^{n+1}}{(n+1)!} = 0$$

or $\lim_{n\to\infty} |R_n(x)| = 0$.

It should be emphasized that there are two phases to finding the Taylor or Maclaurin series for a function. The first is formal – just compute the derivatives $f^{(n)}(a)$ or $f^{(n)}(0)$ and substitute into the appropriate formula; then, to prove the validity of the representation, you must wrestle with the problem of showing that $\lim_{n\to\infty} R_n(x) = 0$.

Example 2. Find the Maclaurin series for $\sin\ x$.

Solution. The even derivatives of $\sin\ x$ at $x = 0$ are zero and its odd derivatives of order $2n+1$ are $(-1)^n$. Substituted in (5), this information gives the Maclaurin series

$$\sin\ x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots \quad (-\infty < x < \infty) \tag{7}$$

The validity of this expansion rests on the observation that if f denotes $\sin\ x$, then $|f^{(n)}(c)| \leq 1$ for any number c and any positive integer n and hence that

$$|R_n(x)| = |f^{(n+1)}(c)|\ \left|\frac{x^{n+1}}{(n+1)!}\right| = |f^{(n+1)}(c)|\ \frac{|x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!}.$$

Therefore $\lim_{n\to\infty} |R_n(x)| = 0$, by (6). ‡

Example 3. Find the Taylor series expansion for $\sin\ x$ about $a = \pi/6$.

Solution. Let $f(x) = \sin\ x$. The derivative pattern is

$$f(x) = \sin\ x,\ \ f'(x) = \cos\ x$$

$$f''(x) = -\sin\ x,\ f^{(3)}(x) = -\cos\ x$$

and, in general,

$$f^{(2n)}(x) = (-1)^n \sin\ x,\ f^{(2n+1)}(x) = (-1)^n \cos\ x.$$

And at $a = \pi/6$,

$$f^{(2n)}(\pi/6) = (-1)^n \frac{1}{2},\ f^{(2n+1)}(\pi/6) = (-1)^n \sqrt{3}/2.$$

Accordingly, the Taylor series is

$$\sin\ x = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{2 \cdot 2!}(x - \pi/6)^2 - \frac{\sqrt{3}}{2 \cdot 3!}(x - \pi/6)^3 + \cdots .$$

The proof that this expansion is valid for all x is exactly as in the previous example because $|f^{(n)}(c) \leq 1$ for all x.

Example 4. Find the Maclaurin series for $(1 + x)e^x$.

Solution. There are two ways to attack this problem.

(A) Put $f(x) = (1 + x)e^x$ and calculate

$$f'(x) = e^x + (1 + x)e^x = (2 + x)e^x,$$

$$f''(x) = e^x + (2 + x)e^x = (3 + x)e^x,$$

$$f'''(x) = e^x + (3 + x)e^x = (4 + x)e^x.$$

The pattern is clear: for each n

$$f^{(n-1)}(x) = (n + 1 + x)e^x.$$

Hence $f^{(n-1)}(0) = n + 1$ for $n \geq 0$ and the Maclaurin series is

$$(1 + x)e^x = 1 + 2x + \frac{3x^2}{2} + \frac{4x^3}{3!} + \frac{5x^4}{4!} + \cdots + \frac{(n + 1)x^n}{n!} + \cdots .$$

This expansion is valid for all x because

$$|R_n(x)| = |(n + 2 + c)e^c \frac{x^{n+1}}{(n + 1)!}| \leq (n + 2)\ \frac{|x|^{n+1}}{(n + 1)!} + |x|e^{|x|}\ \frac{|x|^{n+1}}{(n + 1)!}$$

and both terms on the right tend to zero as $n \to \infty$.

(B) Start with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots ,$$

multiply by x to get

$$xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n + 1)!} + \cdots .$$

Then add:

$$(1+x)e^x = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots) + \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n+1)!} + \cdots\right)$$
$$= 1 + 2x + \left(\frac{1}{2!} + 1\right) x^2 + \left(\frac{1}{3!} + \frac{1}{2}\right) x^3 + \cdots + \left(\frac{1}{n!} + \frac{1}{(n+1)!}\right) x^n + \cdots$$
$$= 1 + 2x + 3\frac{x^2}{2!} + 4\frac{x^3}{3!} + \cdots + (n+1)\,\frac{x^n}{n!} + \cdots.$$

In the preceding example, it is no accident that the two methods yield identical results.

