CHAPTER 1

# Arclength, Parameterized Curves, Ellipses

## 1. Review of Parameterized Curves

We first briefly review parameterized curves. For a more thorough review, reread the two chapters on this topic in the Math 113 notes.

A curve parameterized by time can be thought of as the motion of a particle in the $xy$-plane whose $x$- and $y$-coordinates are each given by a function of $t$: $x = x(t)$, $y = y(t)$. Then $\dot{x}(t) = dx/dt$ is the horizontal velocity, $\dot{y}(t) = dy/dt$ is the vertical velocity, and $dy/dx = \dot{y}(t)/\dot{x}(t)$ is the slope of the particle's path at the particular time $t$. In some situations we have other parameters besides $t$, for example, an angle parameter $\theta$. In that case we can still compute the slope of the curve $dy/dx$ by taking the ratio of $dy/d\theta$ to $dx/d\theta$.

Some basic examples to understand are:

1. **Straight line path from $P = (x_0, y_0)$ to $Q = (x_1, y_1)$.** If you travel from $P$ to $Q$ at constant speed in time $T$, then your $x$- and $y$-coordinates at time $t$ are:

$$x(t) = x_0 + \big((x_1 - x_0)/T\big)t, \text{ and } y(t) = y_0 + \big((y_1 - y_0)/T\big)t\,.$$

2. **Falling bodies (no air resistance).** If the object starts from $(x_0, y_0)$ with initial velocity of $v_0$ directed at an angle $\alpha$ above the horizontal ($\alpha$ is a negative angle if it's being thrown down), then

$$x(t) = x_0 + (v_0 \cos\alpha)t \text{ and } y(t) = -\frac{1}{2}gt^2 + (v_0 \sin\alpha)t + y_0\,.$$

3. **Circular motion.** If the particle is moving around the circle of radius $r$ centered at the point $(h, k)$, and if we let $\theta$ denote the angular position of the particle in radians measured counterclockwise from the positive $x$-direction, then $x(\theta) = h + r\cos\theta$, $y(\theta) = k + r\sin\theta$. If $\theta$ is a known function of $t$, then we can substitute that function here to get $x(t)$, $y(t)$. If the particle's angular position at time $t = 0$ is $\theta_0$ and if it has **constant** angular velocity (in radians/sec) equal to $\pm\omega$ — where we take $\omega$ when it is moving counterclockwise and $-\omega$ when it is moving clockwise — then

$$x(t) = h + r\cos(\theta_0 \pm \omega t), \text{ and } y(t) = k + r\sin(\theta_0 \pm \omega t)\,.$$

Some important special cases are:

1. $h = k = \theta_0 = 0$, so that $x(t) = r\cos(\omega t)$, $y(t) = r\sin(\omega t)$;
2. $\theta_0 = \pi/2$ (i.e., it starts from the positive $y$-direction) and the rotation is clockwise, so that $x(t) = h + r\cos(\frac{\pi}{2} - \omega t) = h + r\sin(\omega t)$, $y(t) = k + r\sin(\frac{\pi}{2} - \omega t) = k + r\cos(\omega t)$;
3. clockwise rotation starting from the *negative* $y$-direction, in which case $x(t) = h - r\sin(\omega t)$, $y(t) = k - r\cos(\omega t)$.

4. **Combined circular motion with some other motion.** Here the particle is whirling around a center which itself is moving. In that case you use the circular motion equations in the preceding paragraph, but with $h$ and $k$ (the coordinates of the center) replaced by the functions $x_{\text{center}}(t)$ and $y_{\text{center}}(t)$ which describe the motion of the center. For example,

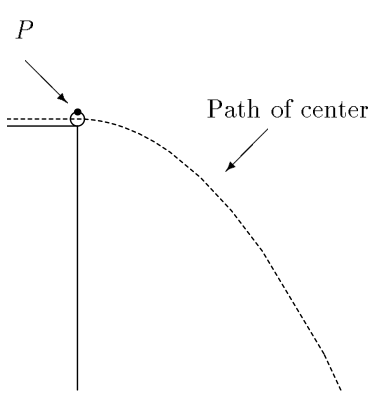
the **trochoid** is the motion of a point $P$ which is stuck at a distance $a$ (where $a$ is between 0 and $r$) from the center of a wheel (where we assume that at time $t = 0$ the wheel passes over the origin with $P$ located directly below the center at that instant). In that case the circular motion of $P$ around the center has radius $a$, and from special case (3) in the preceding paragraph we obtain the equations

$$x(t) = r\omega t - a\sin(\omega t), \text{ and } y(t) = r - a\cos(\omega t)\,.$$

