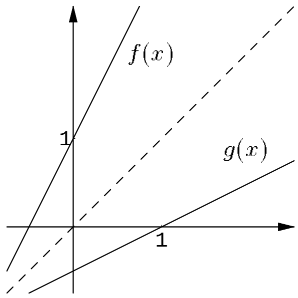
CHAPTER 7

Inverse Trig Functions

Given a function $y = f(x)$, its **inverse function** is the function that "goes backwards." We'll say what we mean by this a little later. To find the inverse function, we (1) solve the equation $y = f(x)$ for x in terms of y, obtaining $x = g(y)$, and then (2) interchange the letters x and y, i.e., write $y = g(x)$. Then we say that $g(x)$ is the inverse function of $f(x)$. We will illustrate inverse functions by a series of examples.

EXAMPLE 7.1. $y = f(x) = 2x + 1$.



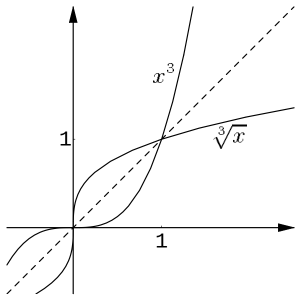
We first solve for x in terms of y: $x = 0.5y - 0.5$. Then we interchange x and y: $y = g(x) = 0.5x - 0.5$. At the right we show the graphs of both $f(x)$ and $g(x)$. Notice that if we imagine a mirror along the line $y = x$, then the two graphs are mirror images of one another. That is always the geometrical relationship between the graph of a function and the graph of its inverse function.

When we say that the inverse function $g(x)$ " goes backwards," we mean that if you compute $f(x)$ and then apply $g(x)$ to that value, i.e., if you compute $g(f(x))$, you will get x back again. The same thing happens if you evaluate the two functions in the other order, applying f to $g(x)$, i.e., $f(g(x))$ is also equal to x:

If $f(x)$ and $g(x)$ are inverse functions of one another, then $g(f(x)) = x$ and $f(g(x)) = x$.

For instance, in Example 7.1, we check that $g(f(x)) = 0.5(2x+1) - 0.5 = x$ and $f(g(x)) = 2(0.5x - 0.5) + 1 = x$.

EXAMPLE 7.2. $y = f(x) = x^3$.



Solving for x, we obtain $x = \sqrt[3]{y} = y^{1/3}$, and so the inverse function is $y = g(x) = \sqrt[3]{x} = x^{1/3}$. Again the graph of $f(x)$ and the graph of $g(x)$ are mirror images of one another with respect to the line $y = x$ (see right). And again we find that applying g to $f(x)$ or applying f to $g(x)$ gets us back to x: $g(f(x)) = \sqrt[3]{x^3} = x$, $f(g(x)) = (\sqrt[3]{x})^3 = x$. We say that taking the cube root is the inverse operation to cubing a number.

EXAMPLE 7.3. $y = f(x) = x^2$.

Here we have a situation that did not exist in the first two examples: solving for x gives two different functions, namely $x = \sqrt{y}$ and $x = -\sqrt{y}$. We do not want this ambiguity, and so we make an agreement to take only the nonnegative square root: $x = \sqrt{y}$. This is equivalent to erasing the left half of the parabola $y = x^2$. Then the inverse function is

$y = \sqrt{x}$. After crossing out the left half of the parabola, the resulting function $y = f(x) = x^2$ for $x \geq 0$ and the inverse function $y = g(x) = \sqrt{x}$ have the same relation to one another as in Examples 7.1 and 7.2. Namely, (1) their graphs are mirror images of one another relative to the line $y = x$; and (2) each function "reverses" the other one. That is, if we take a nonnegative number, square it, and then take the nonnegative square root, we get back to where we started; and if we take a nonnegative number, take its square root, and then square it, we also get back to where we started.

Example 7.3 illustrates two things to think about when dealing with the inverse function of some $y = f(x)$: (1) the possibility that you might have to select only part of $y = f(x)$ (such as the right half of the parabola in Example 7.3) in order to obtain a well-defined inverse function $y = g(x)$; (2) the possibility that the domain does not consist of all x, i.e., only x in a certain range can be substituted in $g(x)$.

1. The Inverse Tangent Function

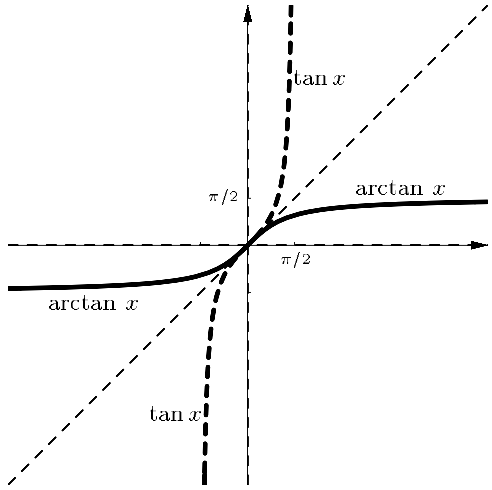
The inverse tangent function, which is called the **arctangent**, goes in the reverse direction from the tangent function. That is, the arctangent of x answers the question: What angle (in radians) has x as its tan? As in Example 7.3, there's some ambiguity here. In fact, in the case of the arctangent, there are infinitely many possible angles whose tangent is x: if θ has the property that $\tan\theta = x$, then so does $\theta + \pi$, $\theta + 2\pi$, $\theta + 3\pi$, $\theta - \pi$, $\theta - 2\pi$, etc., because the tangent function has period π. In other words, the part of the graph of $y = \tan x$ contained between the two asymptotes at $x = -\pi/2$ and $x = \pi/2$ is simply repeated between $x = \pi/2$ and $x = 3\pi/2$, between $x = 3\pi/2$ and $x = 5\pi/2$, and so on — infinitely often in both directions. (Recall that while sine and cosine each has period 2π, the tangent function has period π.) So in constructing the inverse tangent function the first thing we have to do is cross out all but one of the branches of the tangent function. The usual convention is to remove all but the branch near the y-axis, i.e., to look only at angles between $-\pi/2$ and $\pi/2$. After doing that, we obtain a well-defined arctangent function, which will be denoted $y = \arctan\, x$. That is,

$$\arctan\, x = \begin{pmatrix}\text{the angle in radians in the interval from} \\ -\pi/2 \text{ to } \pi/2 \text{ whose tangent is } x\end{pmatrix}$$

Before going farther, we make three remarks about notation.

1. The function defined above is called the " principal value" Arctangent (with a capital "A"). In some situations we might want to consider *any* angle whose tangent is x. In that case we'll write $\arctan x$ with a small "a."
2. In many textbooks and on most calculators, the function $\arctan\, x$ is denoted $\tan^{-1} x$. We shall not use this notation, because it can be confusing. Namely, since $\tan^2 x$ means $(\tan x)^2$, you might erroneously think that $\tan^{-1} x$ is the same as $(\tan x)^{-1}$, which is $\cot x$. This is false. The functions $\arctan x$ and $\cot x$ are completely different: the first *goes backwards* from the tangent function, whereas the second is the *reciprocal* of the tangent function.
3. We are using $\arctan\, x$ to mean the angle *in radians* whose tangent is x. The angle *in degrees* would be $(180/\pi)\arctan\, x$. Be careful when using calculators: most calculators will give you degrees unless you specifically change the mode to radians.

Just as the function $y = \tan x$ has vertical asymptotes at $\pm\pi/2$, the function $y = \arctan x$ has horizontal asymptotes at $\pm\pi/2$. As x goes from $-\infty$ to $+\infty$, its Arctan goes from $-\pi/2$ to $\pi/2$, never taking values below $-\pi/2$ or above $\pi/2$. As expected, the graph of $y = \tan x$ (more precisely, the branch of the graph that we've taken) and the graph of $y = \arctan x$ are mirror images of one another with respect to the line $y = x$.



2. The Derivative of $y = \mathrm{Arctan}\, x$

To find dy/dx, where $y = \arctan x$, we replace the relation $y = \arctan x$ with the equivalent relation expressing x as a function of y: $x = \tan y$. In other words, the definition of Arctan tells us that when $y = \arctan x$, then y is an angle whose tan is x. We then implicitly differentiate the relation $x = \tan y$, i.e., we take d/dx of both sides. On the left we get 1. On the right we must use the chain rule, which says that the derivative of tan(some function of x) is $\sec^2$(this function) times the derivative of this function. Thus, we obtain:

$$1 = (\sec^2 y)\frac{dy}{dx}, \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

The only trouble with this is that, as generally happens with implicit differentiation, our formula for the derivative involves y. We would like to express $\frac{dy}{dx} = \frac{d}{dx}\arctan x$ in terms of x. Now using the trig identity $\sec^2 y = 1 + \tan^2 y$ (this trig identity comes from the identity $1 = \cos^2 + \sin^2$ if you divide through by $\cos^2$), and also the fact that $x = \tan y$, we can replace $\sec^2 y$ in the denominator by $1 + x^2$. Thus,

$$\frac{dy}{dx} = \frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

This is a basic derivative formula.

If we have a function $u(x)$ inside the Arctan, then we combine this formula with the chain rule, i.e., we have

$$\frac{d}{dx}\mathrm{Arctan}(u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

Whenever we have a derivative formula, there's a companion antiderivative formula:

$$\int \frac{dx}{1+x^2} = \arctan x + C.$$

We shall later encounter some practical applications where this integration formula comes in handy.

Whenever you have a new function, such as $\arctan x$, and a new derivative formula, you can combine it with other functions and solve the same type of problems as before — derivatives of products and of functions of functions, tangent line approximation, max/min, related rates, etc.

EXAMPLE 7.4. Find the derivative of $(x+1)\arctan(\sqrt{x})$.

Using the product rule and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}\left((x+1)\arctan(\sqrt{x})\right) &= (x+1)\frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}} + 1\cdot\arctan(\sqrt{x}) \\ &= \frac{1}{2\sqrt{x}} + \arctan(\sqrt{x}).\end{aligned}$$

EXAMPLE 7.5. Using the tangent line approximation and the fact that $\tan 45° = 1$, find the angle in degrees whose tangent is 0.99.

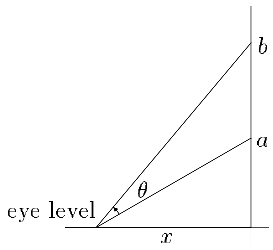
We apply the tangent line approximation formula

$$f(x+\Delta x) \approx f(x) + \Delta x\, f'(x)$$

to the situation: $f(x) = \frac{180}{\pi}\arctan x$, $x = 1$, $\Delta x = -0.01$. We know that $f(1) = 45$, and we want to know: $f(1+(-0.01)) = ?$ Here $f'(x) = \frac{180}{\pi}\cdot\frac{1}{1+x^2} = \frac{180}{\pi}\cdot\frac{1}{1+1} = 90/\pi$. Thus,

$$f(x+\Delta x) \approx 45 + (-0.01)(90/\pi) = 44.71°.$$

EXAMPLE 7.6. Suppose you are viewing a painting whose bottom is a distance a above eye level and whose top is a distance b above eye level. You can see detail in the painting best when it occupies the largest possible angle θ in your field of vision (see the diagram at the right below). How far should you stand back from the wall in order to get the best view?



Using this diagram, we cannot express θ directly as an inverse trig function, since it is not an angle in a right triangle. However, θ is the *difference* between the lower-left angle of the larger right triangle and the lower-left angle of the smaller right triangle. In both right triangles we have letters standing for the two legs. Thus, we can use the arctangent function. That is, θ is equal to the angle $\arctan(b/x)$ of the larger right triangle minus the angle $\arctan(a/x)$ of the smaller right triangle: $\theta = \arctan(b/x) - \arctan(a/x)$. We want to maximize θ as a function of x. Hence, we set $d\theta/dx$ equal to 0:

$$0 = \frac{d\theta}{dx} = \frac{1}{1+(b/x)^2}\left(-\frac{b}{x^2}\right) - \frac{1}{1+(a/x)^2}\left(-\frac{a}{x^2}\right),$$

where we have used the formula for d/dx of $\arctan u$. We now simplify, multiplying the numerators and denominators together in each term and bringing the first fraction (which has a negative sign) over to the left:

$$\frac{b}{b^2+x^2} = \frac{a}{a^2+x^2}.$$

To solve for x, we cross-multiply, then gather together the terms containing an x, and finally factor out common factors on both sides:

$$b(a^2+x^2) = a(b^2+x^2), \qquad bx^2 - ax^2 = ab^2 - ba^2, \qquad (b-a)x^2 = ab(b-a).$$

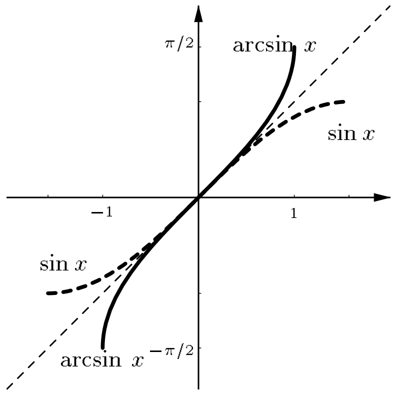
Dividing by $b-a$ and taking the positive square root gives $x = \sqrt{ab}$. From the practical setup it should be clear that this critical point really is a maximum. If you want to check by the second derivative method, just take the second derivative of θ with respect to x at the point $x = \sqrt{ab}$. You will get a negative value.

3. The Inverse Sine Function

The inverse sine function answers the question: what angle (in radians) has sine equal to x? There are two things to watch out for here:

1. As in the case of the tangent, there will be infinitely many angles whose sine is the same. So we must remove all of the sine curve except for one section before we construct the inverse function. The usual convention, as in the case of Arctan, is to take the part of the curve $y = \sin x$ between $x = -\pi/2$ and $x = \pi/2$, where y goes exactly once through all values from -1 to 1. That is, we define the principal value Arcsin of x as follows:
$$\text{arcsin } x = \begin{pmatrix} \text{the angle in radians between} \\ -\pi/2 \text{ and } \pi/2 \text{ whose sine is } x. \end{pmatrix}$$
2. Unlike Arctan, the function Arcsin cannot be applied to all x, but rather only to x between -1 and 1. This is because, unlike the tangent function, which takes all values from $-\infty$ to ∞, the sine function only takes values between -1 and 1, and for x outside of that interval there is no angle whose sine is x. Thus, the *domain* of arcsin x is the interval $[-1, 1]$.

On the right is a graph of the function $y = \sin x$ (between $x = -\pi/2$ and $x = \pi/2$) and the function $y =$ arcsin x (which has domain between $x = -1$ and $x = 1$). As usual, the two graphs are symmetric with respect to the line $y = x$.



4. The Derivative of $y =$ Arcsin x

The procedure is just like in the case of Arctan. We first replace the relation $y =$Arcsin x by the relation $x = \sin y$, and then we take d/dx of both sides, obtaining $1 = (\cos y)\frac{dy}{dx}$, i.e., $dy/dx = 1/\cos y$. We can express the derivative in terms of x, which equals $\sin y$, by means of the trig identity $\cos = \sqrt{1 - \sin^2}$. Thus, $1/\cos y = 1/\sqrt{1 - \sin^2 y} = 1/\sqrt{1 - x^2}$. Hence we have
$$\frac{d}{dx} \text{ arcsin } x = 1\sqrt{1 - x^2}.$$
(Note: When we take the square root in writing cos in terms of sin, in general we have to put a $\pm$ in front of the radical; however, in this case we know that we want the positive square root, because in our definition of the Arcsin function y has to be between $-\pi/2$ and $\pi/2$, and for these values $\cos y$ is positive.)

The companion antiderivative formula is
$$\int \frac{dx}{\sqrt{1 - x^2}} = \text{arcsin } x + C.$$

EXAMPLE 7.7. The tip of a 20-meter span of a drawbridge has been raised 8 meters above the horizontal, and it is being raised at 0.2 m/sec. How fast (in deg/sec) is the angle of the drawbridge increasing?

In this related rates problem, let θ be the angle in degrees, and let x be the height the tip has been raised. Then $\theta = \frac{180}{\pi}\arcsin(x/20)$, and so

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{180}{\pi}\frac{1}{\sqrt{1-(x/20)^2}}\frac{1}{20}\cdot\frac{dx}{dt} = \frac{180}{\pi}\frac{0.05}{\sqrt{1-(8/20)^2}}\cdot 0.2 = 0.625\,\mathrm{deg/sec}.$$

Example 7.8. Evaluate: $\int dx/\sqrt{r^2-x^2}$, where r is a constant.

Here we'll make a u-substitution to transform the integral into a form where we can use the above antiderivative formula, which we now restate using the variable u: $\int du/\sqrt{1-u^2} = \arcsin\, u + C$. The difficulty in our example is that we have r^2 where we'd like to have 1. We can get a 1 there by factoring out r^2 from r^2-x^2, i.e., by writing $r^2-x^2 = r^2\left(1-(x/r)^2\right)$, and then pulling the r^2 out of the radical. Thus,

$$\int\frac{dx}{\sqrt{r^2-x^2}} = \int\frac{1}{r}\cdot\frac{dx}{\sqrt{1-(x/r)^2}}.$$

This suggests the u-substitution $u = x/r$. Then $du = (1/r)dx$, and so in our integral we can replace $(1/r)dx$ by du and the denominator by $\sqrt{1-u^2}$. So we obtain:

$$\int\frac{du}{\sqrt{1-u^2}} = \arcsin\, u + C = \arcsin(x/r) + C,$$

which is our final answer. As always in an antiderivative problem, we can check that this is right by taking the derivative of our answer and seeing that we do in fact get $1/\sqrt{r^2-x^2}$.

5. The Inverse Cosine Function

So far we have dealt with the inverse functions of two of the six trig functions (sine, cosine, tangent, secant, cosecant, cotangent). The other four inverse trig functions are used less frequently, and in any case they can be expressed in a simple way in terms of Arcsin or Arctan. We illustrate this by discussing the function Arccosine.

A basic relationship between the cosine and the sine is that the cosine of an angle is the sine of the complementary angle. This means that if we want to know an angle whose cosine is x, we can take the complementary angle of the angle whose sine is x, i.e., we can take $(\pi/2) - \arcsin\, x$. We shall therefore define the principal value arccosine, denoted $\arccos x$, as follows:

$$\arccos(x) = \frac{\pi}{2} - \arcsin(x) \qquad \text{for } -1 \le x \le 1.$$

Since Arcsin runs from $-\pi/2$ to $\pi/2$, it follows that $\arccos(x)$ runs from π to 0 as x runs from -1 to 1. The derivative of $\arccos(x)$ is easy to compute:

$$\frac{d}{dx}\arccos(x) = 0 - \frac{d}{dx}\mathrm{Arcsin}(x) = -\frac{1}{\sqrt{1-x^2}}.$$

HOMEWORK 6.

