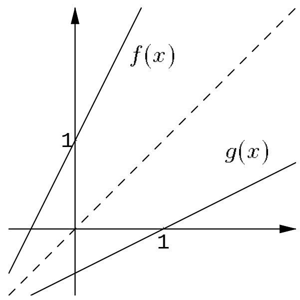
CHAPTER 1

Inverse Trig Functions

Given a function $y = f(x)$, its **inverse function** is the function that "goes backwards." We'll say what we mean by this a little later. To find the inverse function, we (1) solve the equation $y = f(x)$ for x in terms of y, obtaining $x = g(y)$, and then (2) interchange the letters x and y, i.e., write $y = g(x)$. Then we say that $g(x)$ is the inverse function of $f(x)$. We will illustrate inverse functions by a series of examples.

EXAMPLE 1.1. $y = f(x) = 2x + 1$.

We first solve for x in terms of y: $x = 0.5y - 0.5$. Then we interchange x and y: $y = g(x) = 0.5x - 0.5$. At the right we show the graphs of both $f(x)$ and $g(x)$. Notice that if we imagine a mirror along the line $y = x$, then the two graphs are mirror images of one another. That is always the geometrical relationship between the graph of a function and the graph of its inverse function.



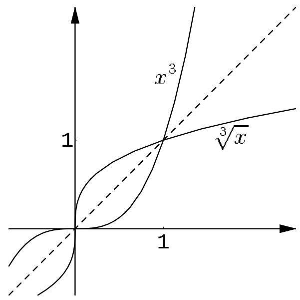
When we say that the inverse function $g(x)$ " goes backwards," we mean that if you compute $f(x)$ and then apply $g(x)$ to that value, i.e., if you compute $g(f(x))$, you will get x back again. The same thing happens if you evaluate the two functions in the other order, applying f to $g(x)$, i.e., $f(g(x))$ is also equal to x:

If $f(x)$ and $g(x)$ are inverse functions of one another, then $g(f(x)) = x$ and $f(g(x)) = x$.

For instance, in Example 1.1, we check that $g(f(x)) = 0.5(2x+1) - 0.5 = x$ and $f(g(x)) = 2(0.5x - 0.5) + 1 = x$.

EXAMPLE 1.2. $y = f(x) = x^3$.

Solving for x, we obtain $x = \sqrt[3]{y} = y^{1/3}$, and so the inverse function is $y = g(x) = \sqrt[3]{x} = x^{1/3}$. Again the graph of $f(x)$ and the graph of $g(x)$ are mirror images of one another with respect to the line $y = x$ (see right). And again we find that applying g to $f(x)$ or applying f to $g(x)$ gets us back to x: $g(f(x)) = \sqrt[3]{x^3} = x$, $f(g(x)) = (\sqrt[3]{x})^3 = x$. We say that taking the cube root is the inverse operation to cubing a number.



EXAMPLE 1.3. $y = f(x) = x^2$.

Here we have a situation that did not exist in the first two examples: solving for x gives two different functions, namely $x = \sqrt{y}$ and $x = -\sqrt{y}$. We do not want this ambiguity, and so we make an agreement to take only the nonnegative square root: $x = \sqrt{y}$. This is equivalent to erasing the left half of the parabola $y = x^2$. Then the inverse function is

$y = \sqrt{x}$. After crossing out the left half of the parabola, the resulting function $y = f(x) = x^2$ for $x \geq 0$ and the inverse function $y = g(x) = \sqrt{x}$ have the same relation to one another as in Examples 1.1 and 1.2. Namely, (1) their graphs are mirror images of one another relative to the line $y = x$; and (2) each function "reverses" the other one. That is, if we take a nonnegative number, square it, and then take the nonnegative square root, we get back to where we started; and if we take a nonnegative number, take its square root, and then square it, we also get back to where we started.

Example 1.3 illustrates two things to think about when dealing with the inverse function of some $y = f(x)$: (1) the possibility that you might have to select only part of $y = f(x)$ (such as the right half of the parabola in Example 1.3) in order to obtain a well-defined inverse function $y = g(x)$; (2) the possibility that the domain does not consist of all x, i.e., only x in a certain range can be substituted in $g(x)$.