The **cycloid** is the special case of this when $a = r$. If the wheel is traveling along at a known linear velocity (say, in m/sec), then the angular velocity $\omega$ (in rad/sec) is obtained by dividing the linear velocity by the radius of the wheel. If you are given $\omega$ in units of revolutions/sec, then multiply by $2\pi$ to get rad/sec.

EXAMPLE 1.1. A wheel of radius 0.2 meters rolls without slipping along the flat roof of a building at speed 6 m/sec. At time $t = 0$ it flies off the roof. The building is 122.5 m high. Let $P$ be the point on the rim of the wheel which is at the top at the instant $t = 0$ when the wheel leaves the roof. Find parametric equations for the motion of $P$. Take the side of the building as the $y$-axis and the ground as the $x$-axis.

When the wheel rolls along the roof, its angular velocity is: (linear velocity)/radius$=$ $6/0.2 = 30$ rad/sec. After leaving the roof it continues spinning at the same rate. So the point $P$ revolves clockwise around the center of the wheel at $\omega = 30$ rad/sec, starting above the center when $t = 0$. This is special case (2) in the paragraph above on circular motion. Thus, the motion of $P$ relative to the center has equations $0.2\sin(30t)$ and $0.2\cos(30t)$. Meanwhile, the center of the wheel starts from



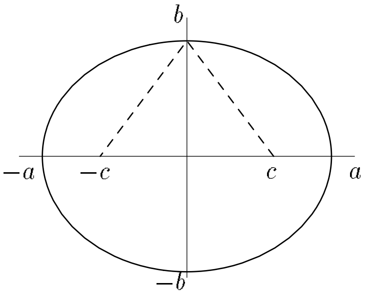

$(0, 122.7)$ with initial horizontal component of velocity 6 m/sec (and zero initial vertical component of velocity). So $x_{\text{center}}(t) = 6t$, $y_{\text{center}}(t) = -4.9t^2 + 122.7$. Finally, we get the equations for $P$ by adding its equations relative to the center to the equations for the coordinates of the center:

$$x(t) = 6t + 0.2\sin(30t), \qquad y(t) = -4.9t^2 + 122.7 + 0.2\cos(30t).$$

## 2. Ellipses

An ellipse is obtained by stretching the circle of radius 1 by a factor of $a$ in one direction and by a factor of $b$ in the perpendicular direction. An ellipse has two axes of symmetry, the **major axis**, having length $2a$, which joins the two farthest opposite points on the circumference, and the **minor axis**, having length $2b$, which joins the two closest opposite points on the circumference. The half-length $a$ is called the **semi-major axis**, and $b$ is called the **semi-minor axis**. If we choose to locate the ellipse in the $xy$-plane so that its center is at the origin, its major axis is in the $x$-direction, and its minor axis is in the $y$-direction, then it has the implicit equation $x^2/a^2 + y^2/b^2 = 1$. (The special case $a = b = r$ is simply the circle of radius $r$, having equation $x^2 + y^2 = r^2$.) In parametric form, the ellipse can be described by the equations $x(\theta) = a\cos\theta$, $y(\theta) = b\sin\theta$, where we go once around the ellipse as $\theta$ goes from 0 to $2\pi$. (One difference with a circle, however: the point $P = (x(\theta), y(\theta))$ is *not* located at an angle $\theta$ from the positive $x$-axis. In fact, if $\alpha$ is the angular position of $P$, then $\tan\alpha = y(\theta)/x(\theta) = (b/a)\tan\theta$, which is not the same as $\tan\theta$ when $a \neq b$; hence $\alpha$ is not the same as $\theta$.)