1. Find d/dx of:
(a) $x^2 \arctan(x/2)$,
(b) $(x+1)/\arcsin(x^2)$,
(c) $\arcsin(\sqrt{1-x})$,
(d) $x \arcsin(x) + \sqrt{1-x^2}$.

2. Find the anti-derivative of
(a) $(100-16x^2)^{-1/2}$,
(b) $(1+2x^2)^{-1}$.

3. Using the tangent line approximation and the fact that $\sin(30°) = 0.5$, find the angle in degrees whose sine is 0.52.

4. Find the volume of the solid formed when the shape bounded by $y = \sqrt{x^{-2}-1}$, the line $y = 1$, and the x- and y-axes is rotated around the y-axis.

5. Find the center of mass of the uniform flat plate bounded by the curve $1/\sqrt{1-x^2}$, the curve $-1/\sqrt{1-x^2}$, the y-axis, and the line $x = 0.5$.

6. Graph and find max/min points of
(a) $y = (\arctan x) - 0.5x$,
(b) $y = \big(\arcsin(x/2)\big) - x$.

7. Sketch and find the area between
(a) $y = 2/(1+x^2)$ and $y = x^2$,
(b) $y = 1/\sqrt{1-x^2}$ and $y = 2$.

8. A rocket launched vertically from a point 2 km from an observer is ascending at 0.20 km/sec at the instant when it is 1 km above the ground. Find the rate in degrees/sec at which its angle of inclination (as seen by the observer) is increasing at that moment.

9. A triangle has two fixed sides of length a and b with included angle θ that is free to widen. Let x be the side opposite θ. (a) Use the law of cosines to express θ as a function of x. (b) At the instant when $x = c$, what is the ratio of the rate at which the angle θ is widening (in rad/sec) to the rate at which the side x is increasing?

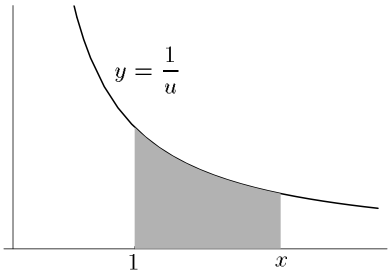
CHAPTER 8

The Natural Logarithm

The **natural logarithm**, denoted $y = \ln x$, is the function defined as follows: $\ln x$ *is the area under the reciprocal function* $1/u$ *from* $u = 1$ *to* $u = x$. Using the integral notation, we write:

$$\ln x = \int_1^x \frac{du}{u}.$$

In the drawing to the right the shaded area is $\ln x$. Here we are showing the case $x > 1$, when the area is positive. If $x < 1$, on the other hand, then the limits of integration are "backwards" (i.e., the top limit of integration is less than the bottom limit of integration). In other words, if $x < 1$, then we traverse the area going from right to left rather than from left to right. In that case the area in question is counted with a negative sign. If $x = 1$, then the area is 0. If $x \leq 0$, then we cannot define $\ln x$, because in that case the reciprocal function $1/u$ has an asymptote (i.e., it goes to infinity) between $u = 1$ and $u = x$. Thus, our first property of $\ln x$ is:



(Property 1)
$$\ln x \begin{cases} > 0 & \text{if } x > 1; \\ = 0 & \text{if } x = 1; \\ < 0 & \text{if } 0 < x < 1; \\ \text{is undefined} & \text{if } x \leq 0. \end{cases}$$

We say that the *domain* of $\ln x$ is *all positive* x.

The next group of properties are the main reason for defining $\ln x$.

(Property 2 a)
$$\frac{d}{dx}\ln x = \frac{1}{x}; \qquad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

The first of these equalities is a direct result of the "fundamental theorem of calculus," which says that the derivative of this type of area function is obtained by removing the integral sign and replacing the dummy variable by the variable in the upper limit of integration: $\frac{d}{dx}\int_1^x \frac{1}{u}du = \frac{1}{x}$. The second equality follows from the chain rule: if $y = \ln u$ and $u = u(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

(Property 2 b)
$$\int \frac{dx}{x} = \ln x + C.$$

This means that $\int_a^b \frac{dx}{x} = \ln b - \ln a$. Property 2(b) follows from 2(a), which says that $\ln x$ is an antiderivative of $1/x$. We shall later see that if we want to allow the possibility of negative limits of integration, we must replace x by $|x|$ inside the ln, i.e., we modify the antiderivative formula as follows:

$$\int \frac{dx}{x} = \ln |x| + C.$$

This antiderivative formula fills in a gap in the antiderivative formula we had before for x^n, namely, the case $n = -1$. For all $n \neq -1$, the antiderivative of x^n is another power function; but the antiderivative of x^{-1} is a different type of function, namely, our "natural logarithm." To summarize, we write:

$$\int x^n \, dx = \begin{cases} \frac{1}{n+1} x^{n+1} + C & \text{if } n \neq -1; \\ \\ \ln |x| + C & \text{if } n = -1. \end{cases}$$

Before discussing further properties of $\ln x$, we give some examples of the use of Properties 2(a) and 2(b).

Example 8.1. Find the derivative of $x \ln^2(\sin x)$.

By the product rule,

$$\begin{aligned} \frac{d}{dx}\big(x \ln^2(\sin x)\big) &= x \frac{d}{dx}\big(\ln^2(\sin x)\big) + 1 \cdot \ln^2(\sin x) \\ &= x \cdot 2 \ln(\sin x) \frac{d}{dx}\big(\ln(\sin x)\big) + \ln^2(\sin x) \\ &= 2x \ln(\sin x) \frac{1}{\sin x} \cos x + \ln^2(\sin x) = 2x \cot x \, \ln(\sin x) + \ln^2(\sin x). \end{aligned}$$

Example 8.2. Find the definite integrals

$$\text{(a)} \int_0^2 \frac{dx}{3x+1}; \qquad \text{(b)} \int_{-2}^{-1} \frac{dx}{x}; \qquad \text{(c)} \int_0^{\pi/4} \tan x \, dx; \qquad \text{(d)} \int_2^4 \frac{2x}{x^2 - 2} dx.$$

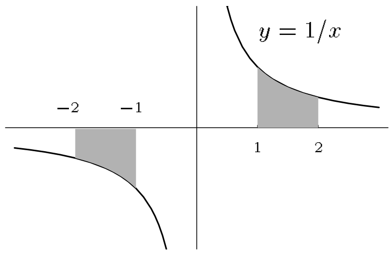
In several of these integrals it is useful to rewrite Property 2(a) with x replaced by u, and then make a u-substitution to transform the integral with respect to x to an integral with respect to u having the form in Property 2(a), i.e., $\int \frac{1}{u} du$.

(a) We make the substitution $u = 3x + 1$, so that $du = 3dx$, and hence $dx = \frac{1}{3} du$. The limits of integration in (a) change as follows: $x = 0 \Longrightarrow u = 3 \cdot 0 + 1 = 1$; and $x = 2 \Longrightarrow u = 3 \cdot 2 + 1 = 7$. Thus, we obtain

$$\int_0^2 \frac{dx}{3x+1} = \frac{1}{3} \int_1^7 \frac{du}{u} = \frac{1}{3} \ln 7.$$

(b) Since our limits of integration are negative, we use the version of the antiderivative formula with the absolute value inside the ln. We obtain:

$$\ln |x| \,\Big|_{-2}^{-1} = \ln |-1| - \ln |-2| = \ln 1 - \ln 2 = -\ln 2.$$



We can verify directly that this is the right answer: the area in question is an integral of the negative branch of the reciprocal function (which is in the third quadrant). But the area there from -2 to -1 is exactly the same as the area under the positive branch of the reciprocal function from 1 to 2, except that it is counted negatively because of being *below* the axis. Thus, the answer $-\ln 2$ agrees with the picture in the graph at the right.

(c) At first glance, this has nothing to do with ln. However, if we rewrite $\tan x$ as $\sin x/\cos x$, and notice that the derivative of $u = \cos x$ is $du = -\sin x dx$, we can transform this integral into $\int du/u$. More precisely, we obtain

$$\int_0^{\pi/4} \frac{\sin x\, dx}{\cos x} = \int_{\cos 0}^{\cos(\pi/4)} \frac{-du}{u} = -\int_1^{\sqrt{2}/2} \frac{du}{u} = -\ln(\sqrt{2}/2).$$

(d) Here we notice that the expression in the denominator has its derivative sitting with dx in the numerator. This means that substituting u for the denominator will result in du/u. That is, we set $u = x^2 - 2$, so that $du = 2xdx$. The limits of integration change from 2 to $2^2 - 2 = 2$ and from 4 to $4^2 - 2 = 14$. So we obtain $\int_2^{14} du/u = \ln(14) - \ln 2$.

We will soon see that the answers to parts (c) and (d) can be simplified.

Before giving the next group of properties of ln, we make a useful observation. We defined $\ln a$ to be the area under $y = 1/x$ from $x = 1$ to $x = a$. But it turns out that for any positive number b this value $\ln a$ is also the area under $y = 1/x$ from $x = b$ to $x = ab$. For example, $\ln 2$ is the area from 1 to 2, and it is also the area from 3 to 6 (or from 1000 to 2000, etc.). Roughly speaking, this is because the latter area is obtained by taking the area from 1 to 2, expanding it horizontally, and then contracting it vertically by the same factor. More precisely, to show that $\ln a$ is equal to $\int_b^{ab} dx/x$, we write $\ln a = \int_1^a dx/x$ and make the u-substitution $u = bx$. Then $du = b\,dx$, and the limits of integration become $b \cdot 1$ to $b \cdot a$, i.e.,

$$\ln a = \int_1^a \frac{dx}{x} = \int_1^a \frac{b\,dx}{b\,x} = \int_b^{ab} \frac{du}{u}.$$

The name of the variable of integration is irrelevant, so that we can equally well write $\ln a = \int_b^{ab} dx/x$. It is now easy to obtain our next basic group of properties.

(Property 3 a) $$\ln(ab) = \ln a + \ln b\,.$$

This follows because $\ln a = \int_b^{ab} dx/x = \ln(ab) - \ln b$.

(Property 3 b) $$\ln(1/a) = -\ln a.$$

This is because, by part (a), we have $\ln(1/a) + \ln a = \ln 1 = 0$.

(Property 3 c) $$\ln(a/b) = \ln a - \ln b\,.$$

This follows immediately from part (a) if we replace b by $1/b$ and use part (b).

(Property 3 d) $$\ln(a^n) = n \ln a\,.$$

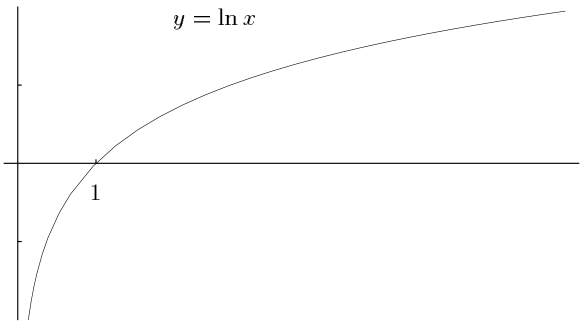
First note that part (a) can be repeated when you have more than two factors: $\ln(abc) = \ln a + \ln(bc) = \ln a + \ln b + \ln c$, and so on. Thus, if n is a positive integer, we can write $\ln(a^n) = \ln(a \cdot a \cdot a \cdots a) = \ln a + \ln a + \ln a + \cdots + \ln a = n \ln a$. If n is a negative integer, one uses a similar argument with a replaced by $1/a$. The formula $\ln(a^n) = n \ln a$ applies for any n (integer or not), but we shall omit the proof in the case when n is not an integer. (But see Problem 2 in the homework for a proof.)

Example 8.3. Simplify the answers to parts (c)–(d) of Example 2.

In part (c) we have $-\ln(\sqrt{2}/2) = -(\ln(2^{1/2}) - \ln 2) = -\left(\frac{1}{2}\ln 2 - \ln 2\right) = \frac{1}{2}\ln 2$. In part (d) we have $\ln(14) - \ln 2 = \ln(14/2) = \ln 7$.

1. Appearance of the Curve $y = \ln x$.

Recall what Property 1 says: the function $y = \ln x$ has domain consisting of all *positive* x, its graph crosses the x-axis at $x = 1$, and the function is positive when $x > 1$ and negative when $0 < x < 1$. It is an *increasing* function, because its derivative is $1/x$, which is always positive for $x > 0$. However, when x is large this derivative is quite small. In other words, the slope of the curve tends to "level off." On the other hand, when x is near 0 the slope is very large, i.e., the curve is steep. At the point $x = 1$ where the curve crosses the x-axis, the slope is 1, i.e., the curve is increasing at a 45° angle as it crosses the axis.



We now ask whether the curve $y = \ln x$ increases without bound as x increases, or whether it is bounded by a horizontal asymptote. It is possible for an increasing curve to be bounded — for example, the curve $y =$Arctan x. It is also possible for a curve whose slopes are becoming smaller and smaller ("leveling off") to increase without bound — for example, the curve $y = \sqrt{x}$. So our question is: does $y = \ln x$ look more like the Arctan curve or the square root curve? The answer to this question is: more like the square root curve, i.e., *there is no horizontal asymptote.* To see this, we can use Property 3(d) with a very large value of n. For example,

$$\ln\left(2^{1000}\right) = 1000 \ln 2 = 693, \quad \ln\left(2^{1000000}\right) = 693147, \quad \ln\left(2^{1000000000}\right) = 693147181.$$

Thus, we can get $\ln x$ to be as large as we want by choosing x to be 2 to a large enough power.

Even though $\ln x$ and $\sqrt{x}$ both approach infinity (fairly slowly) as $x \longrightarrow \infty$, the former function approaches infinity more slowly than the latter one. If we look at the ratio $\ln x/\sqrt{x}$, and set x equal to an extremely large number (like 10^{1000}, for example), we find that

$$\frac{\ln 10^{1000}}{\sqrt{10^{1000}}} = \frac{1000 \ln 10}{10^{500}}.$$

Here the numerator is about 230, while the denominator is a 1 followed by 500 zeros! So the ratio is extremely close to 0, i.e., $\ln x$, while fairly large, is much much smaller than $\sqrt{x}$. The same argument can be used to show that $\ln x$ approaches infinity slower than *any* n-th root function. For example, the ratio $\ln x/\sqrt[10]{x}$ has limit 0 as $x \longrightarrow \infty$. To summarize: *the natural logarithm function grows without bound as x increases, but very, very slowly.*

At the other end, as $x \longrightarrow 0$, the function $\ln x$ approaches $-\infty$. This is because a very small positive number is the reciprocal of a very large positive number, and so its logarithm is the negative of the logarithm of the very large number.

2. Relation with Logarithm-to-the-Base-a

Historically, the first logarithms used were " logarithms to the base 10," denoted $\log_{10}$. The definition of the function $y = \log_{10} x$ is: y is the power to which 10 must be raised to get x, i.e., $10^y = x$. More generally, if a is any fixed number greater than 1 (for example, log to the base $a = 2$ is often used in computer science), then we define $\log_a x$ as the power to which a is raised to get x. That is,

$$y = \log_a x \qquad \text{means that} \qquad a^y = x.$$

It is not hard to relate this "old-fashioned" logarithm to the natural logarithm ln. Namely, if $y = \log_a x$, then to find y in terms of ln we take ln of both sides of the relation $a^y = x$, obtaining $\ln(a^y) = y \ln a = \ln x$, i.e., $y = \ln x/\ln a$. Since $y = \log_a x$, we conclude that

$$\log_a x = \frac{\ln x}{\ln a}.$$

For example, $\log_{10} x = \ln x/\ln(10) = 0.4343 \ln x$. The only difference between the different kinds of logarithm is a constant factor. Thus, there is no problem in taking the derivative of $\log_a$: $\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$.

3. Logarithmic Differentiation

Although it may seem that we can already take the derivative of almost any function for which we can write down a formula, you may have noticed that there is one type of function which we have not been able to differentiate, namely, a function of the form $y = f(x)^{g(x)}$, where $g(x)$ is **not a constant**. Of course, if $g(x)$ is a constant, then we can use the rule for the derivative of $f(x)^n$. But this rule **cannot be used if the exponent is a function of x rather than a constant**. Instead we must use a procedure called **logarithmic differentiation**. As its name implies, it consists of two parts: (1) take the ln of both sides of the equation $y = f(x)^{g(x)}$, and (2) differentiate both sides of the resulting equation. In part (1), be sure to use Property 3(d) of ln, with a replaced by $f(x)$ and n replaced by $g(x)$. We illustrate with the following examples.

EXAMPLE 8.4. Find the derivative of (a) $(\sin x)^x$; (b) $(\ln x)^{\ln x}$.

(a) We first take the ln of both sides of $y = (\sin x)^x$, obtaining $\ln y = \ln\left((\sin x)^x\right) = x\ln(\sin x)$. When taking d/dx of both sides, on the left we use Property 2(a) of ln and on the right we use the product rule:

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\frac{1}{\sin x}\cos x + 1\cdot\ln(\sin x) = x\cot x + \ln(\sin x).$$

Finally, we multiply both sides by y, and express dy/dx in terms of x by replacing y by $(\sin x)^x$:

$$\frac{dy}{dx} = \big(x\cot x + \ln(\sin x)\big)y = \big(x\cot x + \ln(\sin x)\big)(\sin x)^x.$$

(b) We write $y = (\ln x)^{\ln x}$, and $\ln y = \ln\left((\ln x)^{\ln x}\right) = \ln x\,\ln(\ln x)$. Taking d/dx of both sides gives:

$$\frac{1}{y}\frac{dy}{dx} = \ln x\,\frac{d}{dx}\big(\ln(\ln x)\big) + \frac{1}{x}\ln(\ln x) = \ln x\left(\frac{1}{\ln x}\,\frac{1}{x}\right) + \frac{1}{x}\ln(\ln x) = \frac{1+\ln(\ln x)}{x},$$

and hence

$$\frac{dy}{dx} = \frac{1+\ln(\ln x)}{x}\,y = \frac{1+\ln(\ln x)}{x}(\ln x)^{\ln x}.$$

HOMEWORK 7.

