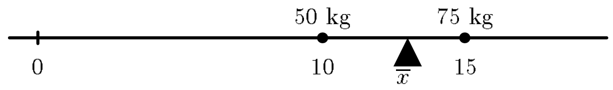
CHAPTER 1

Center of Mass

The seesaw principle. Suppose that a light-weight board 5 meters long is laid out between a point 10 meters from a reference point and a point 15 meters from the reference point. Suppose that someone weighing 50 kg is sitting at one end of the board (at 10 meters from the reference point), and someone weighing 75 kg is sitting at the other end. We want to know where to put the fulcrum so as to balance the board with the people on it. Here we will assume that the weight of the board is negligible, and that the people are point-masses concentrated at 10 and 15 on our axis:



The location of the balancing point $\overline{x}$ is obtained by taking the ratio of the total **moment** M to the total *mass* m. Here m is simply $50 + 75 = 125$ kg. The moment (in the x-direction, relative to the reference point 0) is the sum of (mass$\times x$-coodinate), i.e., $50 \cdot 10 + 75 \cdot 15 = 1625$ kg-m. (The units are kilogram-meters, i.e., mass times distance.) Then $\overline{x} = M/m = 1625/125 = 13$ meters. That is, the balancing point is 3/5 the way from the lighter person to the heavier person. (We would have gotten the same result if we had chosen our reference point differently, for example, if we had chosen one end of the board to be 0.)

If there are a number of point-masses located at various places on the x-axis, the procedure is the same. Suppose, for instance, that we have five masses: a mass m_1 located at position x_1, a mass m_2 located at x_2, and so on. Then

$$\overline{x} = \frac{M}{m} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5}{m_1 + m_2 + m_3 + m_4 + m_5}.$$

We shall call this the "seesaw principle:" *the center of mass is equal to total moment divided by total mass, where moment is computed by adding up each piece of mass with the position of the piece of mass multiplied in.* In reality, mass is usually not all concentrated in a few points, but rather is distributed continuously over some region. That is where integration (rather than taking a finite sum) comes in.

The center of mass is important for understanding the motion of large objects. This is because the vector sum of all external forces on the object is equal to the mass times the acceleration of the center of mass. For example, when a baton is twirled and thrown into the air (Example 11.1 in the Math 124 notes), its center travels along a simple parabolic falling body trajectory (but the path of the tip of the baton is more complicated).

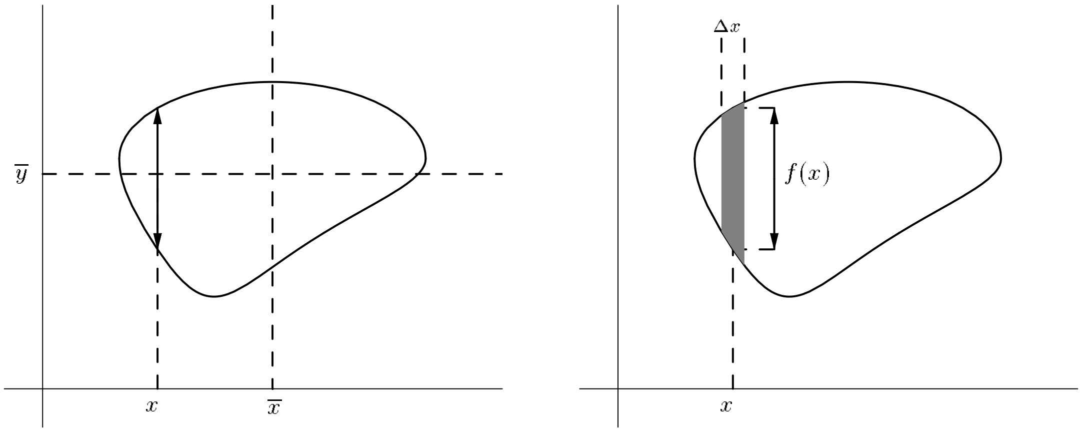
There are many types of center of mass problems, involving three-dimensional shapes with varying densities. However, we shall deal only with problems of a particularly straightforward type: the center of mass of a flat uniform shape (also called its " centroid").