Another description of an ellipse is the locus of all points $P$ for which the sum of the distance from $P$ to two fixed points is a constant. Those two points are called the **foci** (the plural of *focus*). If the ellipse is situated in the standard way, with the major axis along the $x$-axis, then both foci are on the $x$-axis, equidistant from the origin. We let $(c, 0)$ and $(-c, 0)$ denote the coordinates of the foci. See the drawing at the right.




We can get an important relationship between $a$, $b$ and $c$ if we notice that (1) the point $(a, 0)$ on the ellipse is at a distance $a + c$ from the farther focus and $a - c$ from the closer focus, so the sum of the distances is $2a$; and (2) the point $(0, b)$ on the ellipse is at a distance $\sqrt{b^2 + c^2}$ from each of the two foci, so the sum of the distances is $2\sqrt{b^2 + c^2}$. Since these sums of distances must be the same, it follows that $2a = 2\sqrt{b^2 + c^2}$. Dividing by 2 and squaring both sides, we conclude that $a^2 = b^2 + c^2$. This relation enables us to find one of the three constants $a$, $b$, $c$ if we know the other two.

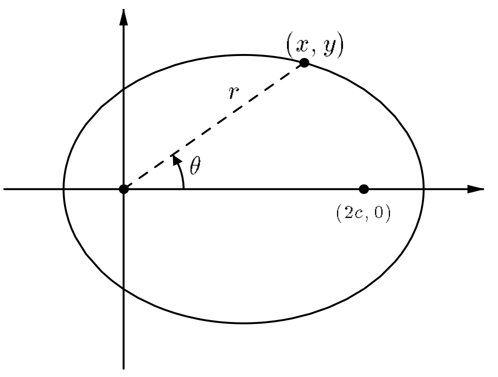
The **eccentricity** of an ellipse is the ratio $e = c/a$, i.e., the fraction of the way from the center to the circumference that a focus is located on the $x$-axis. The eccentricity is always between 0 and 1: $0 \leq e < 1$. If $e = 0$, then $a = b$ and we have a circle. The greater $e$, the farther our ellipse is from looking like a circle. Ellipses with $e$ near 1 are long and thin.

Example 1.2. Find the equation of the ellipse of eccentricity 0.8 whose foci are at $(0, \pm 60)$.

Here the foci are on the $y$-axis rather than the $x$-axis. This means that the major axis is in the $y$-direction. The distance $c$ from the center to a focus is 60, and $e = 0.8 = c/a$. Hence, $a = c/e = 60/0.8 = 75$. Then $b = \sqrt{a^2 - c^2} = 45$. When the major axis is in the $y$-direction, the equation of the ellipse is $x^2/b^2 + y^2/a^2 = 1$, i.e., our answer is $\dfrac{x^2}{2025} + \dfrac{y^2}{5625} = 1$.

We now give a second way to describe an ellipse parametrically. We first move the ellipse horizontally a distance $c$ to the right so that it has a focus at the origin. Then the other focus is at the point $(2c, 0)$. (Sometimes it is more natural to put the origin at a focus of the ellipse rather that at its center. For example, the orbit of a planet as it revolves around the sun is an ellipse with the sun at one focus.) See the diagram on the next page.

If $(x, y)$ are the coordinates of a point on the ellipse, then for our parameter we use the angle $\theta$ that the line from the origin to $(x, y)$ makes with the $x$-axis. (As mentioned above, the $\theta$-parameter in our earlier parametric equations does *not* have this simple geometrical meaning.) Let $r = \sqrt{x^2 + y^2}$ denote the distance of the point from the origin, and let $d$ be its distance from the other focus. Since the sum of the distances of $(x, y)$ from the two foci is equal to $2a$, it follows that $d = 2a - r$. On the other hand, by the law of cosines, $d^2 = r^2 + (2c)^2 - 2r(2c)\cos\theta$. Thus,




$$(2a - r)^2 = r^2 + 4c^2 - 4rc\cos\theta.$$

If we expand the square on the left, then cancel the $r^2$ and divide through by 4, we get $a^2 - ar = c^2 - rc\cos\theta$, and so