1. Find the derivative of:

(a) $\ln(kx)$ (k is a constant); (d) $\ln(\tan x)$;
(b) $\ln^4 x$; (e) $\ln\left((x^2+16)^2\right) - x\arctan(x/4)$.
(c) $\ln^2(\ln x)$;

2. (a) Find the derivative of the functions $\ln(ax) - \ln x - \ln a$ (where a is any constant) and $\ln(x^n) - n\ln x$ (where n is any constant).

(b) Use part (a) to give another derivation of Properties 3(a) and 3(d) of ln.

3. Simplify:

(a) $\ln 30 - \ln 2 - \ln 5$; (d) $\ln\sqrt{6.75}$ (in terms of $\ln 2$ and $\ln 3$);
(b) $\ln\sqrt{8}$ (in terms of $\ln 2$); (e) $\ln(\pi r^2) - 2\ln r$;
(c) $\ln\sqrt[3]{9}$ (in terms of $\ln 3$); (f) $\ln\sqrt{r^2-x^2} - \frac{1}{2}\ln(r-x)$.

4. Find:

(a) $\displaystyle\int_{-1}^{-0.5} \frac{dx}{x}$; (d) $\displaystyle\int_2^3 \frac{dx}{x\ln^2 x}$;
(b) $\displaystyle\int \cot x\, dx$; (e) $\displaystyle\int_0^{2\pi} \frac{\sin(\theta/2)d\theta}{2-\cos(\theta/2)}$.
(c) $\displaystyle\int \frac{dx}{x\ln x}$ (hint: substitute $u = \ln x$);

5. Find the center of mass of the uniform plate contained

(a) between the graphs of $y = 1/x$, $y = -1/x$, $x = 1$ and $x = 2$;
(b) between the graphs of $y = 1/x^2$, $y = -1/x^2$, $x = 1$ and $x = 2$;
(c) between the graphs of $y = 1/(1+x^2)$, $y = -1/(1+x^2)$, $x = 0$ and $x = 1$.

6. Find the area

(a) between the curves $y = 5/(1+x^2)$ and $y = 2/x$;
(b) between the curves $y = 1/\sqrt{1-x^2}$ and $y = 2/(2-x)$.

7. The region bounded by $y = -1/x^2$, $y = -1/4$ and $y = -1$ is revolved around the y-axis to form a container. Find its volume.

8. A bug slides down the curve $y = \ln x$, moving leftward (i.e., its x-coordinate is decreasing) at 10 units/min. Find its speed along the curve when $x = 2$, when $x = 1$, and when $x = 1/2$.

9. By "experiment" (for example, substituting $x = 1/10^{1000}$), find the limit as $x \longrightarrow 0$ (where x is positive) of:

(a) $x^3\ln x$, (b) $x\ln x$, (c) $\sqrt{x}\ln x$, (d) $\sqrt[10]{x}\ln x$.

10. Find the derivative of:

(a) $y = x^x$; (d) $y = x^{\ln x}$;
(b) $y = (\sin x)^{\sqrt{x}}$; (e) $y = (\cos x)^{\operatorname{Arctan} x}$.
(c) $y = (\ln x)^x$;

11. One morning in January it began to snow and kept on snowing at a constant rate. A snow plow began to plow at noon. It plowed clean and at a constant rate (volume per unit time). From one o'clock until two o'clock it went only half as far as it had gone between noon and one o'clock. When did the snow begin falling?

CHAPTER 9

Improper Integrals

An improper integral is the area of a shape that extends out to infinity in some direction. In many cases, such a shape has infinite area, and we say that the improper integral is ∞. However, in certain cases the total area is finite, because the part extending out infinitely far is "pinched closely together," so that the very narrow part going out infinitely far does not contribute much area. This all sounds rather vague. The best way to understand improper integrals is through examples.

EXAMPLE 9.1. Find the area under the reciprocal cube curve $y = 1/x^3$ to the right of the line $x = 1$.

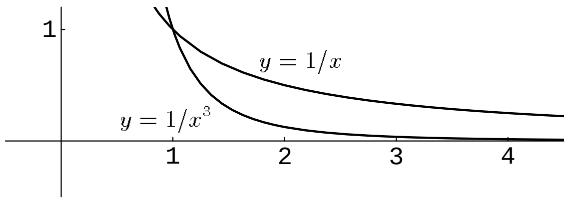
First, suppose that we bound the area on the right at $x = b$, where b is a very large upper limit of integration. Then we compute

$$\int_1^b \frac{dx}{x^3} = -\frac{1}{2}x^{-2}\Big|_1^b = \frac{(1-b^{-2})}{2}.$$

For example, the area from $x = 1$ to $x = 100$ is exactly 0.49995, the area from $x = 1$ to $x = 1000000$ is exactly 0.4999999999995, and so on. Clearly, the limit as $b \longrightarrow \infty$ is 0.5. This is what we mean by the area out to infinity:

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\longrightarrow\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\longrightarrow\infty}\frac{(1-b^{-2})}{2} = 0.5.$$

EXAMPLE 9.2. Find the area under the reciprocal curve $y = 1/x$ to the right of the line $x = 1$.



Here we have

$$\lim_{b\longrightarrow\infty}\int_1^b \frac{dx}{x} = \lim_{b\longrightarrow\infty} \ln b = \infty.$$

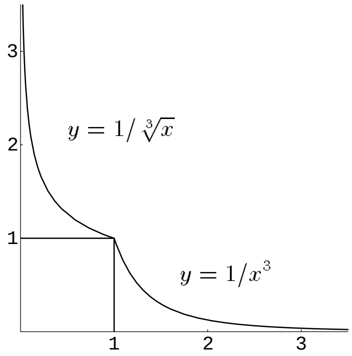
Roughly speaking, the reciprocal cube function pinches together with the x-axis a lot faster than does the reciprocal function, and so we get finite area with $1/x^3$ but not with $1/x$.

EXAMPLE 9.3. Let $y = 1/x^n$, where n is a positive constant not equal to 1. Find the area under this curve to the right of the line $x = 1$.

Here we want the limit as $b \longrightarrow \infty$ of $\int_1^b x^{-n}dx = \frac{1}{-n+1}x^{-n+1}\Big|_1^b = \frac{1}{n-1}\left(1-b^{1-n}\right)$. What happens as $b \longrightarrow \infty$ depends on the exponent of b, which is $1-n$. If this is positive, i.e., if $n < 1$, then the limit is infinite. On the other hand, if $n > 1$, then b^{1-n} approaches 0, and so we obtain the finite area $1/(n-1)$. We can summarize the situation as follows (note that in Example 9.2 we saw that the area is infinite in the special case $n = 1$):

$$\int_1^\infty \frac{dx}{x^n} = \begin{cases} \frac{1}{n-1} & \text{if } n > 1; \\ \infty & \text{if } n \le 1. \end{cases}$$

EXAMPLE 9.4. Find the area bounded by the x- and y-axes, the line $x = 1$, and the reciprocal cube root curve $y = 1/\sqrt[3]{x}$.



Here the region extends out to infinity in the y-direction. But the curve comes very close to the y-axis (its asymptote), so there is very little area enclosed when y is large. For this reason, as we shall see, the total area is finite. The area is the improper integral $\int_0^1 x^{-1/3}dx$. To evaluate this, let us compute the ordinary (*not* improper) integral $\int_a^1 x^{-1/3}dx$ for $a > 0$, and then see what happens when $a \longrightarrow 0$. That is,

$$\int_0^1 x^{-1/3}dx = \lim_{a\longrightarrow 0}\int_a^1 x^{-1/3}dx = \lim_{a\longrightarrow 0}\left(\frac{3}{2}x^{2/3}\Big|_a^1\right) = \lim_{a\longrightarrow 0}\left(\frac{3}{2}\left(1 - a^{2/3}\right)\right) = 3/2,$$

because a raised to any positive power approaches 0 as $a \longrightarrow 0$.

In Example 9.4, suppose we remove from our region the square of unit side that is bounded by the x- and y-axes, the line $x = 1$, and the line $y = 1$. The remaining area is the area to the left of the curve $x = 1/y^3$ above the line $y = 1$. This is exactly the same area as in Example 9.1, except that the roles of x and y are reversed. Thus, we expect the area in Example 9.4 to equal 1 plus the area in Example 9.1. Thus, the value of 3/2 in the last paragraph agrees with the value of 1/2 in Example 9.1.

In the same way as in Example 9.4, we can compute the area bounded by the x- and y-axes, the line $x = 1$, and the curve $y = 1/x^n$. The computation is just like the special case when $n = 1/3$. The result is:

$$\int_0^1 \frac{dx}{x^n} = \begin{cases} \frac{1}{1-n} & \text{if } n < 1; \\ \infty & \text{if } n \ge 1. \end{cases}$$

EXAMPLE 9.5. Find $\int_{-2}^2 dx/\sqrt{4 - x^2}$.

Here the integrand $(4 - x^2)^{-1/2}$ is infinite at *both* endpoints of the interval of integration — the region goes out to infinity along the line $x = -2$ and along the line $x = 2$. To get a region that does not extend to infinity, we integrate between $-a$ and a, where a is a little less than 2. Then we take the limit as $a \longrightarrow 2$. In evaluating the integral, we use Example 7.8 (with r set equal to 2) of the section on the Arcsin. Thus,

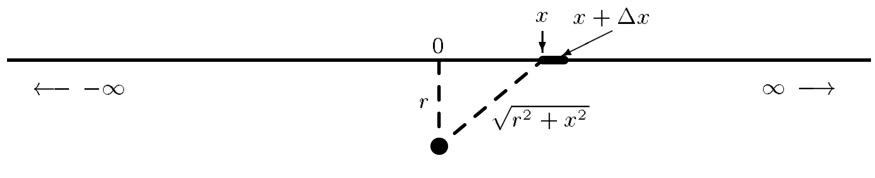
$$\begin{aligned} \int_{-2}^2 \frac{dx}{\sqrt{4 - x^2}} &= \lim_{a\longrightarrow 2}\int_{-a}^a \frac{dx}{\sqrt{4 - x^2}} = \lim_{a\longrightarrow 2} \arcsin(x/2)\Big|_{-a}^a \\ &= \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi. \end{aligned}$$

EXAMPLE 9.6. The region in the fourth quadrant bounded by the y-axis, the line $y = -1$, and the curve $y = -1/x$ is rotated around the y-axis to obtain a container (an infinitely deep funnel). (a) Find the volume of the funnel. (b) Show that the funnel has infinite surface area. Thus, one could fill the funnel entirely with water, but if one cuts the side of the funnel into small pieces and spreads them out on a table, there is no table of finite size that would hold all of the pieces!

(a) We first write x as a function of y: $x = -1/y$. Then in the volume integral $\pi \int_{-\infty}^{-1} dy/y^2$, we make the change of variables $u = -y$, so that $du = -dy$. This transforms the volume to $-\pi \int_{\infty}^{1} dy/y^2 = \pi \int_1^{\infty} du/u^2$ (we used the fact that the sign of a definite integral changes when you reverse the upper and lower limits of integration). The last integral was computed in Example 9.3 (take $n = 2$): it has finite value 1. Thus, the volume of the funnel is π.

(b) We have not developed the formula for the surface area of a volume of revolution. However, we can answer the question in part (b) by observing that the surface area is greater than the area of the flattened surface contained in the third and fourth quadrants below $y = -1$ between the curves $x = 1/y$ and $x = -1/y$. This flat area is equal to $\int_{-\infty}^{-1} \frac{-2}{y} dy = 2 \int_1^{\infty} du/u$ (where we make the same substitution $u = -y$ as before). But the latter improper integral is infinite, as we saw in Example 9.2. Therefore, the funnel has infinite surface area.

EXAMPLE 9.7. Determine how the noise from a freeway drops off as you move farther away. Make the following simplifying assumptions: (1) the amount of noise generated by a segment of the freeway is proportional to the segment's length (i.e., the traffic is uniformly distributed along the freeway); (2) the contribution of a small segment of the freeway to the amount of noise you hear is inversely proportional to the square of your distance from that segment; (3) the freeway is a straight line extending infinitely far in both directions. (We are also assuming that the noise from different parts of the freeway add up without "canceling each other," and we are using a linear scale of noise, rather than a logarithmic scale such as the decibel system.)



Let r be our perpendicular distance from the freeway. Take the freeway to be the x-axis, with 0 located at the point closest to us. We shall find the total noise by the same type of procedure that we used to find definite integral formulas for the area under a curve, the volume of a solid of revolution, and the total moment of a uniform plate. In those cases we had the integral from a to b of the contribution (to area, volume, moment, or whatever) coming from the part between x and $x + \Delta x$ (where we replace Δx by dx when we go from the Riemann sum to the integral).

For the sake of review, the chart on the next page shows the contribution from the part between x and $x + \Delta x$ and the integral that resulted.

TYPE OF PROBLEM	CONTRIBUTION	FORMULA
area under curve	$\approx f(x)\Delta x$	$\int_a^b f(x)dx$
vol of revolution	$\approx \pi f(x)^2 \Delta x$	$\pi \int_a^b f(x)^2 dx$
moment of plate	$\approx D\, x\, f(x)\Delta x$	$D \int_a^b x\, f(x) dx$

So in the freeway problem we want to find the contribution to noise that comes from the tiny segment $[x, x+\Delta x]$. According to assumptions (1) and (2), the noise you hear from this segment is proportional to the length Δx of the segment and inversely proportional to the square of your distance from the segment. In computing this distance, we take the distance from the left endpoint of the segment (since Δx is very small, this is very close to the distance from any other point on the segment); by the Pythagorean theorem, the square of this distance is $r^2 + x^2$. Thus, if we let K denote the constant of proportionality, we find that the contribution is

$$\approx K \frac{\Delta x}{r^2 + x^2}.$$

Thus, the noise coming from the entire highway between a and b is found by summing up these contributions — or rather (after taking the limit as $\Delta x \longrightarrow 0$) taking the integral with Δx replaced by dx: Noise$= \int_a^b \frac{K\,dx}{r^2+x^2}$. Thus,

$$\begin{aligned}
\text{Noise} &= K \int_a^b \frac{dx}{r^2 + x^2} = \frac{K}{r^2} \int_a^b \frac{dx}{1 + (x/r)^2} = \\
&= \frac{K}{r^2} \int_{a/r}^{b/r} \frac{r\, du}{1 + u^2} = \frac{K}{r} \int_{a/r}^{b/r} \frac{du}{1 + u^2} \\
&= \frac{K}{r} \arctan\, u \,\Big|_{a/r}^{b/r} = \frac{K}{r}\left(\arctan\left(\frac{b}{r}\right) - \arctan\left(\frac{a}{r}\right)\right).
\end{aligned}$$

If the freeway had finite length and were contained between a and b, then this would give us a rather complicated expression for the noise from the freeway. It would not be easy to say how the noise drops off as our distance r increases. However, at this point we use simplifying assumption (3), which tells us to take the limit as $a \longrightarrow -\infty$ and $b \longrightarrow \infty$, i.e.,

$$\begin{aligned}
\text{Noise} &= \frac{K}{r}\left(\lim_{b \longrightarrow \infty} \arctan\left(\frac{b}{r}\right) - \lim_{a \longrightarrow -\infty} \arctan\left(\frac{a}{r}\right)\right) \\
&= \frac{K}{r}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{K\pi}{r}.
\end{aligned}$$

Since K and π are constants, we end up with the simple conclusion: *noise from the freeway drops off proportionally to distance.* Thus, if you move twice as far away, the amount of noise drops only by 50%. (We say "only," because in the case of a point source the noise would drop by a factor of 4, i.e., by 75%.)

Even though the simplifying assumptions might seem to detract from the realism of this example, our conclusion roughly agrees with what one observes in practice. For example, real estate agents know that noise from Interstate 5 has a bad effect on property values for quite a distance — much more so than would a "point source" such as a noisy factory.

Notice the role of simplifying assumption (3). Although it might seem odd at first, improper integrals have many practical applications: it often simplifies the calculations to suppose that something extends out to infinity.

HOMEWORK 8.

1. Find the following improper integrals (some of them might be ∞):

(a) $\int_1^\infty dx/(x\sqrt{x})$; (e) $\int_0^\infty dx/(1+x^2)$; (i) $\int_{-10}^{10} dx/\sqrt{100-x^2}$;

(b) $\int_0^1 dx/(x\sqrt{x})$; (f) $\int_{-\infty}^\infty dx/(1+x^2)$; (j) $\int_0^1 x\,dx/\sqrt{1-x^2}$,

(c) $\int_1^\infty dx/\sqrt{x}$; (g) $\int_{-\infty}^\infty dx/(9+x^2)$; (k) $\int_0^\infty x\,dx/(1+x^2)$;

(d) $\int_0^1 dx/\sqrt{x}$; (h) $\int_0^1 dx/\sqrt{1-x^2}$; (l) $\int_0^\infty x\,dx/(1+x^4)$.

2. Let n be a positive number. Find (in terms of n) the volume of the solid obtained by rotating around the y-axis the part of $y=-1/x^n$ below $y=-1$ in the fourth quadrant. For what values of n is this volume infinite?

3. Suppose that $n>2$. Find (in terms of n) the balancing point of the uniform flat plate bounded by the graph of $y=1/x^n$, the graph of $y=-1/x^n$, and the line $x=1$. (The plate extends to the right all the way out to infinity.)

4. Show that the following uniform flat plates have finite mass but do not have center of mass at a finite distance:

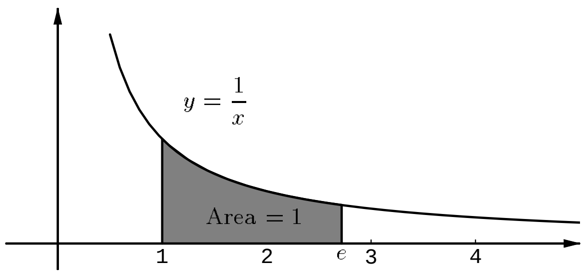
(a) Problem 3 when $1<n\le 2$;

(b) the plate bounded by the graph of $y=1/(1+x^2)$, the graph of $y=-1/(1+x^2)$, and the y-axis. This means that you could lift up the plate, but you could not balance it!

CHAPTER 10

The Number e and the Exponential Function

The number e is a fundamental constant of mathematics, every bit as important as π. It is defined as follows: **e is the number whose natural logarithm is exactly 1**. That is: $\ln e = 1$. In geometric terms, this says that you look at the area under the reciprocal function $y = 1/x$ between 1 and a, and move the upper limit of integration a to the right until you get an area of 1. (See the drawing below). It is not hard to see that e is somewhere between 2 and 3 (see Problem 1 at the end of the section). The value of e to several decimal places is: $e = 2.71828\cdots$. (The exact value cannot be given in decimal form, since e, like π, is an irrational number.)