So suppose we have some shape in the xy-plane, for example, the region contained between two curves (a "top curve" and a "bottom curve") between $x = a$ and $x = b$. Suppose that this shape is made up of a uniform substance having a certain constant density D measured in mass per unit area — say, gm/cm^2. (If the substance has a known density in the sense of mass per volume — say, gm/cm^3 — then D is equal to this volume density times the thickness of the plate.) The mass of the uniform plate is simply D times its area, i.e.,

$$m = D \int_a^b (\text{top function} \;-\; \text{bottom function})dx\,.$$

Let $f(x)$ denote the distance across the plate from top to bottom measured at position x on the x-axis, i.e., $f(x)$ is obtained by subtracting the bottom function from the top function. (In order to keep the notation simple, in this section we will not introduce separate symbols for the top function and the bottom function.) Then

$$m = D \int_a^b f(x)dx\,.$$



The uniform plate will balance at some point with coordinates $(\overline{x}, \overline{y})$. The practical meaning of $\overline{x}$ and $\overline{y}$ is as follows. If you take a razor-edge parallel to the y-axis and put it under the flat plate at a position x which is to the left of $\overline{x}$, then the plate will overbalance to the right; if you put the razor-edge under the plate at a position $x > \overline{x}$, then the plate will overbalance to the left. If you put the razor-edge exactly at the position $\overline{x}$, then the plate will balance. Similarly, if you take a razor-edge parallel to the x-axis and put it under the plate at a position y which is less than or greater than $\overline{y}$, then the plate will overbalance on one side or the other; but if you put it parallel to the x-axis exactly at $\overline{y}$, then the plate will balance. The intersection of the line parallel to the y-axis where the plate balances and the line parallel to the x-axis where it balances is the point $(\overline{x}, \overline{y})$. If you stuck a pencil under the plate at this point, the plate would spin around on the pencil point without falling to any side.

We first find $\overline{x}$, which is the ratio of total moment **in the x-direction** to the total mass. We already know how to find the total mass: $m = D\int_a^b f(x)dx$. To find the moment, we use the same type of procedure as when we derived the integral formula for the area under a curve or the volume of a solid of revolution. Namely, we divide the interval from a

to b into a large number of small subintervals of length $\Delta x = (b-a)/n$. The left endpoints of these subintervals are at a, $a+\Delta x$, $a+2\Delta x, \ldots$ $a+(n-1)\Delta x$. If we assume that the strip above each of the subintervals is a rectangle (this is approximately the case if Δx is small), then the total mass of all of the strips is obtained by summing

$$D\,f(a)\Delta x + D\,f(a+\Delta x)\Delta x + D\,f(a+2\Delta x)\Delta x + \cdots + D\,f(a+(n-1)\Delta x)\Delta x.$$

However, we want the *moment*, which is equal to mass times location in the x-direction. We make the simplifying assumption that the entire strip above a subinterval has the same x-location as the left endpoint of the interval (this is approximately the case if Δx is small); so to get the moment we multiply each term in the above sum by the x-coordinate of the left endpoint. In other words, the moment is approximately

$$\begin{aligned} D\,a\,f(a)\Delta x \quad &+ \quad D(a+\Delta x)f(a+\Delta x)\Delta x + D(a+2\Delta x)f(a+2\Delta x)\Delta x + \\ \cdots \quad &+ \quad D(a+(n-1)\Delta x)f(a+(n-1)\Delta x)\Delta x, \end{aligned}$$

which is the Riemann sum for the function $D\,x\,f(x)$. Thus, in the limit we get the definite integral from a to b of $D\,x\,f(x)$:

$$\text{moment} = D\int_a^b x\,f(x)dx.$$

One way to think of this formula is to look at the contribution to the moment that comes from the part of the region that lies between x and $x+\Delta x$, where we suppose that Δx is very small. This is a thin strip Δx across whose height at the left is $f(x)$. Since Δx is small, the strip is very close to being a rectangle, and so its area is very close to $f(x)\Delta x$, and its mass is very close to $D\,f(x)\Delta x$. Its contribution to moment is then very close to x times $D\,f(x)\Delta x$. Now to find the moment of the entire flat plate, we have to sum up the contributions as x goes from a to b. In the limit as Δx goes to zero this sum becomes an integral, and we write dx instead of Δx. That is, we get $\int_a^b \big(Dx\,f(x)\big)dx$, as before.

Thus, *the total moment of a uniform flat plate is equal to the same type of formula as the total mass, except with x multiplied by $f(x)$ inside the integral.* Finally, to find $\overline{x}$ we divide $M = D\int_a^b x\,f(x)\,dx$ by $m = D\int_a^b f(x)\,dx$. The D's cancel (that is, the location of the balancing point does not depend on whether the flat plate is made of a heavy or light substance), and we conclude that

$$\overline{x} = \frac{\int_a^b x\,f(x)\,dx}{\int_a^b f(x)\,dx}.$$

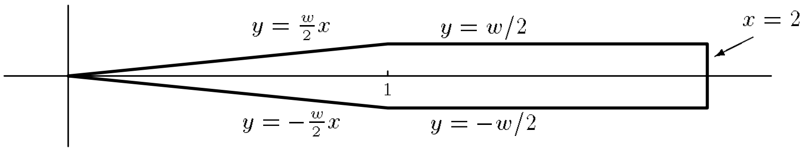
It remains to find a formula for $\overline{y}$. We shift perspectives, and look at the same shape from left to right as we move up the y-axis. Let $g(y)$ denote the distance across the shape from left to right at the position y. To find $g(y)$, we express the right boundary of the shape in the form $x = g_1(y)$ and the left boundary in the form $x = g_2(y)$; then $g(y) = g_1(y) - g_2(y)$. We suppose that in the y-direction the shape is contained between $y = c$ and $y = d$, which will be our limits of integration. (This is what we did when finding the area between two curves in cases when we integrated with respect to y rather than x.) The derivation of a formula for $\overline{y}$ is exactly the same as for $\overline{x}$. The result is:

$$\overline{y} = \frac{\int_c^d y g(y)\,dy}{\int_c^d g(y)\,dy}.$$

Actually, the denominator in this formula has the same value as in the formula for $\overline{x}$ — in both cases it is the area of the plate. The numerators, however, are different.

EXAMPLE 1.1. A uniform flat shape has the form of a rectangular ruler of width w and length 2 feet, with the last foot tapered off to a point, i.e., it is in the shape of a rectangle of length 1 ft surmounted by an isosceles triangle of altitude 1 ft. Find the balancing point.

Our first task is to put this shape in the xy-plane in a convenient way. Let us take the x-axis as the axis of symmetry, and put the pointed end at the origin. (That way the slanted line has the simplest possible equation.) Thus, the shape is bounded by the two lines joining the origin to the points $(1, \pm w/2)$, the two horizontal lines joining $(1, \pm w/2)$ to $(2, \pm w/2)$, and the vertical line joining $(2, -w/2)$ to $(2, +w/2)$. See the drawing below, where the equations of these 5 lines are indicated:



We first observe that, because of the symmetry of the shape, the balancing point is obviously on the x-axis, i.e., $\overline{y} = 0$. Thus, we need only find $\overline{x}$. In general, when finding the center of mass of a symmetrical shape you can reduce the amount of computation required by taking into account that the balancing point must be on the axis of symmetry.