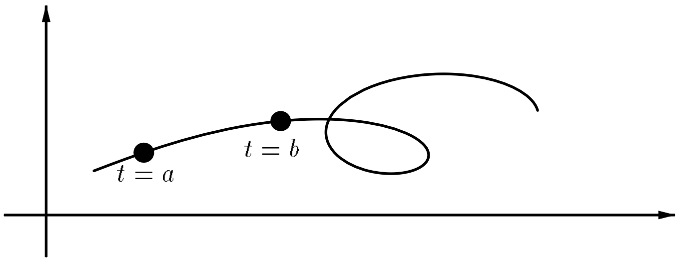
$$r = \frac{a^2 - c^2}{a - c\cos\theta}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we immediately obtain formulas for $x$ and $y$ in terms of the parameter $\theta$:

$$x = \frac{(a^2 - c^2)\cos\theta}{a - c\cos\theta}, \qquad y = \frac{(a^2 - c^2)\sin\theta}{a - c\cos\theta}, \qquad 0 \le \theta \le 2\pi.$$

## 3. Arclength

Suppose we have a curve described parametrically by equations $x = x(t)$, $y = y(t)$. By the **arclength** of the curve between $t = a$ and $t = b$ we mean the actual length of the path (distance traveled). That is, if you were to imagine the path made of string and then straighten out the string and put it along the $x$-axis, this would be the length along the $x$-axis. In what follows we shall use the notation $s(t)$ to mean the distance along the path from some reference point (perhaps the beginning of the path) to the point $(x(t), y(t))$. Then the arclength of the path between $t = a$ and $t = b$ is simply $s(b) - s(a)$.




The derivative $ds/dt$ — the rate at which the particle is traveling along its path — is what we normally call its speed. This is the length of the velocity vector, i.e., the hypotenuse of the triangle made up of the horizontal and vertical components of velocity. Thus,

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This means that the distance function $s(t)$ is an antiderivative of the square root on the right, and so we have

$$\Big(\text{arclength from } t = a \text{ to } t = b\Big) = s(b) - s(a) = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In words, this says that *distance traveled is equal to the area under the graph of speed.*

Example 1.3. Find a definite integral for the distance traveled by the point $P$ in Example 4.1 from when the wheel leaves the roof to when it hits the ground.

The wheel leaves the roof at time $t = 0$. To find out when it hits the ground, we observe that at that instant the center is at height 0.2 (the radius). Setting $0.2 = y_{\text{center}}(t) = -4.9t^2 + 122.7$, we find that $t = \sqrt{122.5/4.9} = 5$ sec. To find the speed, we compute the velocity components $dx/dt = 6 + 6\cos(30t)$, $dy/dt = -9.8t - 6\sin(30t)$, and so the distance traveled is

$$\int_0^5 \sqrt{(6 + 6\cos(30t))^2 + (-9.8t - 6\sin(30t))^2}\, dt.$$

This integral is too complicated to evaluate by the antiderivative method. It must be computed by "numerical integration," i.e., an approximation method. In the next section we shall present one such method, the trapezoid rule.

If your curve is given in terms of a parameter other than time, you can use the same formula to determine the arclength — there is nothing special about the letter $t$ in the above formula. For example, if $\theta$ is your parameter, then the arclength of the curve between $\theta = a$ and $\theta = b$ is

$$\int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Example 1.4. Find the arclength of one arch of the cycloid, i.e., $x(\theta) = r\theta - r\sin\theta$, $y(\theta) = r - r\cos\theta$, between $\theta = 0$ and $\theta = 2\pi$. Express your answer in terms of $r$. This is the actual distance traveled by a pebble in the tread of the wheel each time the wheel makes one revolution. If the bicycle travels 1 mile, how far does the pebble travel?