In Chapter 8, we obtained the following relationship between the natural logarithm and the log-to-the-base-a: $\ln x = \ln a \cdot \log_a x$. In the special case $a = e$, since $\ln e = 1$ this gives us

$$\ln x = \log_e x,$$

in other words, *the natural logarithm is the same as the logarithm to the base e*. That is, $\ln x$ is equal to the power to which e must be raised to get x. This can be expressed in the form $e^{\ln x} = x$, and also in the form $\ln(e^x) = x$. In other words, *the functions* $\ln x$ *and* e^x *are inverse functions.* We shall return to this later.

There is another way to come up with the number e, which we shall explain using bank interest. Suppose that you put an initial amount of money P (called the "principal") in the bank at time $t = 0$. Suppose that the bank pays exactly 100% annual interest (this is not as unreasonable as it may seem, if you happen to live in a country with high inflation), compounded n times per year. If $n = 1$, then the money simply sits in the bank for one year without increasing, at which point you get your 100% interest; that is, you have $2P$ in the bank after the first year is up. If $n = 2$, then after 6 months the money in the bank increases from P to $1.5\,P$ (that is, after half a year you get 50%), and after another 6 months the money in the bank increases from $1.5\,P$ to $1.5 \cdot 1.5\,P = 2.25\,P$ (that is, you get another 50%, this time of a greater amount). In general, if the interest is compounded n times in the year, then every $1/n$ years whatever sum you have in the bank is multiplied by $\left(1+\frac{1}{n}\right)$. After 1 year the amount in the bank is

$$\left(1+\frac{1}{n}\right)^n P.$$

For example, if $n = 12$ (monthly compounding), then you have $2.613\,P$ after 1 year; if $n = 365$ (daily compounding), then you have $2.71457\,P$ after 1 year; if $n = 31536000$ (compounding every second), then you have $2.71828\,P$ after 1 year; and the larger n is (i.e., the more frequently the compounding occurs) the more money you have. However, this amount does not increase without bound. Rather, as $n \longrightarrow \infty$ the amount you have in the bank after 1 year becomes closer and closer to $e\,P = 2.71828\cdots P$. (In the next paragraph we explain why the value gets closer and closer to $e\,P$.) Taking the limit as n goes to infinity is called "continuous compounding." For all practical purposes, compounding every second is equivalent to continuous compounding, and even daily compounding is close to continuous compounding.

To see why the limiting value as $n \longrightarrow \infty$ is eP, we compute the limit as $n \longrightarrow \infty$ of $(1 + (1/n))^n$. Instead of looking at this power, let us take its natural log and find the limit of that:

$$\lim_{n \longrightarrow \infty} \ln\left((1+(1/n))^n\right) = \lim_{n \longrightarrow \infty} \frac{\ln\left(1+\frac{1}{n}\right)}{1/n}.$$

But this is the slope of the chord on the graph of $y = \ln x$ joining the points $(1, 0)$ and $(1 + \frac{1}{n}, \ln(1 + \frac{1}{n}))$. As n increases, this chord approaches the tangent line to the curve at $x = 1$, and so this slope approaches the derivative at $x = 1$, which is 1 (since $\frac{d}{dx}\ln x = \frac{1}{x}$). Thus, as $n \longrightarrow \infty$, the ln of $(1+(1/n))^n$ approaches 1. Thus, $\lim_{n\longrightarrow\infty}(1+(1/n))^n$ is equal to the number whose ln is 1, namely e:

$$\lim_{n \longrightarrow \infty} \left(1+\frac{1}{n}\right)^n = e.$$

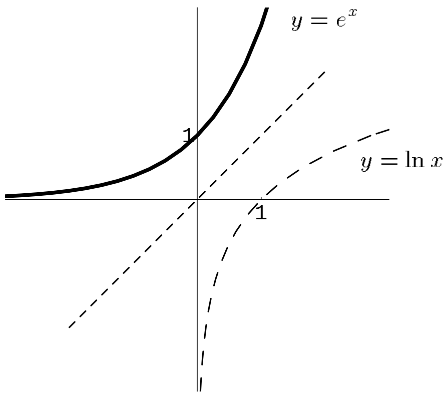
Let us return to the bank interest problem, and ask how much money is in the bank after t years. In t years the interest is compounded nt times, so that the amount in the bank is

$$\left(1+\frac{1}{n}\right)^{nt} P = \left(\left(1+\frac{1}{n}\right)^n\right)^t P.$$

If the money is "compounded continuously," i.e., we take the limit as $n \longrightarrow \infty$, then this amount is simply $e^t P$.

(In practice, most banks compound money daily: $n = 365$. We shall always assume, however, that the money is compounded "continuously," i.e., $n = \infty$. As a practical matter, there is very little difference. For example, even at the high interest rate of 100%, with daily compounding \$1000 grows to \$2714.57 in a year, while with continuous compounding it grows to \$2718.28. With lower interest rates, the difference between daily compounding and continuous compounding is even less.)

The function e^x is called the **exponential function**. It is the inverse function of $\log_e$, which we saw is the natural log function ln. In other words, to say that $y = e^x$ is equivalent to saying that $x = \ln y$. In the graph at the right we show the graphs of e^x (solid line) and $\ln x$ (dotted line). As always with inverse functions, these two graphs are the mirror images of one another with respect to the line $y = x$.



The function $y = f(x) = e^x$ has a very basic property, which we discover when we find its derivative y'. To find this derivative, we start with the relation $x = \ln y$, and take d/dx of both sides:

$$1 = \frac{1}{y}\,\frac{dy}{dx}, \qquad \text{so that} \qquad \frac{dy}{dx} = y, \qquad \text{i.e.,} \qquad \frac{d}{dx}\,e^x = e^x.$$

That is, the function $y = e^x$ has the very special property that *its derivative is the very same function.* It turns out that there is no other function — except for constant multiples Ce^x — that has this property.

1. Some Properties of the Exponential Function

By the law of exponents, if A and B are any numbers or expressions, then

(Property 1) $$e^A \cdot e^B = e^{A+B}; \qquad e^A/e^B = e^{A-B}; \qquad \left(e^A\right)^B = e^{AB}.$$

Because ln and the exponential function are inverse functions,

(Property 2) $$\ln(e^A) = A; \qquad e^{\ln A} = A.$$

By the chain rule,

(Property 3) $$\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x).$$

Since $\frac{d}{dx}e^{kx} = ke^{kx}$ (by (3) with $u(x) = kx$), it follows that

(Property 4) $$\int e^{kx}dx = \frac{1}{k}e^{kx} + C.$$

Example 10.1. Find the center of mass of a uniform plate bounded by the lines $x = 0$, $x = 2$ and the curves $y = 1/2^x$ and $y = -1/2^x$.

By symmetry, the center of mass is on the x-axis. Its x-coordinate is

$$\overline{x} = \frac{\int_0^2 \frac{x}{2^x}dx}{\int_0^2 \frac{dx}{2^x}} = \frac{\int_0^2 x e^{-(\ln 2)\,x}dx}{\int_0^2 e^{-(\ln 2)\,x}dx},$$

where we have used properties (1) and (2) to write $1/2^x$ as $e^{-(\ln 2)\,x}$. We next use property (4) with $k = -\ln 2$ to find the bottom integral, and then the trapezoid rule (with $n = 4$) to find the top integral:

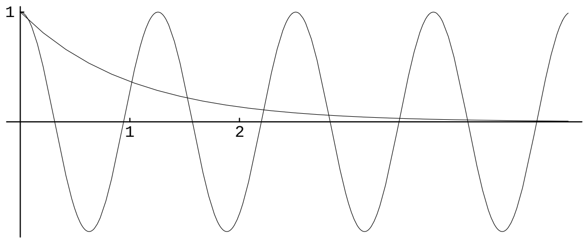
$$\int_0^2 e^{-(\ln 2)\,x}dx = -\frac{1}{\ln 2}e^{-(\ln 2)\,x}\Big|_0^2 = -\frac{1}{\ln 2}\frac{1}{2^x}\Big|_0^2 = \frac{1-(1/4)}{\ln 2} = 1.08202;$$

$$\int_0^2 \frac{x}{2^x}dx \approx \frac{1}{2}\left(0 + \frac{0.5}{\sqrt{2}} + \frac{1}{2} + \frac{1.5}{2\sqrt{2}} + \frac{1}{2}\cdot\frac{2}{2^2}\right) = \frac{5}{16}\sqrt{2} + \frac{3}{8} = 0.81694.$$

(Note: this last integral can be evaluated precisely by the method of integration by parts, which we will get to in about a week; it turns out that the exact value is $-1/(2\ln 2) + 3/(2\ln 2)^2 = 0.83968$.) Finally, $\overline{x} = 0.83968/1.08202 = 0.77603$.

EXAMPLE 10.2. Sketch the curve $f(t) = e^{-t/T}\cos(\omega t)$ for $t > 0$ (where ω and T are constants). For simplicity, take $T = 1$ sec.

To get an idea of what to expect, let's sketch the two functions whose product is $f(t)$. The graph of $y = e^{-t}$ is obtained by reflecting the graph of $y = e^t$ (shown above) about the y-axis. The graph of $\cos(\omega t)$ is obtained by squeezing the graph of $\cos t$ by a factor of ω toward the y-axis (e.g., horizontal contraction by a factor of 5 if $\omega = 5$, horizontal expansion by a factor of 5 if $\omega = 1/5$). The graphs of $y = e^{-t}$ and $y = \cos(5t)$ are shown below:



The effect of multiplying $\cos(\omega t)$ (which oscillates between 1 and -1) by e^{-t} (which steadily decreases from 1 to 0) is to obtain a function which oscillates between maximum and minimum values (*local maxima and minima*) that are steadily decreasing. This type of function $f(t)$, called "damped harmonic oscillation," arises in practice, for example, if you want to describe the motion of an object at the end of a spring which is bouncing back and forth with amplitude that decreases because of friction and air resistance.

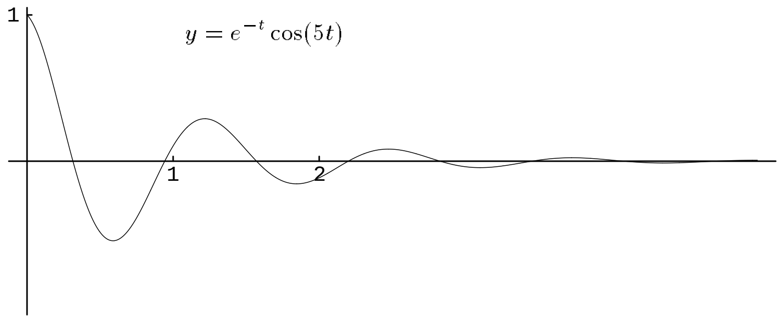
Now let us determine the t-values for which our function $f(t)$ has max/min points and points of inflection. To find the max/min points, we set $f'(t) = 0$:

$$0 = \frac{d}{dt}e^{-t}\cos(\omega t) = -e^{-t}\cos(\omega t) + e^{-t}(-\omega\sin(\omega t)) = -e^{-t}(\cos(\omega t) + \omega\sin(\omega t)).$$

Since e^{-t} is never 0, this derivative vanishes when $\cos(\omega t) + \omega\sin(\omega t) = 0$, i.e. (if we divide through by $\omega\cos(\omega t)$), when $\tan(\omega t) = -1/\omega$. Thus, our solutions t are all positive values (in radians) of $\arctan(-1/\omega)$, divided by ω. For example, if $\omega = 5$, then we successively obtain a minimum at $t = 0.589$, a maximum at $t = 1.217$, a minimum at $t = 1.845$, a maximum at $t = 2.474$, and so on. To find the inflection points, we set $f''(t) = 0$, i.e. (after a little simplification):

$$e^{-t}\big((1-\omega^2)\cos(\omega t) + 2\omega\sin(\omega t)\big) = 0.$$

Thus, the points of inflection occur when $t = \frac{1}{\omega}\arctan\left(\frac{\omega^2-1}{2\omega}\right)$. For example, if $\omega = 5$, then $t = 0.2\arctan 2.4 = 0.235, 0.864, 1.492, 2.120$, etc. The curve $y = e^{-t}\cos(5t)$ is sketched at the top of the next page.


1
$y = e^{-t}\cos(5t)$
1
2

HOMEWORK 9.

1. (a) Let l_1 be the tangent line to the reciprocal function at the point $(2, 0.5)$. Find the area under l_1 between $x = 1$ and $x = 3$.

(b) Let l_2 be the chord joining $(1, 1)$ to $(2.4, 1/2.4)$ on the graph of the reciprocal function. Find the area under l_2.

(c) Explain how your answers to parts (a) and (b) imply that the number e must be between 2.4 and 3.

2. Simplify the following expressions:

(a) $e^{\ln 3+\ln x}$; (c) $e^{\ln x-2\ln y}$; (e) $\ln\left(\sqrt{e^{x^2}}\right)$.

(b) $\ln\left(1/e^{x^2}\right)$; (d) $e^{-\ln(1/x)}$;

3. Solve for y in terms of x:

(a) $e^{0.1y} = x^2$; (c) $x = \ln\sin y$; (e) $\ln(1-y) - \ln y = 0.01x$.

(b) $\ln(y-x) = 2\ln x$; (d) $e^{y^2} = x^2$;

4. Find the derivative of

(a) $e^{2x}\text{Arctan}\, x$; (b) $\dfrac{x}{e^{\sin(\pi x)}}$; (c) e^{e^x}; (d) $x^2e^{-\cos x}$; (e) $e^{\text{Arcsin}\, x}$.

5. The function obtained by averaging the function e^x and the function e^{-x}, i.e., $(e^x+e^{-x})/2$, comes up often enough in applications to have earned the right to have its own name: "hyperbolic cosine." The function $(e^x - e^{-x})/2$ is called the "hyperbolic sine." These are abbreviated "cosh" and "sinh" ("sinh" is pronounced "sinch"):

$$\cosh x = \frac{e^x + e^{-x}}{2}; \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(a) Find derivative formulas for cosh and sinh similar to the derivative formulas for cos and sin. (b) Simplify $\cosh^2 x - \sinh^2 x$. (c) Sketch the hyperbolic sine and cosine.

6. Evaluate:

$$\text{(a)} \int_{\ln 0.04}^{\ln 4} e^{x/2}dx, \qquad \text{(b)} \int_0^1 xe^{-x^2}dx, \qquad \text{(c)} \int_0^\pi e^{\cos x}\sin x\, dx, \qquad \text{(d)} \int_0^\infty \frac{dt}{10^t}.$$

7. Find the volume of the solid obtained by rotating the shape in Example 10.1 in the text around the x-axis.

8. Find the x- and y-coordinates of the maxima/minima and points of inflection, and sketch:

$$\text{(a) } y = e^{\frac{2}{3}\sin(3x)},\ 0 \le x \le \pi/3; \qquad \text{(b) } y = x^2\ln(1/x),\ 0 < x < 1.$$

CHAPTER 11

Exponential Growth and Decay

1. Continuously Compounded Interest

Let's return to the bank interest problem and suppose a more realistic annual interest rate of k (this is a percent written in decimal form, for example 8%= 0.08 — in this problem and many that follow, the units of k are reciprocal time, i.e., $0.08\frac{1}{\text{yr}}$ annual interest rate). We again want to find out how our principal P grows in t years. We again assume continuous compounding. But this time we shall use a different method to derive the function $y(t)$ that tells the amount of money in the bank after t years.

The method is to set up and solve a certain " differential equation." Recall that a **differential equation** is an equation for finding an unknown function in which the derivative of the unknown function (and perhaps its second derivative or higher derivatives) appears. For example, (1) the differential equation $y' = f(x)$, where $f(x)$ is a given function, is satisfied by an antiderivative of $f(x)$; (2) the differential equation $\ddot{y} = -g$ is satisfied by the falling body formula $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$.

Differential equations are fundamental in applications, because often we have to get at a function $y(t)$ indirectly, using some information about the rate of change of $y(t)$ that comes from translating some basic principle (of physics or finances, for instance) into mathematics. For example, the differential equation $\ddot{y} = -g$ is simply the statement that gravity near the surface of the earth imparts a constant acceleration.

In the case of bank interest continuously compounded at rate k, the basic principle is that *at any instant interest is accumulating at a rate in dollars per year equal to k times the current amount in the bank.* For example, if $k = 8\% = 0.08$ and you put \$1000 in the bank, then at first your money increases at the rate of \$80.00 per year; but a little while later, when you have (say) \$1010 in the bank, at that instant it is increasing at the rate of \$80.80 per year. That is, you are "earning interest on the interest" as well as on the original principal. The basic principle in italics can be written mathematically as follows:

$$\frac{dy}{dt} = ky,$$

i.e., the function $y(t)$ telling the number of dollars in the bank at time t years increases at a rate equal to k times the present value of this function.

Our task now is to solve this equation to find an expression for y in terms of t. A useful technique in this and many other differential equations is called " separation of variables." This means that we get everything involving y on one side of the equation and everything involving t on the other. Here the dy and dt are handled just like algebraic quantities. (When discussing the chain rule and also the method of substitution in integrals, we saw that it often makes sense to split up expressions like dy/dx or dy/dt, treating dy, dx, dt separately.) In the above differential equation we separate variables by multiplying both sides by dt and dividing both sides by y. After that the second step is to put an integral sign on both sides:

$$\int \frac{dy}{y} = \int k\,dt.$$

On the left we have an indefinite integral with respect to the variable of integration y, and on the right we have an indefinite integral with respect to t. On the left we get $\ln|y| + C_1$. Since $y(t)$ is a positive function, we drop the absolute value sign. Also, we move the constant of integration to the right of the equality, subtracting it from the constant of integration we get on the right from integrating $\int k\,dt = kt + C_2$. That is, we set $C = C_2 - C_1$. Thus, we have:

$$\ln y = kt + C.$$

Finally, to solve for y we exponentiate both sides (this means: raise e to the expression on each side of the equality). Since $e^{\ln y} = y$, this gives us a formula for y:

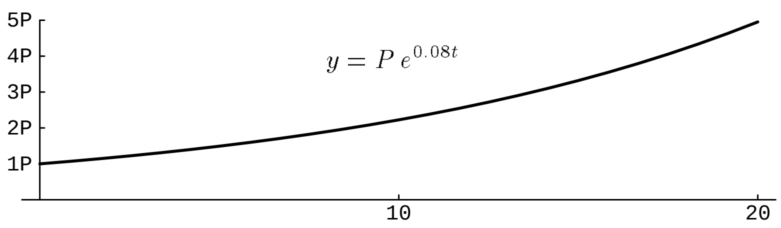
$$y(t) = e^{kt+C} = e^C e^{kt}.$$

We are not done, however, because we must determine the constant e^C in front of e^{kt}. As in acceleration–velocity–distance problems, we do this by using initial information. In the bank interest problem, the initial information is that we had P dollars in the bank at time $t = 0$: $y(0) = P$. Substituting $t = 0$ in the equation $y(t) = e^C e^{kt}$ gives the result: $P = y(0) = e^C e^0 = e^C$. Thus, we conclude that

$$y(t) = Pe^{kt}.$$

(The special case of this formula when $k = 100\% = 1$ was obtained in the last section by a different method.)

