Review Problem Sets: Final Exam

Review Problem Set #1: Final Exam.
(100 points in all; time = 3 hours, checking your work carefully)

1. (20 points) Let $f(x) = \ln x \, - \, 2.5\,\mathrm{Arctan}(0.5x)$, $x > 0$.
   (a) Find all critical points (give both x- and y-coordinates), and use the 2nd derivative test to determine whether they are maximum or minimum points.
   (b) When x approaches 0, what simple function does $f(x)$ become very close to?
   (c) When x approaches ∞, what simple function does $f(x)$ become very close to?
   (d) Use parts (a)–(c) to give a rough sketch of the graph of $f(x)$.

2. (15 points) A container is obtained by rotating around the y-axis a function $x = g(y)$ whose values at regular intervals are given by the following table (the units are cm):

y	0	2	4	6	8	10	12	14
$g(y)$	2.34	2.63	2.90	3.16	3.40	3.63	3.85	4.06

At a certain instant the water level in the container is 12 cm, and it is increasing at the rate of 0.05 cm/sec. At this instant find:
   (a) the volume of water in the container (use the trapezoid rule);
   (b) the rate in cm^3/sec at which water is being poured into the container.

3. (25 points) A wire extending infinitely far in **one direction** (say, along the positive x-axis from the origin) has c coulombs/meter of charge per unit length. A particle with q coulombs of charge is at a distance r from the end of the wire in a direction perpendicular to the wire (i.e., the particle is located at $-r$ on the y-axis). Use the physics principle that the force of repulsion between two point charges is equal to k(product of charges)/(dist)^2, where k is a universal constant.
   (a) Find the magnitude of the force of repulsion caused by the small segment of wire between x and $x+\Delta x$. For simplicity of notation, set $K = kqc$. Then find (i) the component of this force in the direction perpendicular to the wire, and (ii) the component of this force in the direction parallel to the wire. Use parts (i) and (ii) to write an improper integral for the total force perpendicular to the wire and another improper integral for the total force parallel to the wire.
   (b) Find both components of the total force from the wire on the particle.
   (c) Find the direction of the force on the particle.

4. (25 points) You drop a rock (with zero initial velocity) from a bridge. Take $g = 9.8$ m/sec^2. Because of air resistance (which we assume to be proportional to the velocity $v(t)$), the rock would eventually approach a downward terminal velocity of 140 m/sec if it fell indefinitely through the air. However, it does not fall indefinitely: after exactly 5 seconds it hits the water.

(a) Write out the differential equation satisfied by $v(t)$ while the rock falls through the air.

(b) Solve the differential equation in part (a).

(c) How fast is the rock falling when it hits the water?

(d) After the rock hits the water, the water resistance is great enough to slow it down. If it fell indefinitely through the water, its downward velocity would decrease to the terminal velocity of 4.9 m/sec. Find a formula for the rock's velocity while it falls through the water.

(e) How many seconds after the rock hits the water does it slow to a downward velocity of 6 m/sec?

5. (15 points) At time $t = 0$ a container holds 10 liters of water with 1 gram of impurities. Water containing 0.28 gm/liter of impurities enters the container at a rate $r(t)$ that is 2 liters/min when $t = 0$ but decreases steadily by 0.2 liters/min each minute. Water leaves the container at the constant rate of 2 liters/min. Assume that the impurities are at all times uniformly mixed in the water in the container.

(a) Write a formula for the rate of the incoming water at time t.

(b) Find a formula for the volume of water in the container at time t. When will the container be empty?

(c) Set up a differential equation for the amount $y(t)$ of impurity in the water at time t.

Review Problem Set #2: Final Exam.

(100 points; time = 3 hours, checking your work carefully)

1. (20 points) Let $f(x) = x\,e^{-\left(1.5x^2\right)}$.

(a) Find the x- and y-coordinates of all critical points, and use the second derivative test to determine if they are minimum or maximum points.

(b) Find the x- and y-coordinates of all points of inflection.

(c) What happens to $f(x)$ when $x \longrightarrow +\infty$ and when $x \longrightarrow -\infty$?

(d) Use parts (a)–(c) to sketch the curve. Your sketch should show the concavity (upward or downward) correctly, and also the behavior as $x \longrightarrow \pm\infty$.