Theorem 21. If a function f can be represented by a power series

$$f(x) = \sum_{n=0}^{\infty} b_n (x-a)^n \tag{9}$$
$$= b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n + \cdots$$

in some interval about $x = a$, then the series is the Taylor series expansion of f at $x = a$ and the coefficients b_n are given by

$$b_n = \frac{f^{(n-1)}(a)}{n!} \text{ for all } n. \tag{10}$$

Proof. Inside its interval of convergence the power series on the right in (9) represents an infinitely differentiable function that can be differentiated term-by-term. Clearly, $f(a) = b_0$ and from

$$f'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots + nb_n(x-a)^{n-1} + \cdots$$

it follows that $f'(a) = b_1$. Letting $x = a$ in the second derivative

$$f''(x) = 2b_2 + 2\cdot 3b_3(x-a) + \cdots + n(n-1)b_n(x-a)^{n-2} + \cdots$$

gives

$$f''(a) = 2b_2 \text{ or } b_2 = \frac{1}{2}f''(a).$$

From the third derivative,

$$f^{(3)}(x) = 2\cdot 3b_3 + 2\cdot 3\cdot 4b_4(x-a) + \cdots + n(n-1)(n-2)b_n(x-a)^{n-3} + \cdots,$$

we conclude that

$$f^{(3)}(a) = 2\cdot 3b_3 \text{ or } b_3 = \frac{f^{(n-1)}(a)}{3!}.$$

In general,

$$f^{(n-1)}(x) = n(n-1)(n-2)\cdots(2)b_n + (n+1)(n)(n-1)\cdots(2)b_{n+1}(x-a) + \cdots$$

and

$$f^{(n)}(a) = n(n-1)(n-2)\cdots(2)b_n = \frac{f^{(n)}(a)}{n!}. \ \ddagger$$

The point of this theorem is that the Taylor series representation for a function is unique, and that no matter how you arrive at a power series representation for a function, the series in hand *is* the Taylor series. In particular, this means that the series representations of the last section are Maclaurin expansions.

Example 5. Find the Maclaurin series expansion for e^{-x^2}.

Solution. In the series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

replace x by $-x^2$ to get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$$

The power series converges for all x and for all x it converges to e^{-x^2}. Conclusion: It must be the Maclaurin series for e^{-x^2}. ‡

Knowledge of coefficients in a Maclaurin expansion often provides an efficient procedure for computing derivatives at the origin. For example, suppose you want to know the sixth derivative of $f(x) = e^{-x^2}$ at $x = 0$. Straightforward differentiation is tedious; however, according to (10),

$$f^{(6)}(0) = 6!b_6 = 6!\left(-\frac{1}{3}\right) = -120$$

because $-\dfrac{1}{3!}$ is the coefficient of x^6 in the Maclaurin expansion of e^{-x^2}.

PROBLEM SET 20.6

In Problems 1–12 test the series for absolute and for conditional convergence.

1. $1 - 1/3 + 1/9 - 1/27 + \cdots$

2. $1 - 1/\sqrt[2]{2} + 1/\sqrt[3]{3} - 1/\sqrt[4]{4} + \cdots$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n^2}{n^3}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(\ln n)^2}{n^3}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{100 - n}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{n^2 n! 2^n}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 9 \cdots (3n)}{(2n)!}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(\ln 2n)^n}{(2n)^n}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^{3n+1}}{3^{2n+3}}$

11. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n^n}{n^2}$

12. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{(1.02)^n}$

13. For which values of a does the series $\displaystyle\sum_{n=1}^{\infty} \left(a^{-1} - \frac{1}{n}\right)^n$ converge absolutely?

14. Find the values of x for which the series $\displaystyle\sum_{n=1}^{\infty} (x+1)^n (x+2)^n$ is absolutely convergent.

CALCULATOR PROBLEMS

In Problems 15 and 16, let S denote the sum of the series and estimate the error in the approximation $S \cong s_N$ for the indicated value of N.

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n^2}{n^3}; \; N = 15$

16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(3n+7)^5}; \; N = 100$

PROBLEM SET 20.7

For the power series in Problems 1–6, find the interval of convergence; in Problems 7 and 8, find only the radius of convergence.

1. $\sum_{n=0}^{\infty} 3^n x^n$

2. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

3. $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

4. $\sum_{n=1}^{\infty} e^{-n} x^n$

5. $\sum_{n=1}^{\infty} \frac{x^n}{n \ln (n+1)}$

6. $\sum_{n=0}^{\infty} (-1)^n \frac{n^3}{5^n} (x-1)^n$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 6 \cdots (3n)}{(n+1)^3} x^n$

8. $\sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-4)}{n!} x^n$

In Problems 9–11, determine the values of x for which the series converges.