A function of the form $y = Pe^{kt}$ is called **exponential growth**. Its graph is shown below (in the case $k = 8\% = 0.08$).



Effective annual interest rate. By the **effective** interest rate we mean the percent by which the money actually increases if left in the bank 1 year. The increase is from P to $Pe^{k\cdot 1}$, i.e., $P(e^k - 1)$. Thus, the percent increase is $e^k - 1$ (which we usually convert to a percent by multiplying by 100):

$$\text{effective annual interest rate} = e^k - 1.$$

For example, if $k = 0.08$, then the effective rate is $e^{0.08} - 1 = 0.0833 = 8.33\%$. If $k = 100\%$, then the effective rate is $e - 1 = 171.83\%$. In general, if k is small, then the effective rate is only slightly more than k; while if k is large, then the effective rate is much greater, because it takes into account the interest earned on the interest. Sometimes people say "annual interest rate" when they really mean "effective annual interest rate." Sometimes k is called the **instantaneous interest rate**.

Note. *Whenever we say simply "interest rate" we shall mean the instantaneous rate k, not the effective rate $e^k - 1$.*

Doubling time. At time t we have Pe^{kt} in the bank. Let us ask how long it will take for the money to double. This time period, which we call D, must satisfy the equation $Pe^{k(t+D)} = 2Pe^{kt}$. Dividing both sides by Pe^{kt} and then taking the ln of both sides, we obtain $kD = \ln 2$, i.e., $D = (\ln 2)/k$. Notice that the doubling time D does not depend on the starting time t or on the principal P: it depends only on the interest rate. For example, at 8% interest rate the doubling time is $(\ln 2)/0.08 = 8\ 2/3$ years.

2. Other Examples of Exponential Growth.

Any time we have something whose rate of increase is proportional to its present value (i.e., $\frac{dy}{dt} = ky$), we obtain the exponential function $y(t) = y_0e^{kt}$, where k is the constant of proportionality between the rate of increase and the present value, and y_0 is the initial value of our function. Among the examples of this situation are:

Inflation. Rate of increase in cost of living is measured as an annual percent of the current cost of living. If the rate of inflation remains constant, then this cost of living follows the law of exponential growth.

Example 11.1. In the six-month period from June to December 1990 in Nicaragua, the córdoba deflated from an exchange rate of about 200000 per dollar to 2000000 per dollar. Find both the effective annual rate and the instantaneous rate k of inflation of the price in córdobas of an item whose dollar cost remains constant.

Here the inflation was by a factor of 10 in 1/2 year, and hence by a factor of 100 in 1 year. So the cost increased by $99 = 9900\%$. Thus, the effective rate is $99 = e^k - 1$. To find k, we add 1 to both sides and take ln, obtaining $k = \ln 100 = 4.605 = 460.5\%$.

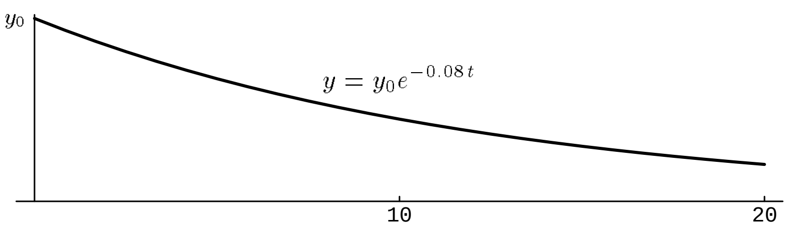
Population. If the birth and death rates remain constant (and there is no immigration or emigration), then the population as a function of time follows the law of exponential growth, in which k is the difference: $k =$ birth rate $-$ death rate.

Example 11.2. If a bacteria culture has a 4% per hour instantaneous birth rate and a 2.5% instantaneous death rate, what is its doubling time?

Here $k = 4\% - 2.5\% = 1.5\%$, and so $D = (\ln 2)/0.015 = 46.2$ hours.

3. Exponential Decay

Here we have a function $y(t)$ which **decreases** at a rate proportional to its current value, with proportionality constant k: $\frac{dy}{dt} = -ky$. Mathematically speaking, nothing much changes: to solve this differential equation we simply replace k by $-k$ in our earlier derivation, thereby obtaining the function $y(t) = y_0e^{-kt}$. However, the graph of y_0e^{-kt} as t increases looks very different from the graph of y_0e^{kt}, namely, it ends up decreasing rapidly toward 0:



Half-life. An exponential decay function $y(t) = y_0 e^{-kt}$ has a " half-life" rather than a doubling time. This is the length of time H that it takes to decrease to half its current value. To find H, we set $y_0 e^{-k(t+H)}$ equal to half of $y_0 e^{-kt}$. Dividing this equation through by $y_0 e^{-kt}$ and then taking ln gives: $-kH = \ln(1/2) = -\ln 2$, and so $H = (\ln 2)/k$. Note that this is the same as the doubling time formula in the case of exponential growth.

4. Examples of Exponential Decay.

Radioactive decay.

EXAMPLE 11.3. Some future visitors to Earth find a planet that was rendered uninhabitable by a catastrophe that occurred many thousands of years before. By reconstructing the accident, they determine how much radioactive Plutonium 239 was released at that time. They also measure the amount of Plutonium 239 remaining, and find that only 1/500 of the radioactive Plutonium is still present. Using the value 24400 years for the half-life of Plutonium 239, determine how much time elapsed between the accident and the arrival of the visitors.

We first find k, using the relation $kH = \ln 2$: $k = \ln 2/24400 = 0.0000284$. Then we find t by setting $y(t) = \frac{1}{500} y_0$, i.e., $\frac{1}{500} y_0 = y_0 e^{-kt}$. Dividing by y_0 and taking ln gives $-kt = \ln(1/500) = -\ln 500$, so that $t = (\ln 500)/0.0000284 = 218800$ years.

Population when the death rate exceeds the birth rate.

"Real" salary (measured in inflation-adjusted dollars, i.e., buying power of what you earn) if the rate of salary increase is less than the inflation rate.

Finally, we should emphasize that real-world phenomena follow the law of exponential growth or decay only if the thing being measured satisfies the differential equation $y' = ky$ or $y' = -ky$. In practice, when we suppose this to be the case we are making a simplifying assumption that may be quite inaccurate. For example, most human or animal populations are influenced by many factors and cannot be described at all accurately by the exponential growth model.

Homework 10.

1. Suppose that interest on money in the bank accumulates at an annual rate of 7% compounded continuously. How much should be invested today so that 20 years from now it will be worth $20000? B

2. Suppose that a sum of money draws interest compounded continuously. How long will it take to double if the annual rate is (a) 6%?, (b) 10%?, (c) 18%? How long will it take to double if the effective annual rate is (a) 6%?, (b) 10%?, (c) 18%?

3. In an experiment to study the growth of bacteria, a medical student measured 5000 bacteria at time 0 and 8000 at time 10 minutes. Assuming that the number of bacteria grows exponentially, how many bacteria will be present after 30 minutes?

4. Suppose that the population of a certain species grows at the instantaneous rate of 2% per year. If the present population is 1000000, what will the population be in 1 year? In 20 years? How long will the population take to double?

5. Assume that the population of the Earth changes at a rate proportional to its current population. Assume further that in A.D. 1650 its population was 250 million, and in A.D. 1950 its population was 2.5 billion. Find an expression giving the population of the Earth at any time. If the greatest population the Earth can support is 25 billion, when will this limit be reached?

6. If you hear on the news that prices are rising exponentially at the effective yearly rate of 12%, what is the instantaneous annual rate of inflation? How long does it take prices to double at this rate?

7. Suppose that the average yearly cost of college in the U.S. (tuition and expenses) is increasing at the instantaneous rate of 10% per year. By what factor will the cost of college increase for a child born now in the 18 years before she goes to college?

8. Suppose that bank interest is compounded continuously at 6%, and inflation on food has instantaneous rate 13%. (a) If you put $10000 in the bank, how much will you have 100 years later? (b) If it now costs $100/week to feed an average family, write a formula for weekly food cost t years later. (c) What is the **total** cost of feeding a family over the next 10 years? (d) What will the weekly food bill be in 100 years? (e) If your great grandchild takes your money in part (a) out of the bank at that time, how long will it feed his or her family?

9. The half-life of Carbon 14 is 5568 years. Archeologists find an ancient table made from a tree which had y_0 grams of Carbon 14 when it was cut. Now the wood has only y_1 grams of Carbon 14. Express the age of the table in terms of y_0 and y_1.

10. A certain radioactive material has a half-life of 2 hours. Find the time needed for a given amount of this material to decay to one-tenth of its original mass.

11. A certain radioactive material is known to decay at a rate proportional to the amount present. A block of this material originally having a mass of 100 grams is observed after 20 years to have a mass of only 80 grams. Find an expression for the mass of the material at any time. Also find the half-life of the material.

12. Suppose that the average yearly real income (i.e., in inflation-adjusted dollars, say 1995-dollars) of a U.S. worker is now decreasing at the instantaneous rate of 1.5% per year, and that this trend continues indefinitely. If the current average salary is $18000, what will an average worker's yearly income (in 1995-dollars) be in 30 years?

13. You are flying in a plane at constant altitude y_0, constant speed v_0 and constant direction. You want to drop a container of relief supplies to a target that is straight ahead under your flight path. You will release the container at the instant when your line of sight to the target makes an angle α below the horizontal. The purpose of this problem is to determine α. Let $v(t)$ denote the horizontal component of the velocity of the container. Assume that air resistance causes a horizontal component of deceleration that is proportional to velocity, with proportionality constant k. Air resistance is the only force acting to change the velocity (since gravity does not affect horizontal motion).

(a) Write a differential equation for $v(t)$, and find a formula for $v(t)$.

(b) Find a formula for the horizontal distance traveled by the container between time 0 and time t.

(c) Suppose that after enough time has elapsed so that the exponential decay term in part (c) is negligible, a parachute opens and the container glides straight down until it hits the ground. Find α.

CHAPTER 12

Techniques of Integration I: Substitution, Powers of Trig Functions

1. Substitution

By a **technique of integration** we mean a method which can be used to find the antiderivative (indefinite integral) of certain types of functions. We first review u-substitutions. We shall use an example to illustrate the thought process that goes into deciding on a substitution of the form $u = u(x)$.

EXAMPLE 12.1. Find the x-coordinate of the center of mass of the uniform plate that is bounded by the x- and y-axes, the line $x = 10$, and the curve $y = 1/(1 + 0.01x^2)$.

We must compute $\overline{x} = \int_0^{10} x\,dx/(1 + 0.01x^2) \Big/ \int_0^{10} dx/(1 + 0.01x^2)$. At first the top and bottom integrals look similar: both have $(1 + 0.01x^2)$ in the denominator. However, they are actually quite different.

In the bottom we have something very close to $\int du/(1 + u^2)$ =Arctan $u + C$. The only problem is the constant 0.01 in front of x^2. So we choose u so that $u^2 = 0.01x^2$, i.e., $u = 0.1x$. Then $x = 10u$, $dx = 10du$. The limits of integration also change: $x = 0 \implies u = 0$, and $x = 10 \implies u = 0.1 \cdot 10 = 1$. Hence, the bottom integral becomes $\int_0^1 10\,du/(1 + u^2) = 10 \operatorname{Arctan} u \,\Big|_0^1 = 10(\pi/4) = 2.5\pi$.

The top integral, on the other hand, has nothing to do with Arctan. The x next to the dx changes everything. Namely, x is essentially the derivative of the denominator. Thus, a substitution u =denominator will lead to something like $\int du/u = \ln|u| + C$. More precisely, we set $u = 1 + 0.01x^2$, so that $du = 0.02x\,dx$, and hence $x\,dx = 50\,du$. This gives us $50\,du/u$ inside the integral. The limits of integration change as follows: $x = 0 \implies u = 1 + 0.01 \cdot 0 = 1$, and $x = 10 \implies u = 1 + 0.01 \cdot 10^2 = 2$. Thus, the top integral becomes $50 \int_1^2 du/u = 50 \ln 2$. Finally, we compute $\overline{x} = 50 \ln 2/2.5\pi = 4.4127$.

2. Odd Powers of Sine and Cosine

The technique of u-substitution can be used to evaluate any integral of the form

$$\int \sin^n(ax + b)\,dx \quad \text{or} \quad \int \cos^n(ax + b)\,dx, \text{ where } n \text{ is a positive odd number.}$$

The method works as follows. Say we want to integrate $\sin^n$. First take out one sin and put it with the dx. What remains will be sin to an *even* power. Express this in terms of cos (using $\sin^2 = 1 - \cos^2$). Now make a u-substitution for the cos. You will end up with only a polynomial to integrate. (If we had $\cos^n$ to integrate, we would do the same thing, with the roles of sin and cos reversed.) We illustrate with an example.

EXAMPLE 12.2. Find $\int_0^1 \sin^5(\pi x)dx$.

We first write

$$\int_0^1 \sin^5(\pi x)dx = \int_0^1 \sin^4(\pi x)\sin(\pi x)dx = \int_0^1 \left(1-\cos^2(\pi x)\right)^2 \sin(\pi x)dx.$$

Next, we make the substitution $u = \cos(\pi x)$, so that $du = -\pi\sin(\pi x)$. Then $\sin(\pi x)dx = -\frac{1}{\pi}du$, and so our integral becomes $-\frac{1}{\pi}\int\left(1-u^2\right)^2du$. The limits of integration change as follows: $x = 0 \Longrightarrow u = \cos 0 = 1$, and $x = 1 \Longrightarrow u = \cos\pi = -1$. To integrate the polynomial, we expand the square: $(1-u^2)^2 = 1 - 2u^2 + u^4$. So we obtain:

$$-\frac{1}{\pi}\int_1^{-1} 1 - 2u^2 + u^4\,du = -\frac{1}{\pi}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_1^{-1} = \frac{16}{15\pi}.$$

It is also possible to use this method to integrate *negative* odd powers of sine and cosine. However, in that case instead of a polynomial in u you end up with a *rational function* (i.e., you have a denominator with a polynomial in it). We will treat some cases of this in part III of techniques of integration.

3. Even Powers of Sine and Cosine

The above technique does not work if you want to integrate something like $\cos^2$ or $\sin^4$. To handle even powers one needs a trig identity, namely, the $\cos(2A)$ formula: $\cos(2A) = \cos^2 A - \sin^2 A$. When combined with the trig identity $1 = \cos^2 A + \sin^2 A$, this gives formulas for either $\cos^2 A$ or $\sin^2 A$ in terms of $\cos(2A)$. If we add these two identities and divide by 2, we get $\cos^2 A$; if we subtract them and divide by 2, we get $\sin^2 A$:

$$\begin{array}{rl} & \cos^2 A + \sin^2 A = 1 \\ + & \underline{\cos^2 A - \sin^2 A} = \underline{\cos(2A)} \\ & 2\cos^2 A = 1 + \cos(2A) \end{array} \qquad \begin{array}{rl} & \cos^2 A + \sin^2 A = 1 \\ - & \underline{\cos^2 A - \sin^2 A} = \underline{\cos(2A)} \\ & 2\sin^2 A = 1 - \cos(2A)\,, \end{array}$$

i.e.,

$$\cos^2 A = \frac{1+\cos(2A)}{2}, \qquad \sin^2 A = \frac{1-\cos(2A)}{2}.$$

We now illustrate the use of these two identities in integration problems.

EXAMPLE 12.3. Find the average value of $\sin^2 x$ between 0 and π.

The average value of a function $f(x)$ on the interval $[a, b]$ is the integral of the function divided by the length of the interval: $\frac{1}{b-a}\int_a^b f(x)dx$. So the average value of $\sin^2 x$ is

$$\begin{aligned} \frac{1}{\pi}\int_0^\pi \sin^2 x\,dx &= \frac{1}{\pi}\int_0^\pi \left(\frac{1-\cos(2x)}{2}\right)dx \\ &= \frac{1}{\pi}\left(\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin(2x)\right)\Big|_0^\pi \\ &= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{1}{4}(0-0)\right) = \frac{1}{2}. \end{aligned}$$

EXAMPLE 12.4. Find the antiderivative of $\cos^4 x$.

We have

$$\int \cos^4 x\,dx = \int \left(\cos^2 x\right)^2\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)^2\,dx$$
$$= \int \left(\frac{1+2\cos(2x)+\cos^2(2x)}{4}\right)\,dx.$$

We now use the identity for $\cos^2(A)$ once more, this time with $A = 2x$: $\cos^2(2x) = (1 + \cos(4x))/2$. In this way we end up replacing all powers of cosine with expressions of the form $\cos(ax)$, which are easy to integrate. After writing

$$1 + 2\cos(2x) + \cos^2(2x) = 1 + 2\cos(2x) + (1+\cos(4x))/2$$
$$= \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)$$

and dividing this by 4, we obtain

$$\int \cos^4 x\,dx = \int \left(\frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right)\,dx$$
$$= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

HOMEWORK 11.

1. Find the x-coordinate of the center of mass of the flat plate bounded by the x- and y-axes, the line $x = 5$, and the curve $y = 1/\sqrt{1 - 0.01x^2}$.