1. The Inverse Tangent Function

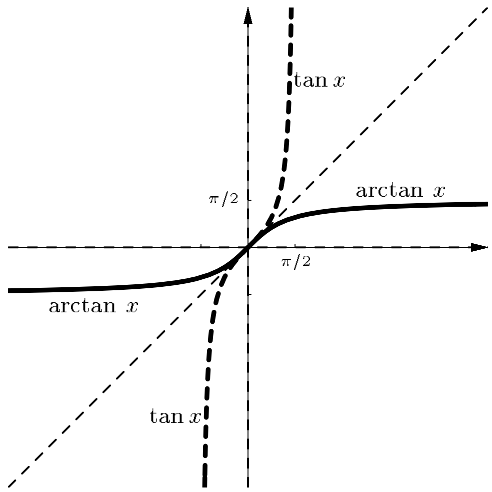
The inverse tangent function, which is called the **arctangent**, goes in the reverse direction from the tangent function. That is, the arctangent of x answers the question: What angle (in radians) has x as its tan? As in Example 1.3, there's some ambiguity here. In fact, in the case of the arctangent, there are infinitely many possible angles whose tangent is x: if θ has the property that $\tan\theta = x$, then so does $\theta + \pi$, $\theta + 2\pi$, $\theta + 3\pi$, $\theta - \pi$, $\theta - 2\pi$, etc., because the tangent function has period π. In other words, the part of the graph of $y = \tan x$ contained between the two asymptotes at $x = -\pi/2$ and $x = \pi/2$ is simply repeated between $x = \pi/2$ and $x = 3\pi/2$, between $x = 3\pi/2$ and $x = 5\pi/2$, and so on — infinitely often in both directions. (Recall that while sine and cosine each has period 2π, the tangent function has period π.) So in constructing the inverse tangent function the first thing we have to do is cross out all but one of the branches of the tangent function. The usual convention is to remove all but the branch near the y-axis, i.e., to look only at angles between $-\pi/2$ and $\pi/2$. After doing that, we obtain a well-defined arctangent function, which will be denoted $y = \arctan x$. That is,

$$\arctan x = \begin{pmatrix}\text{the angle in radians in the interval from} \\ -\pi/2 \text{ to } \pi/2 \text{ whose tangent is } x\end{pmatrix}$$

Before going farther, we make three remarks about notation.

1. The function defined above is called the " principal value" Arctangent (with a capital "A"). In some situations we might want to consider *any* angle whose tangent is x. In that case we'll write $\arctan x$ with a small "a."
2. In many textbooks and on most calculators, the function $\arctan x$ is denoted $\tan^{-1} x$. We shall not use this notation, because it can be confusing. Namely, since $\tan^2 x$ means $(\tan x)^2$, you might erroneously think that $\tan^{-1} x$ is the same as $(\tan x)^{-1}$, which is $\cot x$. This is false. The functions $\arctan x$ and $\cot x$ are completely different: the first *goes backwards* from the tangent function, whereas the second is the *reciprocal* of the tangent function.
3. We are using $\arctan x$ to mean the angle *in radians* whose tangent is x. The angle *in degrees* would be $(180/\pi)\arctan x$. Be careful when using calculators: most calculators will give you degrees unless you specifically change the mode to radians.

Just as the function $y = \tan x$ has vertical asymptotes at $\pm\pi/2$, the function $y =$ arctan x has horizontal asymptotes at $\pm\pi/2$. As x goes from $-\infty$ to $+\infty$, its Arctan goes from $-\pi/2$ to $\pi/2$, never taking values below $-\pi/2$ or above $\pi/2$. As expected, the graph of $y = \tan x$ (more precisely, the branch of the graph that we've taken) and the graph of $y =$ arctan x are mirror images of one another with respect to the line $y = x$.