9. $\sum_{n=1}^{\infty} \frac{1}{x^n}$

10. $\sum_{n=0}^{\infty} \frac{2^n}{(x+1)^n}$

11. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2x-1}{x+2}\right)^2$

12. Construct a power series whose interval of convergence is $(-2, 2)$.

PROBLEM SET 20.8

With one of the series (4), (10), or (11), as a starting point derive a power series expansion for the functions in Problems 1–5. State the interval of convergence.

1. $\frac{1}{(1+x)^2}$

2. $\frac{1}{(1-x)^5}$

3. $\frac{3x^2}{(2-5x)^2}$

4. $\int_0^x \ln (1-t)\, dt$

5. $(2-x) - (1-x) \ln (1-x)$

In Problems 6–8, verify the power series expansion.

6. $\frac{1+x}{1-x} = 1 + 2x + 2x^2 + 2x^3 + \cdots$ if $|x| < 1$.

7. $\frac{x^2+6x+1}{(1-x)^3} = 1 + 3^2 x + 5^2 x^2 + 7^2 x^3 + \cdots$ if $|x| < 1$.

8. $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = x + \frac{1}{2}\frac{x^3}{3} + \frac{1+3}{2 \cdot 4}\frac{x^5}{5} + \frac{1+3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots$ if $|x| < 1$.

CALCULATOR PROBLEMS

In Problems 9–12, use (5), (7), (10), and (11) to estimate the following with an error less than 0.001.

9. $\ln 1.2$

10. $e^{-0.3}$

11. $\sqrt[3]{1.2}$

12. $(0.95)^{-1/4}$

13. Because $\pi/6 = \sin^{-1} 0.5$, the series in Problem 18 can be used to approximate π. Construct a table of its first fifteen partial sums at $x = 1/2$. Compare the resulting approximations to the calculator value for π.

14. Repeated applications of the identity

$$\tan^{-1}\frac{1}{p} = \tan^{-1}\frac{1}{p+q} + \tan^{-1}\frac{1}{p+r}(qr = 1 + p^2)$$

of C.L. Dodgson (Lewis Carroll) leads to the formula

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}.$$

Use this and the Maclaurin expansion for $\tan^{-1} x$ to find the first eight digits in the decimal expansion of π.

PROBLEM SET 20.9

In Problems 1–4, find the Maclaurin series expansion for $f(x)$ and use (3) to establish its validity for all x. The symbols a and b denote constants.

1. $f(x) = e^{ax}$

2. $f(x) = \sinh(x)$

3. $f(x) = \dfrac{x^3}{6} - x + \sin x$

4. $f(x) = \sin^2 x = \dfrac{1 - \cos 2x}{2}$

In Problems 5–8 derive a formula for the general nth derivative $f^{(n)}$ and use it to write the Taylor series (4) for $f(x)$ about $x = a$ You may omit the verification (3).

5. $f(x) = \dfrac{1}{1+x};\ a = 0$

6. $f(x) = \dfrac{1}{2x+3};\ a = -1$

7. $f(x) = x^5;\ a = 1$

8. $f(x) = x^2;\ a = 100$

Use known series expansions from this and the preceding section together with algebraic manipulation and substitution to obtain the following Maclaurin expansions.

9. $x^2 e^{-x} = x^2 - x^3 + \dfrac{x^4}{2!} - \dfrac{x^5}{3!} + \cdots$

10. $(1+x)^{-1/3} = 1 - \dfrac{1}{3}x + \dfrac{1\cdot 4}{3^2\cdot 2}x^2 - \dfrac{1\cdot 4\cdot 7}{3^3\cdot 3!}x^3 + \cdots$

11. $\sin^{-1}\, x - x = \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{x^7}{7} + \cdots$

12. $(1-x)^2)\, \ln\,(1-x^2) = -x^2 + \dfrac{1}{2x^4} + \dfrac{1}{6}x^6 + \dfrac{1}{12}x^8 + \dfrac{1}{20}x^{10} + \cdots$

13. Integrate both sides of

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots (-1)^{n-1}\, t^{n-1} + \frac{t^n}{1+t}$$

from 0 to 1 to obtain

$$\ln\, 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\,\frac{1}{n} + (-1)^n \int_0^1 \frac{t^n}{1+t}\, dt.$$

Then show that

$$R_n = (-1)^n \int_0^1 \frac{t^n}{1+t}\, dt \to 0 \text{ as } n \to \infty,$$

thereby verifying that $\ln\, 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$.

14. Use integration by parts in the formula

$$\tan^{-1}\, x = \int_0^x \frac{1}{1+t^2}\, dt$$

to obtain the series expansion

$$\tan^{-1}\, x = \frac{x}{1+x^2} + \frac{2}{3}\frac{x^3}{(1+x^2)^2} + \frac{2}{3}\cdot\frac{4}{5}\frac{x^5}{(1+x^2)^3} + \cdots$$