2. Find the anti-derivative of
(a) $\sin(2\pi t)/\sqrt{2 + \cos(2\pi t)}$;
(b) $\cos(ax + b)/\sin^8(ax + b)$.

3. Find the volume of the solid obtained by revolving $y = \cos x$ around the x-axis between $x = 0$ and $x = \pi/4$.

4. Find the average value of $\sin^n x$ between 0 and $\pi/2$ when n is
(a) 3, (b) 4, (c) 5, (d) 6, (e) 7.

CHAPTER 13

Techniques of Integration II: Integration by Parts

A general principle is that any derivative rule has a corresponding integration rule. For example, a derivative formula such as $\frac{d}{dx}\sin x = \cos x$ has a corresponding antiderivative formula $\int \cos x\,dx = \sin x + C$. The u-substitution technique of integration comes from the chain rule for differentiation: if you check an antiderivative obtained by u-substitution in order to be sure that its derivative is the function you wanted to integrate, you use the chain rule. For example, in the top integral of Example 12.1 in the last section, to check that $50\ln u = 50\ln(1+0.01x^2)$ is the antiderivative of $x/(1+0.01x^2)$, you use the chain rule:

$$\frac{d}{dx}50\ln u = \frac{50}{u}\frac{du}{dx} = \frac{50}{1+0.01x^2}(0.02x) = \frac{x}{1+0.01x^2}.$$

The product rule for derivatives also gives rise to a very useful integration rule, called **integration by parts**. To get this rule, we recall the product rule, which says that the derivative of $u(x)v(x)$ is $u(x)v'(x) + u'(x)v(x)$. We shall omit the (x), and write simply $(uv)' = uv' + u'v$. Now solve this equality for uv', and then take the antiderivative of both sides. Since the antiderivative of $(uv)'$ is just uv, we obtain: $\int uv' = uv - \int u'v$. This is the integration by parts rule in shorthand form. Written out completely, it is:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx \qquad \text{(indefinite integral form)}.$$

For definite integrals, we put in the limits of integration:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)dx \qquad \text{(definite integral form)}.$$

At first, this might seem to be useless: the rule says that our integral is equal to uv minus some other integral, and so we still have an integral to compute. However, for an important class of functions $f(x)$ it is possible to choose $u(x)$ and $v(x)$ in such a way that $f(x) = u(x)v'(x)$ and the integral on the right — namely, $\int u'(x)v(x)dx$ — is easier to evaluate than $\int f(x)dx$. This section is devoted to explaining how this is done. We first give a series of examples, and then we make up a table summarizing the main uses of integration by parts.

Example 13.1. Find $\int x\cos x\,dx$, and then find the x-coordinate of the center of mass of the flat plate bounded by the x- and y-axes and the cosine curve in the first quadrant.

We choose $u(x) = x$ and $v'(x) = \cos x$. The first thing we must do is find $v(x)$, which is an antiderivative of $\cos x$. So we take $v(x) = \sin x$. In an integration by parts problem, $v'(x)$ must always be a function whose antiderivative you can find right away. (The integration by parts would work just as well if we took any other antiderivative of the form $\sin x + C$; but to avoid unnecessary work we shall always choose $v(x)$ by setting the constant of integration equal to 0.) Using $u(x) = x$ and $v(x) = \sin x$ in the integration by parts rule, we obtain:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx = x \sin x - (-\cos x) + C = x \sin x + \cos x + C.$$

It is instructive to check our answer by taking the derivative of $x \sin x + \cos x$. As always in an integration by parts problem, our answer has two parts: a product of two functions (namely $x \sin x$) and another function which can be thought of as a "correction term" (namely, $\cos x$). When we take the derivative of the product, the product rule gives us two terms. One of them is our original function inside the integral, and the other is a term we don't want. The derivative of the "correction term" precisely cancels out the term we don't want. We see how this works when we check our answer to Example 13.1:

$$\frac{d}{dx}(x \sin x + \cos x) = (x \cos x + 1 \cdot \sin x) - \sin x = x \cos x.$$

Finally, $\overline{x}$ in this example is equal to

$$\int_0^{\pi/2} x \cos x \, dx \Big/ \int_0^{\pi/2} \cos x \, dx = (x \sin x + \cos x) \Big|_0^{\pi/2} \Big/ \sin x \Big|_0^{\pi/2}$$
$$= (\pi/2 - 1)/1 = \frac{\pi}{2} - 1 = 0.5708.$$

Example 13.2. Find $\int xe^{5x} dx$.

Again choose $u(x) = x$. Let $v'(x) = e^{5x}$, so that $v(x) = \frac{1}{5}e^{5x}$. Then

$$\int xe^{5x} dx = x \cdot \frac{1}{5}e^{5x} - \int 1 \cdot \frac{1}{5}e^{5x} dx = \frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x} dx = \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x} + C.$$

Example 13.3. Find the improper integral $\int_0^\infty x^{10}e^{-x} dx$.

We shall apply the definite integral version of integration by parts. Having upper limit of integration ∞ means that we can take a finite upper limit of integration b and then let $b \longrightarrow \infty$.

Before doing this problem, we shall digress for a minute, and show that $x^n e^{-kx}$ always approaches 0 as $x \longrightarrow \infty$, for any n and any positive constant k. Look at the ln of this expression: $\ln(x^n e^{-kx}) = \ln(x^n) + \ln(e^{-kx}) = (n \ln x) - kx$. That is, we get a constant multiple of $\ln x$ minus a constant multiple of x. When x is large $\ln x$ is much smaller than x, and so eventually this difference must approach $-\infty$. Hence, our original expression $x^n e^{-kx}$, which is $e^{n \ln x - kx}$, is e to a power that approaches $-\infty$, and this approaches zero, as claimed.

We shall apply this limit fact when we evaluate expressions of the form $x^n e^{-x}$ at infinity (i.e., at b and then take $b \longrightarrow \infty$). That is, when we evaluate $x^n e^{-x}$ at infinity we get 0. In our integral we take $u(x) = x^{10}$ and $v'(x) = e^{-x}$, so that $v(x) = -e^{-x}$. Then

$$\int_0^\infty x^{10}e^{-x} \, dx = -x^{10}e^{-x} \Big|_0^\infty - \int_0^\infty 10x^9 \left(-e^{-x}\right) \, dx = 0 - 0 + 10\int_0^\infty x^9 e^{-x} \, dx.$$

The last integral is like the one we started with; however, the power of x has been reduced by 1. So we must apply integration by parts again, this time with $u(x) = x^9$. The result is:

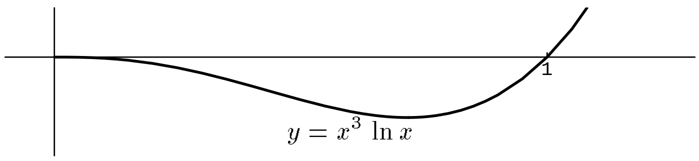
$\int_0^\infty x^9 e^{-x}dx = 9\int_0^\infty x^8 e^{-x}dx$. Continuing in this way, we obtain

$$\int_0^\infty x^{10}e^{-x}dx = 10\int_0^\infty x^9 e^{-x}dx = 10\cdot 9\int_0^\infty x^8 e^{-x}dx$$
$$= \cdots = 10\cdot 9\cdot 8\cdots 2\cdot 1\cdot \int_0^\infty e^{-x}dx.$$

But $\int_0^\infty e^{-x}dx = -e^{-x}\Big|_0^\infty = 1$, so that our final answer is $10{\cdot}9{\cdot}8{\cdot}7{\cdot}6{\cdot}5{\cdot}4{\cdot}3{\cdot}2{\cdot}1 = 3628800$. (The product of all numbers from 1 to 10 is called "10 factorial," and is written 10!.)

Example 13.4. Find $\int_0^1 x^3 \ln x\, dx$.

First note that the function $x^3\ln x$ is negative between 0 and 1 (because $\ln x$ is negative there), has value 0 when $x = 1$, and approaches the limit 0 as $x \longrightarrow 0$. (For the latter fact, see Problem 9(a) in the section on ln.) Thus, the function we're integrating in Example 13.4 has the following appearance:



We now use integration by parts. If we try to proceed as in Example 13.3, setting $u(x)$ equal to x^3, we run into a problem: we would then have to take $v'(x)$ equal to $\ln x$, and we do not yet know an antiderivative of $\ln x$. In an integration problem with $\ln x$, it makes sense to reverse our choice of u and v. That is, we set $u(x) = \ln x$ and $v'(x) = x^3$, so that $v(x) = x^4/4$. This gives us:

$$\int_0^1 x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x\,\Big|_0^1 - \int_0^1 \frac{1}{4}x^4\cdot\frac{1}{x}\,dx.$$

Notice what happened in the integral at the right: we have nothing more than a power of x to integrate. The point is that integration by parts can be viewed as a trade-off, where in the integral on the right you get to replace u by its derivative u' (which should be a simpler function), but at the expense of replacing v' by its antiderivative v. In this problem we had to replace x^3 by a higher power x^4, but in return we got to replace $\ln x$ by the much simpler function $1/x$ which is its derivative. Returning to our computation, we find that $x^4\ln x$ gives 0 when evaluated at either 1 or 0, and hence

$$\int_0^1 x^3\ln x\, dx = -\frac{1}{4}\int_0^1 x^3\, dx = -\frac{1}{4}\cdot\frac{1}{4} = -\frac{1}{16}.$$

Example 13.5. Find the antiderivative of $\ln x$.

At first it might seem that the integral $\int \ln x\, dx$ should have nothing to do with integration by parts, since the function inside the integral is not a product. However, we can treat this integral in the same way as we treated $\int x^3\ln x\, dx$ in Example 13.4. Namely, we view $\ln x$ as the product $1\cdot\ln x = x^0\ln x$, and we set $u(x) = \ln x$ and $v'(x) = 1$, so that $v(x) = x$. Then:

$$\int \ln x\, dx = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - \int 1\, dx = x\ln x - x + C.$$

EXAMPLE 13.6. Find the antiderivative of arcsin x.

This antiderivative (and also the antiderivative of Arctan) can be computed in the same way as the antiderivative of $\ln x$ in Example 13.5. Namely, we take $u(x) = \arcsin x$, $v'(x) = 1$, $v(x) = x$, so that $u'(x) = 1/\sqrt{1-x^2}$, and

$$\int \arcsin x\, dx = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} dx = x \arcsin x - \int \frac{x\, dx}{\sqrt{1-x^2}}.$$

The last integral can be evaluated using a u-substitution. That is, for the rest of the problem we forget about integration by parts, and use the letter u for an entirely different purpose. Namely, if we set u equal to the expression under the radical, we see that du is essentially what we have in the numerator. More precisely, $u = 1 - x^2$, $du = -2xdx$, so that

$$\begin{aligned}\int \arcsin x\, dx &= x \arcsin x + \int \frac{du}{2\sqrt{u}} \\ &= x \arcsin x + \sqrt{u} + C = x \arcsin x + \sqrt{1-x^2} + C.\end{aligned}$$

The reader should check this answer by taking its derivative and seeing that, after cancelation, you get arcsin x.

Before doing more examples, we give a table that summarizes the most common uses of integration by parts. They all relate to functions which are the product of a power of x and either an exponential function, sine or cosine, or else one of the inverse functions ln, Arcsin, Arctan.

1. How to Choose u and v

I. In the following cases choose $u(x) = x^n$ and $v'(x)$ =the other function. Here n is a positive integer, and k, a, and b are constants.

$$\int f(x)dx, \text{ where } f(x) \text{ is a product } x^n \cdot \begin{cases} e^{kx}, \\ \sin(ax+b), \\ \cos(ax+b). \end{cases}$$

II. In the following cases choose $v'(x) = x^n$ and $u(x)$ = the other function. Here n is a positive integer or 0, m is a positive integer, and a and b are constants.

$$\int f(x)dx, \text{ where } f(x) \text{ is a product } x^n \cdot \begin{cases} \ln(ax+b), \\ \ln^m(ax+b), \\ \arcsin(ax+b), \\ \arctan(ax+b). \end{cases}$$

One type of problem in part II which has not yet occurred in the examples is $x^n \cdot \ln^m(ax+b)$. We now give an example of this.

EXAMPLE 13.7. Suppose that the shape in Example 13.4 is revolved around the x-axis to form a solid. Find its volume.

Since $\left(x^3 \ln x\right)^2 = x^6 \ln^2 x$, we choose $u(x) = \ln^2 x$ and $v'(x) = x^6$. Then $v(x) = x^7/7$, and $u'(x) = 2(\ln x)/x$. We have:

$$\begin{aligned}\text{Vol} &= \pi \int_0^1 x^6 \ln^2 x \, dx = \pi \left(\frac{1}{7} x^7 \ln^2 x \, \Big|_0^1 - \int_0^1 \frac{1}{7} x^7 \cdot 2(\ln x) \frac{1}{x} \, dx \right) \\ &= \pi \left(0 - 0 - \frac{2}{7} \int_0^1 x^6 \ln x \, dx \right).\end{aligned}$$

We now have to use integration by parts a second time. (In general, to integrate $x^n \ln^m x$, you'll need to use integration by parts m times, each time lowering the power of $\ln x$ by one.) We obtain:

$$\begin{aligned}\text{Vol} &= -\frac{2\pi}{7} \int_0^1 x^6 \ln x \, dx = -\frac{2\pi}{7} \left(\frac{1}{7} x^7 \ln x \, \Big|_0^1 - \int_0^1 \frac{1}{7} x^7 \cdot \frac{1}{x} \, dx \right) \\ &= \frac{2\pi}{7} \cdot \frac{1}{7} \int_0^1 x^6 \, dx = \frac{2\pi}{343} = 0.01832.\end{aligned}$$

EXAMPLE 13.8. Find both coordinates of the center of mass of the flat plate bounded by the x- and y-axes and the curve $y = e^{-x}$ in the first quadrant (out to infinity).

The denominator in $\overline{x}$ and $\overline{y}$ is the area, which is $\int_0^\infty e^{-x} dx = 1$, as we saw in Example 13.3. Thus,

$$\overline{x} = \int_0^\infty x e^{-x} dx = -x e^{-x} \, \Big|_0^\infty - \int_0^\infty \left(-e^{-x}\right) \cdot 1 \, dx = 0 - 0 + \int_0^\infty e^{-x} dx = 1.$$

To compute $\overline{y}$, we must first write x in terms of y: $y = e^{-x} \implies x = -\ln y$. In the y-direction, the shape is contained between $y = 0$ and $y = 1$. Thus,

$$\overline{y} = \int_0^1 y(-\ln y) dy = -\left(\frac{1}{2} y^2 \ln y \, \Big|_0^1 - \int_0^1 \frac{1}{2} y^2 \cdot \frac{1}{y} \, dy \right) = \frac{1}{2} \int_0^1 y \, dy = \frac{1}{4}.$$

So the center of mass is at $(1, 0.25)$.

HOMEWORK 12.

1. Find the anti-derivative of:
(a) $x \sin(ax + b)$; (b) $x^2 \cos x$; (c) arctan x; (d) $x^3 e^x$; (e) x arctan x.

2. Find the center of mass of the uniform flat plate bounded by
(a) the x- and y-axes and the curve e^{kx} in the second quadrant (out to $-\infty$);
(b) the curves $y = -x^2 \ln x$ and $y = x^2 \ln x$;
(c) the curves $y = -x^n \ln x$ and $y = x^n \ln x$ (express your answer in terms of n);
(d) the curves $y = \sin^2(\pi x)$ and $y = -\sin^2(\pi x)$ to the left of $x = 1/4$.

3. Find the volume of the container obtained by rotating $x = y \sin y$ around the y-axis between $y = 0$ and $y = \pi/3$.

CHAPTER 14

Techniques of Integration III: Rational Functions

In this section we treat integration of **rational functions**. A "rational function" is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1}$$

are all rational functions of x. There is a general technique called " partial fractions" for integrating a rational function. The algebraic steps in that technique are rather cumbersome if the polynomial in the denominator has degree more than 2 (except in the case when the denominator has the form $(ax+b)^n$, a case we shall treat below). However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only in the two cases which are most frequently encountered: (a) the denominator has the form $(ax+b)^n$; (b) the denominator is a quadratic polynomial ax^2+bx+c.

In the case when the denominator has the form $(ax+b)^n$, the u-substitution $u=ax+b$ will always work. Just replace the denominator by u^n, replace x in the numerator by $(u-b)/a$, and replace dx by $\frac{1}{a}du$.

Example 14.1. Find

$$\int_0^1 \frac{t^3 dt}{(3-2t)^2}.$$

We set $u=3-2t$, so that $t=(3-u)/2$, $dt=-\frac{1}{2}du$. The limits of integration change: $t=0 \Longrightarrow u=3$, and $t=1 \Longrightarrow u=1$. So we obtain:

$$\begin{aligned} -\frac{1}{2}\int_3^1 \left(\frac{3-u}{2}\right)^3 \frac{du}{u^2} &= -\frac{1}{16}\int_3^1 \frac{(3-u)^3}{u^2}\,du = \frac{1}{16}\int_1^3 \frac{27-27u+9u^2-u^3}{u^2}\,du \\ &= \frac{1}{16}\int_1^3 \left(\frac{27}{u^2}-\frac{27}{u}+9-u\right)du \\ &= \frac{1}{16}\left(-\frac{27}{u}-27\ln u+9u-\frac{1}{2}u^2\right)\Big|_1^3 = 2-\frac{27}{16}\ln 3. \end{aligned}$$

We now proceed to the case when the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral. This will lead to a denominator in which x^2 has coefficient 1, i.e., a denominator of the form x^2+bx+c. There are three possible cases, depending on how many real roots the quadratic has. Recall the quadratic formula used to solve $ax^2+bx+c=0$, which when $a=1$ takes the form $r_1, r_2 = (-b\pm\sqrt{b^2-4c})/2$. The sign of the discriminant b^2-4c tells us whether there is (a) 1 double root (when the discriminant is 0), (b) two distinct roots (when the discriminant is positive), or (c) no real roots (when the discriminant is negative). In the first case, the denominator takes the form $(x-r)^2$, and we can integrate the rational function by the method explained above (substituting $u = x - r$). The remainder of this section is concerned with the other two possibilities.

1. Quadratic Denominator That Factors

We first make an algebraic observation about what happens when we subtract as follows:

$$\frac{1}{x-r_1} - \frac{1}{x-r_2} = \frac{1}{x-r_1}\cdot\frac{x-r_2}{x-r_2} - \frac{1}{x-r_2}\cdot\frac{x-r_1}{x-r_1} = \frac{r_1-r_2}{(x-r_1)(x-r_2)}.$$

We shall use this in the reverse order: instead of combining $1/(x-r_1)$ and $1/(x-r_2)$ into a single fraction, we shall split up $1/(x-r_1)(x-r_2)$ into a difference of two fractions. Namely, we divide through by $r_1 - r_2$ in the above equality:

$$\frac{1}{(x-r_1)(x-r_2)} = \frac{1}{r_1-r_2}\left(\frac{1}{x-r_1} - \frac{1}{x-r_2}\right).$$

This basic algebraic identity will allow us to split up our integral into two separate integrals, in each of which the denominator has the simpler form $x - r$. Here are some examples.