2. The Derivative of $y =$ Arctan x

To find dy/dx, where $y =$ arctan x, we replace the relation $y =$ arctan x with the equivalent relation expressing x as a function of y: $x = \tan y$. In other words, the definition of Arctan tells us that when $y =$ arctan x, then y is an angle whose tan is x. We then implicitly differentiate the relation $x = \tan y$, i.e., we take d/dx of both sides. On the left we get 1. On the right we must use the chain rule, which says that the derivative of tan(some function of x) is sec^2(this function) times the derivative of this function. Thus, we obtain:

$$1 = (\sec^2 y)\frac{dy}{dx}, \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

The only trouble with this is that, as generally happens with implicit differentiation, our formula for the derivative involves y. We would like to express $\frac{dy}{dx} = \frac{d}{dx}$ arctan x in terms of x. Now using the trig identity $\sec^2 y = 1 + \tan^2 y$ (this trig identity comes from the identity $1 = \cos^2 + \sin^2$ if you divide through by $\cos^2$), and also the fact that $x = \tan y$, we can replace $\sec^2 y$ in the denominator by $1 + x^2$. Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\,\text{arctan } x = \frac{1}{1+x^2}.$$

This is a basic derivative formula.

If we have a function $u(x)$ inside the Arctan, then we combine this formula with the chain rule, i.e., we have

$$\frac{d}{dx}\,\text{Arctan}(u) = \frac{1}{1+u^2}\,\frac{du}{dx}.$$

Whenever we have a derivative formula, there's a companion antiderivative formula:

$$\int \frac{dx}{1+x^2} = \text{arctan } x + C.$$

We shall later encounter some practical applications where this integration formula comes in handy.

Whenever you have a new function, such as arctan x, and a new derivative formula, you can combine it with other functions and solve the same type of problems as before — derivatives of products and of functions of functions, tangent line approximation, max/min, related rates, etc.

EXAMPLE 1.4. Find the derivative of $(x+1)\arctan(\sqrt{x})$.

Using the product rule and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\left((x+1)\arctan(\sqrt{x})\right) &= (x+1)\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}+1\cdot\arctan(\sqrt{x})\\ &= \frac{1}{2\sqrt{x}}+\arctan(\sqrt{x}).\end{aligned}$$

EXAMPLE 1.5. Using the tangent line approximation and the fact that $\tan 45° = 1$, find the angle in degrees whose tangent is 0.99.

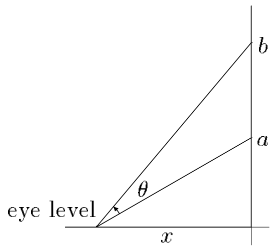
We apply the tangent line approximation formula

$$f(x+\Delta x)\approx f(x)+\Delta x\, f'(x)$$

to the situation: $f(x)=\frac{180}{\pi}\arctan x$, $x=1$, $\Delta x=-0.01$. We know that $f(1)=45$, and we want to know: $f(1+(-0.01)) = ?$ Here $f'(x)=\frac{180}{\pi}\cdot\frac{1}{1+x^2}=\frac{180}{\pi}\cdot\frac{1}{1+1}=90/\pi$. Thus,

$$f(x+\Delta x)\approx 45+(-0.01)(90/\pi)=44.71°.$$

EXAMPLE 1.6. Suppose you are viewing a painting whose bottom is a distance a above eye level and whose top is a distance b above eye level. You can see detail in the painting best when it occupies the largest possible angle θ in your field of vision (see the diagram at the right below). How far should you stand back from the wall in order to get the best view?



Using this diagram, we cannot express θ directly as an inverse trig function, since it is not an angle in a right triangle. However, θ is the *difference* between the lower-left angle of the larger right triangle and the lower-left angle of the smaller right triangle. In both right triangles we have letters standing for the two legs. Thus, we can use the arctangent function. That is, θ is equal to the angle $\arctan(b/x)$ of the larger right triangle minus the angle $\arctan(a/x)$ of the smaller right triangle: $\theta=\arctan(b/x)-\arctan(a/x)$. We want to maximize θ as a function of x. Hence, we set $d\theta/dx$ equal to 0:

$$0=\frac{d\theta}{dx}=\frac{1}{1+(b/x)^2}\left(-\frac{b}{x^2}\right)-\frac{1}{1+(a/x)^2}\left(-\frac{a}{x^2}\right),$$

where we have used the formula for d/dx of $\arctan u$. We now simplify, multiplying the numerators and denominators together in each term and bringing the first fraction (which has a negative sign) over to the left:

$$\frac{b}{b^2+x^2}=\frac{a}{a^2+x^2}.$$

To solve for x, we cross-multiply, then gather together the terms containing an x, and finally factor out common factors on both sides:

$$b(a^2+x^2)=a(b^2+x^2),\qquad bx^2-ax^2=ab^2-ba^2,\qquad (b-a)x^2=ab(b-a).$$

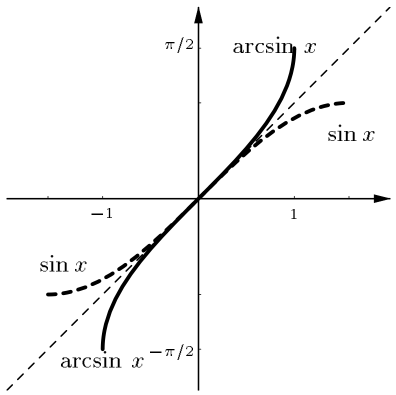
Dividing by $b-a$ and taking the positive square root gives $x=\sqrt{ab}$. From the practical setup it should be clear that this critical point really is a maximum. If you want to check by the second derivative method, just take the second derivative of θ with respect to x at the point $x=\sqrt{ab}$. You will get a negative value.

3. The Inverse Sine Function

The inverse sine function answers the question: what angle (in radians) has sine equal to x? There are two things to watch out for here:

1. As in the case of the tangent, there will be infinitely many angles whose sine is the same. So we must remove all of the sine curve except for one section before we construct the inverse function. The usual convention, as in the case of Arctan, is to take the part of the curve $y = \sin x$ between $x = -\pi/2$ and $x = \pi/2$, where y goes exactly once through all values from -1 to 1. That is, we define the principal value Arcsin of x as follows:
$$\arcsin\ x = \begin{pmatrix}\text{the angle in radians between}\\ -\pi/2 \text{ and } \pi/2 \text{ whose sine is } x.\end{pmatrix}$$
2. Unlike Arctan, the function Arcsin cannot be applied to all x, but rather only to x between -1 and 1. This is because, unlike the tangent function, which takes all values from $-\infty$ to ∞, the sine function only takes values between -1 and 1, and for x outside of that interval there is no angle whose sine is x. Thus, the *domain* of arcsin x is the interval $[-1, 1]$.

On the right is a graph of the function $y = \sin x$ (between $x = -\pi/2$ and $x = \pi/2$) and the function $y =$ arcsin x (which has domain between $x = -1$ and $x = 1$). As usual, the two graphs are symmetric with respect to the line $y = x$.



4. The Derivative of $y =$ Arcsin x

The procedure is just like in the case of Arctan. We first replace the relation $y =$Arcsin x by the relation $x = \sin y$, and then we take d/dx of both sides, obtaining $1 = (\cos y)\frac{dy}{dx}$, i.e., $dy/dx = 1/\cos y$. We can express the derivative in terms of x, which equals $\sin y$, by means of the trig identity $\cos = \sqrt{1 - \sin^2}$. Thus, $1/\cos y = 1/\sqrt{1 - \sin^2 y} = 1/\sqrt{1 - x^2}$. Hence we have
$$\frac{d}{dx}\ \arcsin\ x = 1\sqrt{1 - x^2}.$$
(Note: When we take the square root in writing cos in terms of sin, in general we have to put a $\pm$ in front of the radical; however, in this case we know that we want the positive square root, because in our definition of the Arcsin function y has to be between $-\pi/2$ and $\pi/2$, and for these values $\cos y$ is positive.)

The companion antiderivative formula is
$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin\ x + C.$$

EXAMPLE 1.7. The tip of a 20-meter span of a drawbridge has been raised 8 meters above the horizontal, and it is being raised at 0.2 m/sec. How fast (in deg/sec) is the angle of the drawbridge increasing?

In this related rates problem, let θ be the angle in degrees, and let x be the height the tip has been raised. Then $\theta = \frac{180}{\pi}\arcsin(x/20)$, and so

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{180}{\pi}\,\frac{1}{\sqrt{1-(x/20)^2}}\,\frac{1}{20}\cdot\frac{dx}{dt} = \frac{180}{\pi}\,\frac{0.05}{\sqrt{1-(8/20)^2}}\cdot 0.2 = 0.625\,\mathrm{deg/sec}.$$

EXAMPLE 1.8. Evaluate: $\int dx/\sqrt{r^2-x^2}$, where r is a constant.