2. (15 points) As a bug crawls up the y-axis, the sun shines from the left from an angle θ above the horizontal, casting the bug's shadow on the x-axis. At a certain instant you measure the following information: the bug's position a on the y-axis, its velocity $\dot{a}$ (in units/min), the position b of its shadow on the x-axis, and the shadow's velocity $\dot{b}$ (in units/min). Express the following in terms of these measured values: (a) the angle θ; (b) the rate **in DEGREES per minute** at which the sun is rising.

3. (15 points) The following table shows the horizontal and vertical velocities (in m/sec) of a particle at intervals of 2 sec from $t = 0$ to $t = 8$. Using the trapezoid rule, find the distance the particle traveled along its trajectory during the 8-second period.

t	0	2	4	6	8
$\dot{x}$	10.0	9.8	9.4	9.0	8.4
$\dot{y}$	0.0	−8.0	−15.2	−22.0	−28.2

4. (15 points) A container contains a volume V of liquid. At time $t = 0$, the liquid contains y_0 grams of impurity. Fresh water is pumped into the container at rate r, and the well-mixed liquid is drained from the container at the same rate.

(a) Find a differential equation for the amount $y(t)$ of impurity at time t.

(b) Find a formula for the length of time it takes for the amount of impurity to be reduced to 1% of its original amount.

5. (25 points) At time $t = 0$ a rhinoceros farm has 8000 rhinos. The instantaneous annual population growth rate (birth rate minus death rate) is 12%. The farm sells rhinos steadily at the rate of 1500 each year.

(a) Write a differential equation for the number $y(t)$ of rhinos the farm has at time t.

(b) Solve the differential equation in part (a). Please show all of your steps clearly.

(c) In how many years will the farm have no more rhinos left?

(d) How many rhinos will the farm have sold during this time?

(e) Make up an installment buying problem which leads to the identical mathematics as this rhino population problem (i.e., the exact same differential equation and the same initial value of $y(t)$). Write out your problem clearly (use complete sentences).

6. (10 points) At time $t = 0$ a rocket-car traveling at 100 m/sec starts to decelerate. Its deceleration is caused by two things: a constant braking force that produces a negative acceleration of -8 m/sec^2, and a small parachute which causes an air resistance equal to 0.05 sec^{-1} times the velocity with which the air is rushing against the car. If there were no wind, the latter velocity would be simply the velocity of the car. At first there is no wind, but at time $t = 0$ a wind starts blowing in the opposite direction to the car's motion. The wind increases steadily by 5 m/sec each second.

(a) Write a differential equation for the velocity $v(t)$ of the car.

(b) Arrange the differential equation in part (a) in the form $dv/dt = k(t)y + l(t)$.

Review Problem Set #3: Final Exam.

(100 points in all, time = 3 hours, checking your work carefully)

1. (20 points) At time $t = 0$ an object is located 50 meters above your eye level (directly overhead). At that instant it is thrown at an initial velocity of 20 m/sec directed 30° **below** the horizontal. Neglect air resistance, and take $g = 9.8$ m/sec^2.

(a) Find parametric equations for the object's motion.

(b) Express the angle θ at which you observe the object (i.e., the angle above the horizontal of your line of sight to the object) as a function of t.

(c) Find the rate of change of θ in rad/sec when $t = 2$.

2. (20 points) At time $t = 0$ you start driving at the speed limit of 55 miles/hr. As your confidence builds up, noticing no cop for miles around, you gradually start accelerating. Your acceleration is

$$a(t) = \frac{24}{\sqrt{1 - t^2}} \text{ miles/hr}^2.$$

(a) Find the velocity $v(t)$.

(b) After a half-hour you are arrested. What is your velocity at that time?

(c) Find the distance you traveled in the half-hour before you're arrested.

3. (25 points) Let $f(x) = \frac{1}{x} + \ln x,\ x > 0$.

(a) Find the x- and y-coordinates of any max/min points and any points of inflection.

(b) For x very close to 0, what does $f(x)$ do?

(c) As x approaches ∞, what does $f(x)$ do?

(d) Sketch the graph of $f(x)$.

(e) Find the arclength of the curve between $x = 1/2$ and $x = 1$, using the trapezoid rule with $n = 4$ subdivisions.