Example 14.2. Find

$$\int_1^\infty \frac{dx}{2x^2+x}.$$

Since $2x^2 + x = 2(x^2 + 0.5) = 2(x-0)(x-(-0.5))$, we take $r_1 = 0$, $r_2 = -0.5$. Then

$$\frac{1}{2x^2+x} = \frac{1}{2}\cdot\frac{1}{x^2+0.5} = \frac{1}{2}\cdot\frac{1}{0-(-0.5)}\left(\frac{1}{x-0} - \frac{1}{x-(-0.5)}\right) = \left(\frac{1}{x} - \frac{1}{x+0.5}\right).$$

Thus, our integral is

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+0.5}\right) dx = \big(\ln x - \ln(x+0.5)\big)\Big|_1^\infty .$$

(We used the formula $\int dx/(ax+b) = \frac{1}{a}\ln|ax+b| + C$, and we omitted the absolute value signs, because both x and $x + 0.5$ are positive between 1 and ∞.) To evaluate this antiderivative at ∞ means to take the limit of $\ln b - \ln(b+0.5)$ as $b \longrightarrow \infty$. Both $\ln b$ and $\ln(b+0.5)$ approach ∞, so we have the indeterminate form $\infty - \infty$. Write $\ln b - \ln(b+0.5) = \ln\left(\frac{b}{b+0.5}\right)$. Notice that $b/(b+0.5)$ has limit 1 as $b \longrightarrow \infty$. Thus, its ln has limit $\ln 1 = 0$. In other words, evaluating $\ln x - \ln(x+0.5)$ at ∞ gives 0. Thus, we conclude that $\int_1^\infty dx/(2x^2+x) = -(\ln 1 - \ln 1.5) = \ln 1.5$.

Example 14.3. Find the x-coordinate of the center of mass of the uniform flat plate bounded by the x- and y-axes, the line $x = 2$, and the curve $y = 1/(9-x^2)$.

Here $\overline{x}$ is the ratio of the integral $\int_0^2 x\,dx/(9-x^2)$ to the integral $\int_0^2 dx/(9-x^2)$. We first factor out the coefficient of x^2, which is -1, and write $9 - x^2 = -(x^2-9) = -(x+3)(x-3)$. Thus,

$$\begin{aligned}\frac{1}{9-x^2} &= -\frac{1}{(x-(-3))(x-3)} = -\frac{1}{(-3)-3}\left(\frac{1}{x-(-3)} - \frac{1}{x-3}\right)\\ &= \frac{1}{6}\left(\frac{1}{x+3} - \frac{1}{x-3}\right).\end{aligned}$$

So the bottom integral is

$$\frac{1}{6}\int_0^2 \left(\frac{1}{x+3} - \frac{1}{x-3}\right) dx = \frac{1}{6}\left(\ln(x+3) - \ln|x-3|\right)\Big|_0^2 .$$

(We need absolute values because $x-3$ is negative for x between 0 and 2.) The result is $\frac{1}{6}\ln 5$.

The top integral is the same, except with an x, i.e.,

$$\frac{1}{6}\int_0^2 \left(\frac{x}{x+3} - \frac{x}{x-3}\right) dx,$$

and we could integrate each of these terms without difficulty. However, in this particular integral there happens to be an easier way: notice that in the original integral $\int_0^2 x\,dx/(9-x^2)$ the derivative of $9-x^2$ is essentially what we have in the numerator. Hence, it is more efficient simply to substitute $u = 9 - x^2$, thereby transforming the top integral as follows ($du = -2x\,dx$, and the limits of integration become $9-0^2 = 9$ to $9-2^2 = 5$):

$$-\frac{1}{2}\int_9^5 \frac{du}{u} = -\frac{1}{2}(\ln 5 - \ln 9) = \frac{1}{2}(\ln 9 - \ln 5) = \frac{1}{2}\ln 1.8.$$

Finally, $\overline{x} = \frac{1}{2}\ln 1.8 \,/\, \frac{1}{6}\ln 5 = 3\ln 1.8/\ln 5 = 1.0956$.

EXAMPLE 14.4. Find the antiderivative of $x^4/(x^2-x-2)$.

If the numerator has degree 2 or more, then you can simplify the integral by first performing a long division of polynomials. In this case you get x^4 divided by x^2-x-2 is x^2+x+3 with a remainder of $5x+6$. Thus, the integral is

$$\int \left(x^2 + x + 3 + \frac{5x+6}{x^2-x-2}\right) dx.$$

To handle the fraction, we factor $x^2-x-2 = (x-2)(x+1)$. (If you can't see the factorization by inspection, then use the quadratic formula to determine the roots, which in this case are 2 and -1.) Then

$$\frac{5x+6}{x^2-x-2} = (5x+6)\cdot\frac{1}{2-(-1)}\left(\frac{1}{x-2} - \frac{1}{x+1}\right) = \frac{1}{3}\left(\frac{5x+6}{x-2} - \frac{5x+6}{x+1}\right).$$

Thus, the problem boils down to integrating $(5x+6)/(x-2)$ and $(5x+6)/(x+1)$. These integrals are done by the method described at the beginning of this section. For example, substituting $u = x-2$ in the first integral, we have:

$$\begin{aligned}\int \frac{5x+6}{x-2}\,dx &= \int \frac{5(u+2)+6}{u}du = \int \left(5 + \frac{16}{u}\right) du \\ &= 5u + 16\ln|u| + C = 5(x-2) + 16\ln|x-2| + C.\end{aligned}$$

The integral of $(5x+6)/(x+1)$ is similar.

2. Quadratic Denominator That Does Not Factor

As a preliminary procedure, we first factor out the coefficient of x^2, so that the quadratic denominator has the form x^2+bx+c. If the numerator has degree 2 or greater, we then use long division of polynomials to write our rational function in the form

$$\text{polynomial} + \frac{\alpha x + \beta}{x^2+bx+c}.$$

Next, we "complete the square" in the denominator: $x^2+bx+c = \left(x+\frac{1}{2}b\right)^2 + (c - \frac{1}{4}b^2)$, and we set $r = \sqrt{c - \frac{1}{4}b^2}$, so that $x^2+bx+c = \left(x+\frac{1}{2}b\right)^2 + r^2$. After a u-substitution of the form $u = x + \frac{1}{2}b$, we can transform the integral to a sum of two integrals of the form

$$\int \frac{u\,du}{u^2+r^2} = \frac{1}{2}\ln(u^2+r^2)+C, \qquad \text{and} \qquad \int \frac{du}{u^2+r^2} = \frac{1}{r}\text{Arctan}\left(\frac{u}{r}\right)+C.$$

We illustrate with two examples. In Example 14.5 we will already have the denominator in the form x^2+1, so we do not make a u-substitution. In Example 14.6 we will have to do some algebra to get the integral in the two standard forms $\int u\,du/(u^2+r^2)$ and $\int du/(u^2+r^2)$.

Example 14.5. Integrating $x^n\text{Arctan}\,x$.

Applying integration by parts in accordance with the table in the last section (setting $v'(x) = x^n$ and $u(x) = \text{Arctan}\,x$), we obtain

$$\int x^n \text{Arctan}\,x\,dx = \frac{x^{n+1}}{n+1}\text{Arctan}\,x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}\,dx.$$

To evaluate the last integral, divide x^2+1 into x^{n+1}, obtaining a polynomial quotient and a remainder. It turns out that the remainder will be x or $-x$ if n is an even number, and 1 or -1 if n is an odd number. Thus, after integrating the polynomial quotient, you will be left with either $\int x\,dx/(x^2+1) = \frac{1}{2}\ln(x^2+1)+C$, or else $\int dx/(x^2+1) =$Arctan $x + C$.

Example 14.6. Find the antiderivative of $x^3/(x^2+2x+5)$.

First, x^3 divided by x^2+2x+5 is $x-2$ with a remainder of $-x+10$. Next, completing the square gives $x^2+2x+5 = (x+1)^2+4 = (x+1)^2+2^2$, and so we substitute $u = x+1$ (then $x = u-1$, $dx = du$). Thus,

$$\begin{aligned}
\int \frac{x^3dx}{x^2+2x+5} &= \int \left(x-2+\frac{-x+10}{x^2+2x+5}\right)\,dx \\
&= \frac{1}{2}x^2 - 2x + \int \frac{-x+10}{(x+1)^2+4}\,dx \\
&= \frac{1}{2}x^2 - 2x + \int \frac{-(u-1)+10}{u^2+4}\,du \\
&= \frac{1}{2}x^2 - 2x - \int \frac{u\,du}{u^2+4} + 11\int \frac{du}{u^2+4} \\
&= \frac{1}{2}x^2 - 2x - \frac{1}{2}\ln(u^2+4) + \frac{11}{2}\text{Arctan}\left(\frac{u}{2}\right)+C \\
&= \frac{1}{2}x^2 - 2x - \frac{1}{2}\ln(x^2+2x+5) + \frac{11}{2}\text{Arctan}\left(\frac{x+1}{2}\right)+C.
\end{aligned}$$

HOMEWORK 13.

1. Find

(a) $\int_0^1 dx/(4-x^2)$; (e) $\int_0^1 x^4 dx/(4-x^2)$; (h) $\int_0^\infty dx/(x^2+10x+25)$;

(b) $\int_0^1 x^2 dx/(4-x^2)$; (f) $\int_0^1 x^4 dx/(4+x^2)$; (i) $\int_0^\infty dx/(x^2+10x+29)$;

(c) $\int_0^1 x^3 dx/(4+x^2)$; (g) $\int_0^\infty dx/(x^2+10x+21)$; (j) $\int_0^1 dx/(2x^2-x-3)$.

(d) $\int_1^\infty dx/(x^2+3x)$;

2. Find the center of mass of a uniform flat plate bounded by:

(a) the lines $x=0$ and $x=2$ and the curves
$y=1/(x^2+3x+2)$ and $y=-1/(x^2+3x+2)$;

(b) the line $x=1$, the curve $y=x^2 \mathrm{Arctan}\, x$, and the curve $y=-x^2 \arctan x$.

3. Find the anti-derivative of $\sec x=(\cos x)^{-1}$ by the method used before for odd powers of $\cos x$. That is, write $(\cos x)^{-1}=(\cos x)^{-2}\cos x=\cos x/(1-\sin^2 x)$. After making the appropriate u-substitution, you will have a rational function to integrate.

CHAPTER 15

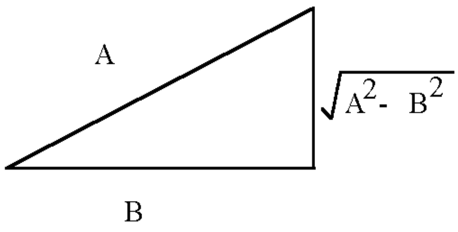
Techniques of Integration IV: Inverse Trig Substitution

The technique of " inverse trig substitution" is used when you want to integrate a function involving the square root of a quadratic polynomial, such as $\sqrt{25-x^2}$, $\sqrt{x^2-2x}$, $\sqrt{x^2+2x+2}$, etc. It is an "inverse" substitution because, instead of writing a new variable u as a function of x (i.e., $u = u(x)$), you write x in terms of a new variable θ (i.e., $x = x(\theta)$), where the new variable θ stands for an angle in a right triangle. Here is the procedure:

1. Take the quadratic which is under the radical and write it either as a sum of squares $A^2 + B^2$ or as a difference of squares $A^2 - B^2$, where one of the terms A or B is a constant and the other is either x or a multiple of x (or at worst an expression of the form $ax + b$). In the three examples above, we select A and B as follows:
$$25 - x^2 = 5^2 - x^2, \qquad x^2 - 2x = (x-1)^2 - 1^2, \qquad x^2 + 2x + 2 = (x+1)^2 + 1^2.$$
2. Draw a right triangle, and label two of the sides A and B. If the quadratic under the radical is $A^2 + B^2$, then take the two sides to be the legs of the right triangle. If the quadratic is $A^2 - B^2$, then take A to be the hypotenuse and B to be one of the legs. In this way the radical itself (i.e., $\sqrt{A^2+B^2}$ or else $\sqrt{A^2-B^2}$) will be the third side.
3. Let θ be one of the acute angles of the right triangle (either one will do). Then write everything in the integral — x, dx, and the square root — in terms of the new variable θ (also change the limits of integration, if you're evaluating a definite integral). Hopefully, the new integral with θ as the variable of integration will be relatively easy to evaluate.

We now give some examples of the use of this method.

EXAMPLE 15.1. Find the area in the first quadrant that is bounded by the x- and y-axes, the line $x = 3$, and the circle of radius 6 centered at the origin.



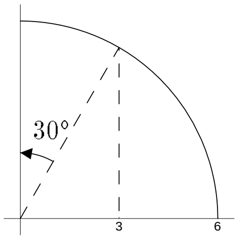
To evaluate $\int_0^3 \sqrt{36-x^2}dx$, we go through the three steps above. (1) Set $A = 6$ and $B = x$, so that the radical is $\sqrt{A^2-B^2}$. (2) Draw a right triangle with the sides and angle labeled as shown (see the figure at the right)
(3) Write $x = 6\sin\theta$, so that $dx = 6\cos\theta\, d\theta$. In addition, the square root is the adjacent side, so that $\sqrt{36-x^2} = 6\cos\theta$. Since $\theta = \arcsin(x/6)$, the limits of integration become

$\arcsin(0) = 0$ and $\arcsin(3/6) = \pi/6$. Thus,

$$\begin{aligned}\int_0^3 \sqrt{36-x^2}\,dx &= \int_0^{\pi/6} (6\cos\theta)6\cos\theta\,d\theta = 36\int_0^{\pi/6}\cos^2\theta\,d\theta\\ &= 36\int_0^{\pi/6}\frac{1+\cos(2\theta)}{2}\,d\theta = 36\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/6}\\ &= 3\pi + 9\sin(\frac{\pi}{3}) = 3\pi + \frac{9}{2}\sqrt{3}.\end{aligned}$$

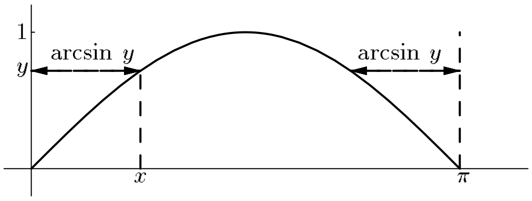
We could have actually obtained the same result by some simple geometry. The region in question is made up of a 30° sector of a circle (i.e., 1/12 of a circle of radius 6) and a triangle of base 3 and altitude $3\sqrt{3}$ (see the figure at the right). So the area is

$$\frac{1}{12}\pi\cdot 6^2 + \frac{1}{2}3\cdot 3\sqrt{3} = 3\pi + \frac{9}{2}\sqrt{3}.$$



Example 15.2. Find the center of mass of a uniform flat plate in the shape of one hump of the sine curve.

By considerations of symmetry, $\overline{x}$ is clearly $\pi/2$. To find $\overline{y}$, we must find the distance across from left to right at height y. From the diagram at the right, it should be clear that this distance is $\pi - 2\arcsin(y)$. The bottom integral in $\overline{y}$ is just the area under $y = \sin x$ from 0 to π, which is 2. So it remains to compute the top integral



$$\int_0^1 y(\pi - 2\arcsin y)\,dy = \frac{\pi}{2} - 2\int_0^1 y\arcsin y\,dy\,.$$

Using integration by parts, we transform the last integral as follows:

$$\int_0^1 y\arcsin y\,dy = \frac{1}{2}y^2\arcsin y\,\Big|_0^1 - \int_0^1 \frac{1}{2}y^2\frac{1}{\sqrt{1-y^2}}dy = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{y^2dy}{\sqrt{1-y^2}}.$$

We now use an inverse trig substitution, drawing our triangle so that 1 is the hypotenuse and y is the leg opposite the angle θ. We have $y = \sin\theta$, $dy = \cos\theta\,d\theta$, $\sqrt{1-y^2} = \cos\theta$, and the limits of integration become 0 to $\pi/2$. Putting this all together, we find that the top integral in $\overline{y}$ is equal to

$$\begin{aligned}\frac{\pi}{2} - 2\int_0^1 \arcsin y\,dy &= \frac{\pi}{2} - 2(\frac{\pi}{4} - \frac{1}{2}\int_0^1 (y^2dy/\sqrt{1-y^2}))\\ &= \int_0^1 (y^2dy/\sqrt{1-y^2}),\end{aligned}$$

which with the substitution $y = \sin\theta$ becomes

$$\int_0^{\pi/2} \frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta} = \int_0^{\pi/2} \sin^2\theta\,d\theta$$
$$= \int_0^{\pi/2} \frac{1-\cos(2\theta)}{2} d\theta = \left(\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right)\Big|_0^{\pi/2} = \pi/4\,.$$

We conclude that $\overline{y} = \pi/8 = 0.3927$.

Using this type of procedure — integration by parts followed by inverse trig substitution — one can integrate $x^n \arcsin x$ for any positive integer n.

Example 15.3. Find the antiderivative of $1/\sqrt{x^2-2x}$.

As shown above, we can write $x^2 - 2x$ as a difference of two squares:

$$x^2 - 2x = (x-1)^2 - 1^2\,.$$

So we choose the hypotenuse of the right triangle to be $x-1$ and the leg opposite θ to be 1 (see the drawing at the right). Then $\sin\theta = 1/(x-1)$ and so $x = 1 + (1/\sin\theta)$, $dx = -(\cos\theta/\sin^2\theta)\,d\theta$, and the radical is $\cot\theta = \cos\theta/\sin\theta$. Hence,

$$\int \frac{dx}{\sqrt{x^2-2x}} = \int \frac{-(\cos\theta/\sin^2\theta)d\theta}{\cos\theta/\sin\theta} = -\int \frac{d\theta}{\sin\theta}.$$

This integral is very similar to Problem 3 in the last homework set (just reverse the roles of sin and cos). The result (taking into account the minus sign in front of the integral) is: $\frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} + C$. Finally, we can write the answer in terms of x by reading $\cos\theta$ off the triangle: $\cos\theta = \sqrt{x^2-2x}/(x-1)$. (Note: The answer can be simplified algebraically, using the properties of ln; the result is: $\ln|x - 1 + \sqrt{x^2-2x}| + C$.)

Example 15.4. Find $\int x\,dx/\sqrt{x^2+2x+2}$.

