CHAPTER 1

Related Rate Volume Problems

Suppose we have a container that is described by rotating some function $x = g(y)$ around the y-axis between $y = c$ and $y = d$, and suppose that the container is filled with water to a height h. Then the volume of water is given by the definite integral

$$V(h) = \int_c^{c+h} \pi g(y)^2 dy.$$

We write $V(h)$ to indicate that the volume of the water is a function of its depth h.

In this situation one type of question we might want to answer is: given the rate at which water is being pumped into the container, what is the rate at which the water level is rising? In symbols this says: given dV/dt, find dh/dt. Alternately, we might know dh/dt (by observing how fast the water level is rising) and want to find dV/dt. These are called "related rate" problems. According to the chain rule, the two rates dV/dt and dh/dt are related as follows:

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}.$$

Since $V(h)$ is given as an integral with h in the upper limit of integration, we can use the "fundamental theorem of calculus" to find dV/dh, **without ever having to actually compute an integral**. To see how to do this, let's review what the fundamental theorem says, using the present notation. Suppose we have a function defined by an integral

$$V = \int_c^{u(h)} f(y) dy.$$

Then

$$\frac{dV}{du} = f(u(h))$$

(this says that *the derivative of an integral with respect to the upper limit of integration is obtained by removing the integral sign and substituting the upper limit of integration in place of the dummy variable in the integrand*). Thus

$$\frac{dV}{dh} = \frac{dV}{du} \cdot \frac{du}{dh} = f(u(h))\frac{du}{dh}.$$

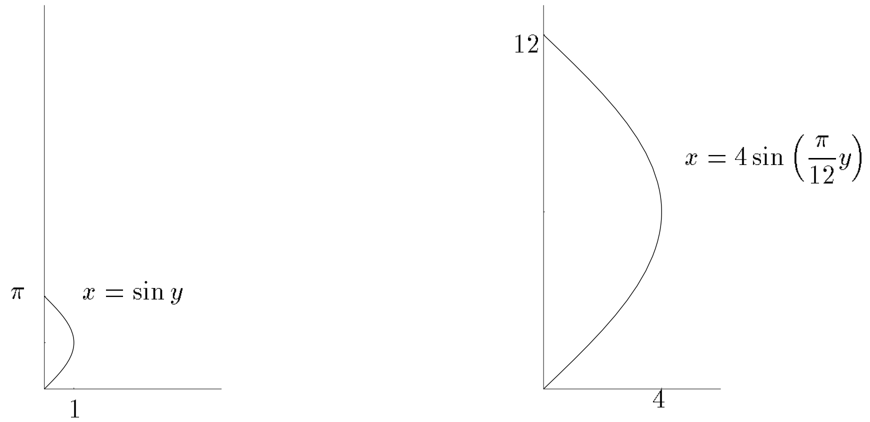
In our situation with the volume of revolution we have $u(h) = h + c$, so that $du/dh = 1$, and the integrand $f(y)$ is $\pi g(y)^2$. Thus, the fundamental theorem tells us that

$$\frac{dV}{dh} = \pi g(c+h)^2, \qquad \text{and so} \qquad \frac{dV}{dt} = \pi g(c+h)^2 \frac{dh}{dt}.$$

Another way to say this is as follows: *the ratio of volume rate to height rate is the cross-sectional area*, i.e., the area of the top surface of the liquid at the instant in question. This principle also applies when the liquid is filling a container whose cross-sections are not necessarily circles.

EXAMPLE 1.1. A container is formed by rotating the function $x = 4\sin(\pi y/12)$ around the y-axis between $y = 0$ and $y = 6$ (the units are feet). At a certain instant the container is filled to a depth of 4 ft, and water is being poured in at the rate of 5 ft^3/sec. How fast is the water level rising?

It's generally a good idea (even if not absolutely essential) to draw a picture of what we're dealing with. As explained earlier in the section on sinusoidal functions, the graph of $x = 4\sin(\pi y/12)$ is obtained from the graph of $x = \sin y$ by stretching in the x-direction by a factor of 4 and in the y-direction by a factor of $12/\pi$ (so that its period expands from 2π to 24):



Taking this function from $y = 0$ to $y = 6$ means taking the part until x reaches its first maximum at the point (4,6). (See the picture at the right). According to the above formula (here $c = 0$), we have

$$\frac{dV}{dt} = \pi\big(4\sin(\pi h/12)\big)^2\frac{dh}{dt}.$$

At the instant in question $h = 4$ and $dV/dt = 5$, so we have

$$5 = 16\pi\sin^2(\pi/3)\frac{dh}{dt} = 16\pi(0.5\sqrt{3})^2\frac{dh}{dt} = 12\pi\frac{dh}{dt},$$

so that

$$\frac{dh}{dt} = \frac{5}{12\pi} = 0.1326\frac{\text{ft}}{\text{second}} = 1.59\frac{\text{inch}}{\text{second}}.$$

In this problem, notice that we never had to evaluate an integral. If we had instead been asked to find the volume of water in the container when $h = 4$, then we would have had to compute $\pi\int_0^4 16\sin^2(\pi y/12)\,dy$. At this point we do not have a technique for finding the anti-derivative of $\sin^2$ (later we will get a method for doing this type of integral). The point is: there is a big difference between asking about the *rates* at which the height and volume are increasing and asking about the actual values of the height and volume. In the latter case you need to evaluate an integral, whereas in the former case you do not.

HOMEWORK 1.

1. The graph of $y = \sqrt{x}$ is rotated around the y-axis to form a container into which liquid is being poured at the rate of 0.1 cubic units per minute.

(a) How many minutes before the container is filled to a height of 1?

(b) At that point ($h = 1$), how fast is the water level rising?

2. Same as Problem 1, but with the graph of $y = x^4$ rotated around the y-axis.

3. Sketch the graph of $x = \tan(\pi y/20)$. This graph is rotated around the y-axis to form a vat (the units are feet). Liquid is poured into the vat in such a way that the height of liquid is increasing steadily at 0.2 ft/sec. At what rate (in ft^3/sec) is liquid being poured when the height is 5 ft?

4. A hemispherical vat of radius 2 meters is being filled with a liquid, starting at time $t = 0$ sec, in such a way that the depth of the liquid increases at the steady rate of 4 cm/sec. What is the rate of flow of the liquid (in m^3/sec) after t seconds (expressed in terms of t)?

5. Same as Problem 4, except that the vat is obtained by rotating around the y-axis the bottom half of the ellipse $x^2/9 + y^2/4 = 1$.

6. With the vat in Problem 5, suppose now that it is being filled at the steady rate of 0.15 m^3/sec. Find the rate at which the depth of the liquid in the container is increasing at the moment when its depth is 0.5 m.

7. Suppose that the tetrahedron in Problem 9 of the previous chapter is situated upside down with its vertex at the origin and its base perpendicular to the y-axis at $y = H$. If the tetrahedron is being filled with water, find the ratio of dV/dt to dh/dt at the instant when the water has depth h at the deepest point. Express your answer in terms of h, H, and s.