4. (25 points) A vat holds 5 m^3 of liquid. At time $t = 0$ the liquid contains y_0 grams of impurity. One pipe carries water with k gm/m^3 of impurity into the vat at rate r m^3/min, and a second pipe carries the uniformly mixed impurities from the vat at the same rate r. In this problem r is **not constant**, but rather varies sinusoidally according to the following formula: $r(t) = 3 + 2\sin(\pi t/10)$.

(a) Write a differential equation for the amount $y(t)$ of impurity in the vat at time t. The constant k should appear in your differential equation.

(b) Solve the differential equation. Show all your work clearly. The initial value y_0 (as well as k) should appear in your final formula for $y(t)$.

(c) Does the amount of impurity in the vat increase or decrease with time?

5. (10 points) A country with a population of 100 million has instantaneous growth rate (birth rate minus death rate) of 2.5% and a net immigration that equals 1.0 million people/year at time $t = 0$ and increases steadily by 100000 people/year each year. Set up a differential equation for the population $y(t)$ (measured in units of millions of people).

Review Problem Set #4: Final Exam.

(100 points in all, time = 3 hours, checking your work carefully)

1. (15 points) Suppose you are 5 feet tall, and the sun casts a shadow y feet long when it is at an angle θ above the horizon.

(a) Express the angle θ in terms of the length y of your shadow.

(b) Shortly before sunset, your shadow is 25 feet long, and it is lengthening at the rate of 0.5 feet/min. At what angular speed, **expressed in deg/min**, is the sun setting?

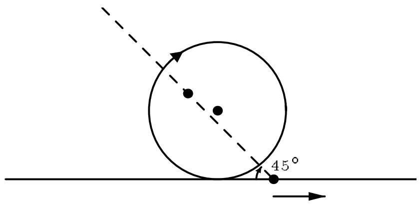
2. (15 points) Find any critical points of the function $f(x) = x^x,\ x > 0$; then use the second derivative test to determine whether the point(s) is (are) maxima or minima.

3. (10 points) You measure the slope of the tangent line to the curve $y = f(x)$ at intervals of 0.5 from $x = 0$ to $x = 3$, and find the following values:

x	0	0.5	1.0	1.5	2.0	2.5	3.0
slope	1.00	0.91	0.74	0.56	0.41	0.21	0.15

Use the trapezoid rule to find the arclength of the graph of $y = f(x)$ between $x = 0$ and $x = 3$.

4. (20 points) At time $t = 0$ a little stick gets lodged in a spoke of a wheel at a point half way between the center of the wheel and the point where the wheel is touching the ground. The wheel has radius 2 meters and is traveling from left to right along the x-axis (see the diagram at the right) at 5 m/sec. The sun is shining from behind the wheel at an angle of 45°, casting the stick's shadow on the x-axis. Find the speed with which the shadow is moving along the x-axis when the stick has gone one third the way around the wheel (as in the diagram). (To find the coordinates of the stick, use: $x = r\theta - a\sin\theta,\ y = r - a\cos\theta$.)



5. (20 points) An item is bought using a loan of y_0 dollars, which is paid back at the steady rate of a dollars/year. The instantaneous interest rate is k (assume continuous compounding), and the term of the loan is T.

(a) Give the differential equation for the amount $y(t)$ still owed after t years.

(b) Solve the differential equation for $y(t)$.

(c) Derive a formula for a in terms of k, y_0, and T.

(d) Now suppose that the interest rate is 10%, and the term is 20 years. At what instant t is exactly half of the money going for interest (i.e., half of the a dollars per year being paid at that instant) and half for principal?

6. (10 points) In a certain population the instantaneous deathrate always **exceeds** the instantaneous birthrate by k. Meanwhile, individuals are immigrating to the region. The initial rate of immigration is I_0 individuals per year, but this rate is increasing exponentially in such a way so as to double every D years. Write a differential equation for the population $y(t)$.

7. (10 points) At time $t = 0$ you borrow y_0 dollars to make a purchase. The instantaneous annual interest rate is 12%, compounded continuously. You will pay back the loan initially

at the rate of α dollars per month, but the monthly amount you pay will decrease steadily by 0.01α each month, so that after 100 months your payments will decrease to zero. Set up a differential equation for the amount $y(t)$ that you owe after t months.

(b) Find $y(t)$. Your answer should involve the constants y_0 and α.