As mentioned earlier, we can write $x^2 + 2x + 2$ as a sum of two squares: $x^2 + 2x + 2 = (x+1)^2 + 1^2$. So we choose $x+1$ to be one leg (say, the leg opposite θ) and 1 to be the other leg of the right triangle. Then $x + 1 = \tan\theta$, and so $x = \tan\theta - 1$, $dx = \sec^2\theta\,d\theta$. In addition, $\sqrt{x^2+2x+2}$ is the hypotenuse of the triangle, which is $\sec\theta$. So our integral becomes

$$\int \frac{(\tan\theta - 1)\sec^2\theta\,d\theta}{\sec\theta} = \int \tan\theta\sec\theta\,d\theta - \int \sec\theta\,d\theta.$$

The function inside the first integral happens to be the derivative of $\sec\theta$. (If you didn't notice this, you could convert the integral to sines and cosines, i.e., to $\int \sin\theta\,d\theta/\cos^2\theta$, and use the substitution $u = \cos\theta$.) The second integral is Problem 3 of the last homework set. Putting this together (and using the fact that $\sec\theta = \sqrt{x^2+2x+2}$ and $\sin\theta = (x+1)/\sqrt{x^2+2x+2}$), we obtain the following answer:

$$\int \frac{x\,dx}{\sqrt{x^2+2x+2}} = \sqrt{x^2+2x+2} + \frac{1}{2}\ln\frac{\sqrt{x^2+2x+2}+x+1}{\sqrt{x^2+2x+2}-(x+1)} + C.$$

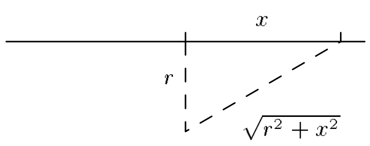
Example 15.5. (Force of repulsion from a line source of electric charge.)

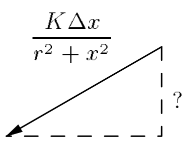
Suppose that we have a long straight wire with an electric charge uniformly distributed along it. Our purpose is to determine how the force of repulsion exerted on a similarly charged particle drops off as the particle moves away from the wire.

The basic physical law needed here is: *the force of repulsion between two particles of charge q_1 and q_2 has magnitude $k\,q_1q_2/distance^2$, where k is a universal constant.* Before doing this example, we should review Example 9.7 in the section on improper integrals (noise from the freeway). There also we had an inverse square law. As in the noise example, we set up our wire as the x-axis, taking 0 to be the point at the foot of the perpendicular drawn from the charged particle. If c stands for the amount of charge per unit distance along the wire, then the contribution to the force of repulsion coming from the small segment of wire between x and $x+\Delta x$ is $k\,q_1(c\Delta x)/\text{distance}^2 = K\Delta x/(r^2+x^2)$, where we have set $K = kq_1c$. (See the diagram in the freeway example.)

However, there is one basic difference between the two examples. In the freeway problem we were dealing with a **scalar** (amount of noise) rather than force, which is a **vector** (it has both magnitude and direction). This means that the force contributed by the segment of the wire between x and $x+\Delta x$ points at an angle away from the vertical. What is that angle? You should convince yourself that the angle is $\arctan(x/r)$. The second diagram to the right below shows the force vector contributed by the part of the wire between x and $x+\Delta x$.

The horizontal component of this force (i.e., the component parallel to the wire) is canceled by the force contributed from the opposite segment (the segment between $-x$ and $-x-\Delta x$). So the contribution to the force of repulsion (the component pointing away from the wire) is only a portion of $K\Delta x/(r^2+x^2)$. The portion can be determined by similar triangles (see the diagram at the right). Comparing the two triangles, we find that the side "?" which we want to find has the same ratio to $K\Delta x/(r^2+x^2)$ as r does to $\sqrt{r^2+x^2}$. Thus, the contribution of the small segment is equal to:





$$\frac{r}{\sqrt{r^2+x^2}}\cdot\frac{K\Delta x}{r^2+x^2} = \frac{Kr\Delta x}{(r^2+x^2)^{3/2}}.$$

Proceeding as in the freeway example and making the simplifying assumption that the wire extends from $-\infty$ to ∞, we obtain the following definite integral for the force of repulsion: $\int_{-\infty}^{\infty} Kr\,dx/\left(r^2+x^2\right)^{3/2}$. This is an integral that can be evaluated by an inverse trig substitution, taking x and r to be the legs of the right triangle (say, x is the leg opposite θ). Then $x = r\tan\theta$, $dx = r\sec^2\theta\,d\theta$, $\sqrt{r^2+x^2} = r\sec\theta$. The limits of integration become

$-\pi/2$ to $\pi/2$, so we obtain:

$$\int_{-\infty}^{\infty} \frac{Kr\,dx}{\left(\sqrt{r^2+x^2}\right)^3} = \int_{-\pi/2}^{\pi/2} \frac{Kr^2\sec^2\theta\,d\theta}{(r\sec\theta)^3} = \frac{K}{r}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \frac{2K}{r}.$$

Hence we conclude (as in the freeway example) that the force drops off proportionally to the distance (not the square of the distance).

1. A Final Word

Before leaving the subject of techniques of integration, we should point out that sometimes you have a choice of techniques, and your answer using one technique might at first glance seem to be different from your answer if you use a different technique.

Example 15.6. Find $\int \sin x\cos x\,dx$.

First method. Substitute $u = \sin x$, so that $du = \cos x\,dx$. We obtain: $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$.

Another method. Substitute $u = \cos x$, so that $du = -\sin x\,dx$. Then we get: $-\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$.

A third method. Use the trig identity for the sine of twice an angle: $\sin(2x) = 2\sin x\cos x$. We can thus rewrite our integral as $\int \frac{1}{2}\sin(2x)dx = -\frac{1}{4}\cos(2x) + C$.

Why three different answers? The point is that the three answers are not really different. Namely, the second answer can be written $-\frac{1}{2}(1-\sin^2 x)+C = \frac{1}{2}\sin^2 x + C - \frac{1}{2}$; and the third answer (if we use the cosine double angle formula $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$) can be written as follows: $-\frac{1}{4}(1-2\sin^2 x) + C = \frac{1}{2}\sin^2 x + C - \frac{1}{4}$. So the second and third answers differ from the first only by constants. But remember that the meaning of "$+C$" in the answer to an indefinite integral is that if you add any constant to the answer, you still have an antiderivative, i.e., the new function is still a correct answer.

HOMEWORK 14.

1. Use inverse trig substitution to find $\int dx/\sqrt{r^2 - x^2}$, where r is a constant (we evaluated this earlier using a u-substitution).

2. Find the anti-derivative of

(a) $1/\sqrt{x^2 + r^2}$,

(b) $1/\sqrt{x^2 - r^2}$.

3. Find the center of mass of the uniform flat plate bounded by the curve $y = x$ arcsin x, the curve $y = -x$ arcsin x, and the line $x = 0.5$.

4. Find the x-coordinate of the center of mass of the uniform flat plate bounded by the x- and y-axes, the line $x = 2$, and the curve $y = 1/\sqrt{x^2 + 6x + 13}$.

5. Find $\int_0^\infty dx/\left((x+2)\sqrt{x^2 + 4x + 5}\right)$.

CHAPTER 16

Differential Equations I: Separation of Variables

The differential equations in this section will be of a slightly more complicated form than in the section on exponential growth and decay, where we treated the differential equation $y' = \pm ky$ ($+k$ leading to exponential growth, and $-k$ leading to exponential decay). Namely, we shall deal with differential equations of the form $y' = \pm k(y-b)$, where b is some constant. Before discussing the applications, let's solve the differential equation $\frac{dy}{dt} = k(y-b)$ by the same method — separation of variables — that we used before. We first multiply both sides by dt and divide through by $y-b$, and we then put an integral sign in front of both sides:

$$\begin{aligned}\int \frac{dy}{y-b} &= \int k\,dt\\ \ln|y-b| + C_1 &= kt + C_2\\ \ln|y-b| &= kt + C \quad (\text{where } C = C_2 - C_1)\\ |y-b| &= e^{kt+C} = e^C e^{kt}\\ y-b &= \pm e^C e^{kt}.\end{aligned}$$

To find the constant $\pm e^C$ in front of e^{kt}, we suppose that we have the initial information $y(0) = y_0$; then substituting $t=0$ on both sides gives us: $y_0 - b = \left(\pm e^C\right) e^0 = \pm e^C$, and so we can write $y - b = (y_0 - b)e^{kt}$, i.e.,

The differential equation $\frac{dy}{dt} = k(y-b)$ has solution $y(t) = b + (y_0 - b)e^{kt}$.

If our differential equation is of the form $\frac{dy}{dt} = -k(y-b)$ with a negative constant of proportionality $-k$, then the solution is exactly the same with k replaced by $-k$. Thus,

The differential equation $\frac{dy}{dt} = -k(y-b)$ has solution $y(t) = b + (y_0 - b)e^{-kt}$.

1. Installment Buying

Large purchases (cars, houses, furniture, etc.) are often made according to the following system. When you make the purchase (time $t = 0$), you pay only a small fraction of the price (called a "downpayment") and agree to pay the remaining amount — which we shall denote y_0 — over a certain period of time. In essence, you are borrowing the y_0 dollars from the seller (or a mortgage company in the case of a house). We shall let T denote the term of this loan (in years), i.e., the time during which you continue to make payments, and we shall let a denote the rate in dollars per year at which you pay money during this time T. (If you make monthly payments, then a equals 12 times the amount you pay each month.) If the seller is willing to loan you the y_0 interest free, then the situation is very simple: $a = y_0/T$. But in practice you must pay back not only y_0 to make up the purchase price, but also interest on the money you owe. Let k be the instantaneous rate of interest.

In order to treat this problem by a differential equation, we shall make two simplifying assumptions:

1. The interest is compounded continuously. (This is the same simplifying assumption that we made in the section on exponential growth.)
2. Your payments are being made continuously (rather than at the end of the month, as usually happens in practice). For example, in the case of monthly payments of \$300, this would mean, roughly speaking, that instead of waiting until the end of the month, you arrange to have your bank's computer transfer one penny from your account to the seller's account every 86.4 seconds.

Even though both of these simplifying assumptions seem to diminish the realism of our treatment of installment buying, in practice the inaccuracy introduced is very small — an error of considerably less than 1%.

From a practical point of view, perhaps the most useful thing to figure out is the monthly payment $a/12$, which depends on y_0 (which is purchase price minus downpayment), the term T, and the interest rate k. But there are other questions which might also interest us. In any case, the way to approach the matter is to introduce the function

$$y(t) = \text{the amount you still owe at time } t.$$

This amount $y(t)$ starts out at $y(0) = y_0$, and steadily decreases until at time $t = T$ you've paid off the loan, i.e., $y(T) = 0$. This function $y(t)$ satisfies a differential equation, which we first state in words: *the instantaneous rate of change of the amount $y(t)$ that you owe is the result of two contributions: (1) a positive contribution caused by interest, which accumulates at a rate equal to k times the current amount you owe, and (2) a negative contribution of a dollars/year, as you make payments to decrease your debt.*

Written mathematically, the first contribution is ky and the second contribution is $-a$. Thus, the function $y(t)$ satifies the differential equation

$$\frac{dy}{dt} = ky - a.$$

If we rewrite the right side in the form $k(y - \frac{a}{k})$, we see that this is the same type of differential equation that we already solved at the beginning of the section. We can simply take the solution we already found, where we substitute $b = a/k$. The result is:

$$y(t) = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{kt}.$$

We now want to interpret this result to obtain useful information. First, let us find the rate of payment a (or the monthly payment rate $a/12$) as a function of y_0, T, and k. The basic fact we use is that the amount you owe must be reduced to 0 when $t = T$. That is,

$$0 = y(T) = \frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{kT}.$$

To solve this for a, we do a little algebra. We start out by dividing through by e^{kT}, to make things look slightly simpler (no parentheses needed):

$$0 = \frac{a}{k}e^{-kT} + y_0 - \frac{a}{k}.$$

(Recall that the reciprocal of e^{kT} is e^{-kT}.) Since a is the unknown, we move all the terms with a over to the left, and factor out a:

$$\frac{a}{k}\left(1 - e^{-kT}\right) = y_0.$$

Now it is easy to solve, by multiplying both sides by k and dividing both sides by $1-e^{-kT}$:

$$a = \frac{ky_0}{1-e^{-kT}}.$$

EXAMPLE 16.1. A few years ago the State of Washington had a 12% ceiling on interest rates, while neighboring Oregon had no such limit. During this time an ad for a \$450 dinette set running in Pacific Northwest newspapers said: no money down; residents of Washington take 32 months to pay at 12% interest; residents of Oregon make the same monthly payments as in Washington, although their interest rate will be 18%. Find the monthly payment that residents of both states make, and also find out how much beyond the 32 months must Oregonians continue to make payments because of their higher interest rate.

In Washington, $T = 32/12$ years, $k = 0.12$, and $y_0 = 450$. Hence,

$$\frac{a}{12} = \frac{1}{12}\cdot 0.12\cdot 450 \Big/ \left(1-e^{-0.12\cdot 32/12}\right) = \$16.43.$$

In Oregon, we now know the value of a (namely, $12\cdot 16.4323 = \$197.19$), and we want to find T. So we return to the equation $y(T) = 0$ above, and instead of solving for a we solve for T. Moving the term with T over to the left, we have

$$\left(\frac{a}{k}-y_0\right)e^{kT} = \frac{A}{k}.$$

We divide both sides by $((a/k)-y_0)$, simplify the fraction on the right by multiplying by k/k, and then take the ln of both sides. The result is

$$kT = \ln\left(\frac{a}{a-ky_0}\right), \quad \text{i.e.,} \quad T = \frac{1}{k}\ln\left(\frac{a}{a-ky_0}\right).$$

Substituting $k = 0.18$, $a = 197.19$, and $y_0 = 450$, we obtain: $T = 2.9386$ years $= 35.26$ months. That is, Oregonians must continue paying for about 3 1/4 months more.

The next example deals with home mortgages. By a "mortgage" we shall always mean a fixed-rate, fixed-payment mortgage, i.e., the standard kind. Adjustable-rate mortgages and other variants will not be examined here.

EXAMPLE 16.2. To buy a house for \$71,500.00, you make a downpayment of 20% and take a 30-year mortgage for the remaining 80% at 11% interest. Find your monthly payments $a/12$, your yearly rate of payment a, and the total amount you end up paying for the house. Also compute how much of the money paid in the first year is tax-deductible, and how much of the money paid in the 30th year is tax-deductible. (Note: money paid toward interest is deductible from U.S. federal income tax, while money that goes toward reducing the amount you owe is not.)

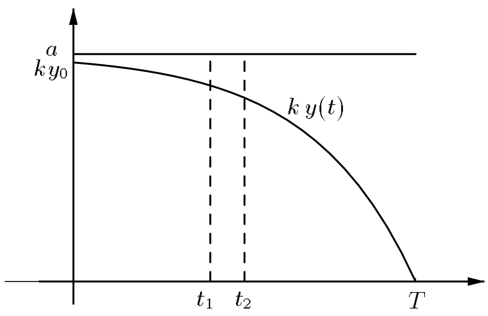
Here $y_0 = 0.8\cdot 71500 = 57200$, and so

$$a = \frac{0.11\cdot 57200}{1-e^{-0.11\cdot 30}} = \$6532.96; \qquad \frac{a}{12} = \$544.41.$$

The amount you end up paying for the house is equal to the downpayment plus aT, i.e., $\$14300 + 30\cdot\$6532.96 = \$210289$, which is about three times the price of the house! (This isn't as bad as it seems, for at least two reasons: (1) a significant part of the \$138789 that goes for interest is being paid by the government, in the form of reduced income tax, and (2) the dollars that you pay 20 or 30 years later are dollars whose real value has been reduced by inflation.) Next, we want to find out how much of the \$6532.96 paid between $t = 0$ and $t = 1$ is tax-deductible, and how much of the \$6532.96 paid between $t = 29$ and $t = 30$ is tax-deductible. The simplest way to do this is to ask how much is *not* tax-deductible, and that is the amount which has gone to decrease your indebtedness. Thus, the amount in the first year that is not tax-deductible is

$$\begin{aligned} y(0) - y(1) &= \left(\frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{k\cdot 0}\right) - \left(\frac{a}{k} + \left(y_0 - \frac{a}{k}\right) e^{k\cdot 1}\right) \\ &= \left(y_0 - \frac{a}{k}\right)\left(1 - e^k\right) = \left(\frac{a}{k} - y_0\right)\left(e^k - 1\right) \\ &= \left(\frac{6532.96}{0.11} - 57200\right)\left(e^{0.11} - 1\right) = \$254.71. \end{aligned}$$

The remaining $6532.96 - 254.71 = \$6278.25$ is tax-deductible. In the 30th year, on the other hand, the non-tax-deductible amount is $y(29) - y(30) = y(29) - 0 = y(29) = \$6185.64$, and so only \$347.32 is deductible in the last year.



What is happening is that over the term of the mortgage the way your a dollars per year payment divides up is changing. At the beginning, when you still owe a lot of money, most of the a goes toward interest payment, and only very little goes toward reducing the principal. As time goes on, more and more of the a goes toward reducing the principal, and less and less of it goes toward interest payment. This can be represented visually by a graph (see the figure at the right) that depicts what our original differential equation is saying. The graph shows the horizontal line a along with the curve $ky(t)$. The differential equation says that at any instant, of the a dollars/year that you pay, $ky(t)$ of it must go toward interest, and the remaining $a - ky$ goes toward reducing the amount you owe. In symbols, $dy/dt = -(a - ky)$ (the minus sign appears because we are *decreasing* y). Over any time interval from t_1 to t_2, the amount you pay is $a(t_2 - t_1)$, which is the area under the horizontal line from t_1 to t_2. Of that area, the area under the curve $ky(t)$ is the amount that goes toward interest (the tax-deductible amount), and the area between the curve and the horizontal line is the amount that goes toward paying off the principal.

Recall that we made two simplifying assumptions at the very beginning. Since Example 16.2 is based on an actual purchase, we can show the actual values computed by the mortgage company, based on daily compounding and payments made at the end of each month. Our error turns out to be quite small: the true monthly payments for this mortgage were \$545.12. We were off by \$0.71, which is only 0.13% of the total.

Below we show the table summarizing a statement sent us by the mortgage company for the 8th year. This statement gives the same type of information as the above graph between $t = 7$ and $t = 8$. The first column has the constant value $a/12$ that is paid each month. The fourth column shows how much of this has gone toward the principal, and the sixth column shows the difference, which has gone toward interest. The sum of the amounts in the sixth column is the "interest total" given at the bottom — the amount that is tax-deductible.