To compute $\overline{x}$, we find the following formulas for the distance across from top to bottom (i.e., top function minus bottom function):

$$f(x) = \begin{cases} wx, & \text{if } 0 \le x \le 1; \\ w, & \text{if } 1 \le x \le 2. \end{cases}$$

Thus, the numerator (the moment integral) in the formula for $\overline{x}$ is

$$\int_0^1 wx^2 dx + \int_1^2 wx\, dx = w/3 + 3w/2 = 11w/6\,.$$

The denominator can be found either by integration or simply by using the formulas for the area of a rectangle and triangle: this gives us $3w/2$. Thus, $\overline{x} = (11w/6)/(3w/2) = 11/9 = 1.222$. In other words, the balancing point is in the rectangular part of the shape a distance $2/9$ the way from the base of the triangle to the opposite side of the rectangle.

Two observations should be made about this answer. (1) The width w canceled out; this means that the balancing point does not depend on whether the shape is wide or narrow (just as it does not depend on whether it is made of a heavy or light substance). (2) A rough check on the correctness of the answer is that we would expect an answer between 1 and 1.5, since a razor-edge placed at the base of the triangle would cause the shape to overbalance to the right, whereas a razor-edge place in the middle of the rectangular part would case the shape to overbalance to the left.

EXAMPLE 1.2. Find the center of mass of a uniform flat plate in the shape of a quarter circle of radius 1 (say, the quarter of the unit circle that is in the first quadrant of the xy-plane) on top of a square of side 1 (i.e., the unit square in the corner of the fourth quadrant).

Here we do not have symmetry to help us, so we must compute both $\overline{x}$ and $\overline{y}$. The denominator in both cases is the area of our shape, which is

$$\left(\frac{1}{4}\pi \cdot 1^2\right) + \left(1^2\right) = (\pi/4) + 1\,.$$

The distance from top to bottom is $f(x) = 1 + \sqrt{1-x^2}$. Thus, in computing $\overline{x}$, we find that the moment integral is

$$\int_0^1 x\left(1+\sqrt{1-x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 x\sqrt{1-x^2}dx\,.$$

The first of these two integrals is 0.5. The second integral can be evaluated by the method of substitution, since the derivative of what's under the radical is essentially $x\,dx$. That is, we substitute $u = 1-x^2$; then $du = -2xdx$, and so the integral becomes $-0.5\int\sqrt{u}\,du$. As for the limits of integration, $x = 0$ implies $u = 1-0^2 = 1$, and $x = 1$ implies $u = 1-1^2 = 0$. Thus, the moment integral is

$$0.5 - 0.5\int_1^0 \sqrt{u}\,du = 0.5 - 0.5(-2/3) = 5/6, \text{ so that } \overline{x} = \frac{5/6}{(\pi/4)+1} = 0.4667.$$

To compute $\overline{y}$, we see that the distance across our shape from the y-perspective (i.e., from left to right) is

$$g(y) = \begin{cases} 1, & \text{if } -1 \le y \le 0; \\ \sqrt{1-y^2}, & \text{if } 0 \le y \le 1. \end{cases}$$

Thus, the numerator of the $\overline{y}$ formula is $\int_{-1}^0 y\,dy + \int_0^1 y\sqrt{1-y^2}dy$. The first of these integrals is $-1/2$, while the second integral happens to be exactly the same integral that we already computed in the last paragraph, except with a different letter as the variable of integration. That is, $\int_0^1 y\sqrt{1-y^2}dy$ has the same value as $\int_0^1 x\sqrt{1-x^2}dx$, namely, 1/3. We conclude that

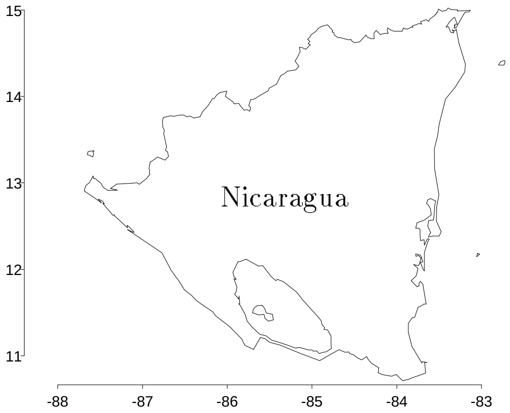
$$\overline{y} = \frac{-\frac{1}{2}+\frac{1}{3}}{\frac{\pi}{4}+1} = -0.09335.$$

The reader should sketch a picture of the shape, and check that $(0.4667, -0.09335)$ is a reasonable location for the balancing point.