Here $dx/d\theta = r(1 - \cos\theta)$ and $dy/d\theta = r\sin\theta$. Before taking the integral, it's helpful to simplify the sum of the squares of the derivatives. We do this by factoring out $r^2$ and then using the trig identity $\sin^2 + \cos^2 = 1$:

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= r^2\Big((1 - \cos\theta)^2 + \sin^2\theta\Big)\\ &= r^2\Big(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\Big) = r^2(2 - 2\cos\theta).\end{aligned}$$

We must integrate the square root of this: $\int_0^{2\pi} r\sqrt{2 - 2\cos\theta}\, d\theta$. It is not obvious how to do this integral by the antiderivative method. For example, a $u$-substitution of the form $u = 2 - 2\cos\theta$ would **not** work. (Such a substitution would work if we had a $\sin\theta$ term sitting beside the square root — but unfortunately we don't have that.) Perhaps we should wait until the trapezoid rule, and be satisfied with an approximate value. However, there is a trick that will make the integrand take a simpler form. It uses the double angle formula

$\cos(2A) = 1 - 2\sin^2(A)$. (We will later find that this formula comes up in integration problems in other situations — see the section on integrating powers of sine and cosine.) If we replace $A$ by $\frac{1}{2}\theta$ in this double angle formula, and then move the cosine to the right and the $\sin^2$ to the left and multiply through by 2, we obtain: $4\sin^2(\theta/2) = 2 - 2\cos(\theta)$. Taking the square root of both sides, we get: $\sqrt{2 - 2\cos\theta} = 2\sin(\theta/2)$. (Note: In general when we take square roots we have to put a $\pm$ in front of the radical; however, in this case, since we have assumed that $\theta$ is in the range between 0 and $2\pi$, and so $\theta/2$ is between 0 and $\pi$, we know that $\sin(\theta/2)$ is positive.) Thus, the arclength of the cycloid is

$$\int_0^{2\pi} 2r\sin(\theta/2)\,d\theta = -2r \cdot \frac{1}{1/2}\cos(\theta/2)\,\Big|_0^{2\pi} = -4r(\cos\pi - \cos 0) = 8r.$$

Thus, the pebble travels a distance $8r$ during the time the wheel makes one revolution, i.e., during the time the bicycle travels $2\pi r$. Thus, the pebble travels $8/2\pi$ times as far as the bicycle, and so when the bicycle travels 1 mile the wheel travels $8/2\pi = 4/\pi = 1.27$ miles.

Finally, we consider the case when a function is given in the explicit form $y = f(x)$ between $x = a$ and $x = b$. We can still use the same arclength formula, because the explicit form of a curve is a special case of parameterized equations: simply choose $x$ as your parameter, and take the two equations $x = x$, $y = f(x)$. Then $dx/dx = 1$ and $dy/dx = f'(x)$. Using $x$ in place of $\theta$ in the above arclength formula, we obtain

$$\Big(\text{arclength of the curve } y = f(x) \text{ from } x = a \text{ to } x = b\Big) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

In most cases — even when dealing with simple basic functions — these arclength formulas lead to integrals that cannot be computed by the antiderivative method, and we must use numerical integration.

Example 1.5. Find the length of one arch of the sine curve.

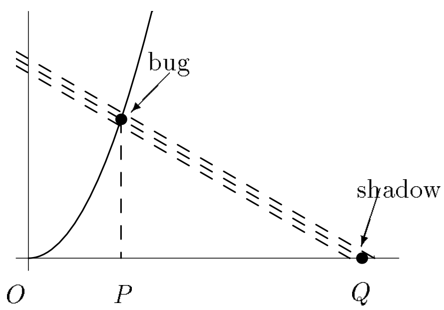
Here we want the arclength of $y = f(x) = \sin x$ from $x = 0$ to $x = \pi$. By the previous formula, since $dy/dx = \cos x$ we have

$$\text{arclength} = \int_0^{\pi} \sqrt{1 + \cos^2 x}\,dx.$$

This integral cannot be computed by the antiderivative method. In the next section we will get an approximate value by the trapezoid rule.

Sometimes instead of the arclength itself, one is interested in the *rate* at which arclength is increasing — i.e., the speed of the particle. For example, given the horizontal component of velocity, we might ask for $ds/dt$; or given $ds/dt$, we might want to determine $dx/dt$ or $dy/dt$. Here is a slightly more complicated example of a related rate problem.

Example 1.6. A bug is crawling up the $y = x^2$ parabola. Meanwhile, the sun is shining from the left from an angle of 30° above the horizontal, casting a shadow on the $x$-axis (see the figure). When the bug reaches the point $(1.5, 2.25)$, its shadow is moving at the rate of 0.3 units/min. Find the bug's horizontal and vertical components of velocity and its speed.