Here we'll make a u-substitution to transform the integral into a form where we can use the above antiderivative formula, which we now restate using the variable u: $\int du/\sqrt{1-u^2} = \arcsin\ u + C$. The difficulty in our example is that we have r^2 where we'd like to have 1. We can get a 1 there by factoring out r^2 from r^2-x^2, i.e., by writing $r^2-x^2 = r^2\left(1-(x/r)^2\right)$, and then pulling the r^2 out of the radical. Thus,

$$\int \frac{dx}{\sqrt{r^2-x^2}} = \int \frac{1}{r}\cdot\frac{dx}{\sqrt{1-(x/r)^2}}.$$

This suggests the u-substitution $u = x/r$. Then $du = (1/r)dx$, and so in our integral we can replace $(1/r)dx$ by du and the denominator by $\sqrt{1-u^2}$. So we obtain:

$$\int \frac{du}{\sqrt{1-u^2}} = \arcsin\ u + C = \arcsin(x/r) + C,$$

which is our final answer. As always in an antiderivative problem, we can check that this is right by taking the derivative of our answer and seeing that we do in fact get $1/\sqrt{r^2-x^2}$.

5. The Inverse Cosine Function

So far we have dealt with the inverse functions of two of the six trig functions (sine, cosine, tangent, secant, cosecant, cotangent). The other four inverse trig functions are used less frequently, and in any case they can be expressed in a simple way in terms of Arcsin or Arctan. We illustrate this by discussing the function Arccosine.

A basic relationship between the cosine and the sine is that the cosine of an angle is the sine of the complementary angle. This means that if we want to know an angle whose cosine is x, we can take the complementary angle of the angle whose sine is x, i.e., we can take $(\pi/2) - \arcsin\ x$. We shall therefore define the principal value arccosine, denoted $\arccos x$, as follows:

$$\arccos(x) = \frac{\pi}{2} - \arcsin(x) \qquad \text{for } -1 \le x \le 1.$$

Since Arcsin runs from $-\pi/2$ to $\pi/2$, it follows that $\arccos(x)$ runs from π to 0 as x runs from -1 to 1. The derivative of $\arccos(x)$ is easy to compute:

$$\frac{d}{dx}\arccos(x) = 0 - \frac{d}{dx}\mathrm{Arcsin}(x) = -\frac{1}{\sqrt{1-x^2}}.$$

HOMEWORK 1.

1. Find d/dx of:
(a) $x^2 \arctan(x/2)$,
(b) $(x+1)/\arcsin(x^2)$,
(c) $\arcsin(\sqrt{1-x})$,
(d) $x \arcsin(x) + \sqrt{1-x^2}$.

2. Find the anti-derivative of
(a) $(100-16x^2)^{-1/2}$,
(b) $(1+2x^2)^{-1}$.

3. Using the tangent line approximation and the fact that $\sin(30°) = 0.5$, find the angle in degrees whose sine is 0.52.

4. Find the volume of the solid formed when the shape bounded by $y = \sqrt{x^{-2}-1}$, the line $y = 1$, and the x- and y-axes is rotated around the y-axis.

5. Find the center of mass of the uniform flat plate bounded by the curve $1/\sqrt{1-x^2}$, the curve $-1/\sqrt{1-x^2}$, the y-axis, and the line $x = 0.5$.

6. Graph and find max/min points of
(a) $y = (\arctan x) - 0.5x$,
(b) $y = \big(\arcsin(x/2)\big) - x$.

7. Sketch and find the area between
(a) $y = 2/(1+x^2)$ and $y = x^2$,
(b) $y = 1/\sqrt{1-x^2}$ and $y = 2$.

8. A rocket launched vertically from a point 2 km from an observer is ascending at 0.20 km/sec at the instant when it is 1 km above the ground. Find the rate in degrees/sec at which its angle of inclination (as seen by the observer) is increasing at that moment.

9. A triangle has two fixed sides of length a and b with included angle θ that is free to widen. Let x be the side opposite θ. (a) Use the law of cosines to express θ as a function of x. (b) At the instant when $x = c$, what is the ratio of the rate at which the angle θ is widening (in rad/sec) to the rate at which the side x is increasing?