Example 1.3. Find the centroid of Nicaragua (see the map on the next page). As an approximation to the actual shape of the country, suppose that it is a quadrilateral with vertices at the points

11° northern latitude & 86° western longitude,
11° northern latitude & 83° western longitude,
15° northern latitude & 83° western longitude,
13° northern latitude & 88° western longitude.

Also suppose that Nicaragua is a plane shape, i.e., disregard the mountains, curvature of the earth, etc.



Here our first problem is to put the shape in the xy-plane in a convenient way. We could choose the conventional longitude/latitude coordinates as our xy-coordinates. But that would be a little cumbersome, because the origin of that coordinate system is at the Equator along the Greenwich Meridian, which is far from Nicaragua. Also note that "west longitude" means the negative direction from the Greenwich Meridian, so we would be using fairly large negative numbers for the x-coordinates of the corners (namely, -86, -83, -88). A better idea is to choose the origin at the intersection of the two slanted sides, which is ($88°$ west, $13°$ north). Then the other three corners have the following coordinates relative to this origin: $(2,-2)$, $(5,-2)$, $(5,2)$. The two slanted sides have equations $y = 0.4x$ (or $x = 2.5y$) and $y = -x$ (or $x = -y$). The horizontal side is $y = -2$, and the vertical side is $x = 5$. Thus, the distance $f(x)$ from top to bottom and the distance $g(y)$ from left to right are:

$$f(x) = \begin{cases} 1.4x, & \text{if } 0 \le x \le 2; \\ 0.4x + 2, & \text{if } 2 \le x \le 5; \end{cases} \qquad g(y) = \begin{cases} 5 + y, & \text{if } -2 \le y \le 0; \\ 5 - 2.5y, & \text{if } 0 \le y \le 2. \end{cases}$$

The rest of this example is routine, and the details will be left to the reader. The answer is $(3.10°, -0.31°)$. This must be added to the longitude $-88°$ and the latitude $13°$ of the origin. Thus, the centroid of Nicaragua is at $12.7°$ northern latitude & $84.9°$ western longitude, i.e., about 70 miles east of Matagalpa, on the Rio Grande.

HOMEWORK 1.

1. Find the coordinates of the center of mass of a uniform flat plate that is bounded by: (a) the line $x + 2y = 6$, the x-axis, and the y-axis; (b) the parabola $y = x^2$, the x-axis, and the line $x = 2$; (c) the right half of the circle $x^2 + y^2 = r^2$ and the y-axis; (d) the top half of the ellipse $x^2 + 4y^2 = 4$ and the two lines joining the point $(0, -4)$ to the points where the ellipse meets the x-axis.[1]

2. Find the centroid of Nevada. Suppose that Nevada is a trapezoid with corners at the four points

$42°$ northern latitude & $120°$ western longitude,
$42°$ northern latitude & $114°$ western longitude,
$39°$ northern latitude & $120°$ western longitude,
$35°$ northern latitude & $114°$ western longitude.

3. Use the trapezoid rule with $n = 6$ subdivisions to find the center of mass of a uniform flat plate in the shape bounded by the curve $y = \tan x$, the curve $y = -\tan x$, and the line $x = \pi/4$.

[1] To compute the area of the plate in part (d), use the following general principle: If a shape such as the unit circle is expanded horizontally by a factor of a and vertically by a factor of b, then its area is multiplied by a factor of ab.