Let $u(t)$ denote the location of the bug's shadow on the $x$-axis at time $t$. We know that $du/dt = 0.3$, and we want to find $dx/dt$, $dy/dt$, and $ds/dt = \sqrt{(dx/dt)^2 + (dy/dt)^2}$. We have a simple relationship between $y$ and $x$, namely $y = x^2$, but we need to find a relationship between $u$ and $x$. To do that we observe the geometry of the shadow. Let $O$ be the origin, $P$ be the point on the $x$-axis directly under the bug, and $Q$ be the location of the bug's shadow on the $x$-axis. Then $u = \overline{OP} + \overline{PQ} = x + y\cot(30°) = x + y\sqrt{3} = x + x^2\sqrt{3}$. This gives us a relation between $u$ and $x$. Taking $d/dt$ of this relation, we obtain:

$$0.3 = \frac{du}{dt} = \frac{d}{dt}(x + x^2\sqrt{3}) = \frac{dx}{dt} + 2x\sqrt{3}\frac{dx}{dt} = (1 + 2 \cdot 1.5\sqrt{3})\frac{dx}{dt}.$$

Solving for $dx/dt$ gives us $dx/dt = 0.3/(1 + 3\sqrt{3}) = 0.0484$ units/min. Next, $dy/dt = (dy/dx)(dx/dt) = 2x(dx/dt) = 2 \cdot 1.5 \cdot 0.0484 = 0.1452$ units/min, and the speed is $\sqrt{0.0484^2 + 0.1452^2} = 0.1531$ units/min.

HOMEWORK 1.

**1.** Find the equation of an ellipse centered at the origin if

(a) it passes through (2,0) and also through (0,1/2);

(b) it passes through (10,0) and has foci at ($\pm$6,0);

(c) it has foci at ($\pm$7,0) and eccentricity 0.28;

(d) it passes through (1,0) and also through (0,3);

(e) it has foci at (0,$\pm$3) and passes through (4,0).

**2.** Find the foci and eccentricity of the ellipses

(a) $x^2/25 + y^2/16 = 1$; (b) $x^2/49 + y^2/625 = 1$; (c) $4x^2 + 9y^2 = 1$.

**3.** A particle travels at constant velocity in a straight line from the point (1,3) to the point (7,11) in 2 seconds.

(a) Find parametric equations $x(t)$, $y(t)$ for the motion of the particle, and find the distance the particle travels using the formula for the arclength of a parameterized curve.

(b) Find the equation of the line the particle travels along in the form $y = mx + b$, and then find the distance the particle travels using the formula for the arclength of an explicit function $y = f(x)$.

(c) Find the distance the particle travels by simply using the distance formula between its starting point and stopping point. Your answers to parts (a), (b) and (c) should be the same.

**4.** Let $n$ be greater than 1.

(a) Write a definite integral for the arclength of $y = x^n$ from $x = 0$ to $x = b$.

(b) There is one (and only one) case when this integral can be easily evaluated by the anti-derivative method, namely, when $n = 3/2$. In that case use a suitable substitution to evaluate the integral and find a formula in terms of $b$ for the arclength in part (a).

**5.** Write a definite integral for

(a) the total distance traveled by a ball that is thrown from a 38.8-meter high window at 14 m/sec at an angle of 45° above the horizontal;

(b) the distance one-quarter the way around an ellipse having semi-major axis $a$ and semi-minor axis $b$, i.e. the part of the circumference that is in the first quadrant (use parametric equations for the ellipse);

(c) the distance traveled by the pebble in Example 4.4 if the pebble is caught in a spoke half-way from the center to the rim.

**6.** A bug is crawling up the arch of the sine curve $y = \sin x$ at speed 0.5. When the bug reaches the point $(\pi/6, 0.5)$, find

(a) the rate at which its $x$-coordinate is increasing;

(b) the velocity of its shadow on the $x$-axis, if the sun is shining from the left from an angle of 21.8° above the horizontal;

(c) the same as part (b), but no longer assume that the angle from which the sun is shining remains constant. Suppose that the units of time are hours, and the sun is setting at the rate of 15° per hour.